Piecewise regular solutions to scalar balance laws with singular nonlocal sources

L. Bociu, E. Ftaka, K. T. Nguyen, and J. Schino

Department of Mathematics, North Carolina State University
e-mails: lvbociu@ncsu.edu, eftaka@ncsu.edu, khai@math.ncsu.edu, jschino@ncsu.edu

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Abstract

The present paper establishes a local well-posed result for piecewise regular solutions with single shock of scalar balance laws with singular integral of convolution type kernels. In a neighborhood of the shock curve, a detailed description of the solution is provided for a general class of initial data.

Keyword. Piecewise smooth solutions, scalar balance law, singular kernels, shock

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1 Introduction

We consider a scalar balance law in one space dimension with a singular source term

$$u_t + f(u)_x \;=\; \mathbf{G}[u], \tag{1.1}$$

where $u : [0,\infty[\times\mathbb{R} \to \mathbb{R}$ is the state variable, $f : \mathbb{R} \to \mathbb{R}$ is a C^4 strictly convex flux, i.e.,

$$\theta \cdot f(x_1) + (1-\theta)\cdot f(x_2) \;>\; f(\theta\cdot x_1 + (1-\theta)\cdot x_2), \quad \theta \in]0,1[, \quad x_2 \neq x_1, \tag{1.2}$$

and $\mathbf{G}$ is a singular integral of convolution type defined by convolution with a kernel K that is locally integrable on $\mathbb{R}\backslash\{0\}$, in the sense that

$$\mathbf{G}[g](x) \;=\; \lim_{\varepsilon\to 0+}\int_{|y-x|>\varepsilon} K(x-y)\cdot g(y)\, dy. \tag{1.3}$$

We work under the following assumptions on $\mathbf{G}$, as it is typically done in applications:

(H1) $\mathbf{G} : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R})$ is a bounded operator;

(H2) The kernel K is in $\mathcal{C}^2(\mathbb{R}\backslash\{0\})$ and satisfies

$$\left|K^{(i)}(x)\right| \;\leq\; \frac{C}{|x|^{i+1}} \qquad \text{for all } i = 0,1,2. \tag{1.4}$$

Under the assumption **(H2)**, the $\mathbf{L}^\infty$ bound on the Fourier transform of K ensures the continuity condition **(H1)** (see e.g in [15]). In particular, assumption **(H1)** holds if the kernel K takes the form of

$$K = K_1 + K_2, \qquad \text{with } K_2 \in \mathbf{L}^1(\mathbb{R}),$$

and the singular part K_1 is odd and satisfies **(H2)**. Equation (1.1) has an interesting structure since the scalar conservation law generates a contractive semigroup on $\mathbf{L}^1(\mathbb{R})$, but the operator $\mathbf{G}$ may be discontinuous and unbounded as an operator on $\mathbf{L}^1(\mathbb{R})$. In the archetypal case when

$$f(u) = \frac{u^2}{2}, \qquad K(x) = \frac{1}{\pi x},$$

equation (1.1) is well-known as the Burger-Hilbert equation which was introduced by Biello and Hunter in [1] as a model for surface waves with constant frequency. A lower bound on the maximal time of existence for smooth solutions was studied in [1, 12, 13], the formation of singularities and the local asymptotic behavior of a solution up to the time when a new shock is formed in finite time was investigated in [6, 17], and the global existence of entropy weak solutions was proved in [3], together with a partial uniqueness result. Recently, piecewise regular solutions with a single shock for the "well-prepared" initial data have been constructed in [4, 16]. To complete an asymptotic description of a solution to the Burgers-Hilbert equation in a neighborhood of a point y_0 where two shocks interact in [5], the result was extended to a bigger class of initial data

$$u(0,x) = \overline{w}(x-y_0) + \left(c_1 \cdot \chi_{]-\infty,y_0[} + c_2 \cdot \chi_{]y_0,+\infty[}\right) \cdot \psi(x-y_0),$$

for some $\overline{w} \in H^2(\mathbb{R}\backslash\{0\})$, constants $c_1, c_2 \in \mathbb{R}$, and $\psi(x) \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\})$ being a fixed even function with compact support, smooth outside the origin and satisfying

$$\psi(x) = \frac{2}{\pi} \cdot |x| \ln|x| \qquad \text{for all} \quad |x| \le 1.$$

In the present article, we study the unique piecewise regular solution with a single shock of (1.1) with a general class of initial data of the form

$$u(0,x) = \overline{w}(x-y_0) + \bar{v}(x-y_0), \qquad \overline{w} \in H^2(\mathbb{R}\backslash\{0\}), \tag{1.5}$$

with for some $3/4 < \alpha < 1$,

$$\begin{split}\bar{v}(x) \in \mathcal{X}_\alpha \doteq \Bigg\{ v \in \mathcal{C}_c^0(\mathbb{R}) \cap \mathcal{C}^4(\mathbb{R}\backslash\{0\}) : \quad \sup_{x\neq 0} \frac{|x|^{-\alpha}}{1+|x|^{-\alpha}} \big|v(x)\big| < +\infty, \\ \sup_{x\neq 0} \frac{|x|^{i-\alpha}}{1+|x|^{i-\alpha}} \big|v^{(i)}(x)\big| < +\infty \qquad i \in \{1,2,3,4\} \Bigg\}.\end{split} \tag{1.6}$$

Intuitively, $\mathcal{X}_\alpha$ is the space of functions that have an arbitrarily large derivative as $x \to 0\pm$ and grow like $|x|^\alpha$. However, this does not lead to the formation of additional shocks. Indeed, one expects that characteristics fall into the large shock before the blow-up of the gradient. As in [4, 5], the solutions are more regular than the usual weak entropy solutions and can be determined by integrating along characteristics. These correspond to the "broad solutions" considered in [2, 14].

Below we recall the notion of piecewise regular solutions for scalar balance laws.

Definition 1.1 *A function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is called a* **piecewise regular solution** *of (1.1) if there exist finitely many shock curves $y_1(t), \ldots, y_n(t)$ such that the following holds.*

(i) The map $t \mapsto u(t,\cdot) \in H^1(\mathbb{R}\backslash\{y_1(t),\ldots,y_n(t)\}) \cap H^2_{loc}(\mathbb{R}\backslash\{y_1(t),\ldots,y_n(t)\})$ satisfies

$$\sup_{t\in[0,T]} \Big(\|u(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{y_1(t),\ldots,y_n(t)\})} + \|u(t,\cdot)\|_{H^2(\mathbb{R}\backslash(\bigcup_{i=1}^n [y_i(t)-\delta, y_i(t)+\delta]))} \Big) < \infty$$

for every $\delta > 0$ sufficiently small.

(ii) For each $i = 1, \ldots, n$, the Rankine-Hugoniot conditions hold:

$$u_i^-(t) \doteq u(t, y_i(t)-) > u(t, y_i(t)+) \doteq u_i^+(t), \tag{1.7}$$

$$\dot{y}_i(t) = \frac{f(u_i^-(t)) - f(u_i^+(t))}{u_i^-(t) - u_i^+(t)}. \tag{1.8}$$

(iii) Along every characteristic curve $t \mapsto x(t)$ such that $\dot{x}(t) = f'(u(t,x(t)))$, one has

$$\frac{d}{dt} u\big(t, x(t)\big) = \mathbf{G}[u](x(t)). \tag{1.9}$$

We note that in the above definition, as well as throughout the sequel, the upper dot denotes a derivative with respect to time.

The remainder of the paper is structured as follows. Our main theorem is presented in 2, along with the main steps of the proof. Section 3 develops various key a priori estimates on the source term which are necessary in the remaining steps of the proof. In Section 4 we construct the unique local piecewise regular solution to [(1.1),(1.5)] as the limit of a convergent sequence of approximations. We conclude with an Appendix which contains some basic estimates on the singular kernel, the corrector term, and related functions.

2 Main result

We establish the local existence and uniqueness of a piecewise regular solution with a single shock to the general scalar balance law with nonlocal singular sources (1.1) for a large class of initial data defined in (1.5)-(1.6):

$$\begin{cases} u_t + f(u)_x = \mathbf{G}[u], \\ u(0,x) = \overline{w}(x - y_0) + \bar{v}(x - y_0), \quad \text{with } \overline{w} \in H^2(\mathbb{R}\backslash\{0\}), \text{ and } \bar{v} \in \mathcal{X}_\alpha. \end{cases}$$

Our main theorem is presented below.

Theorem 2.1 *Given $y_0 \in \mathbb{R}$ and $\bar{v} \in \mathcal{X}_\alpha$ with $\alpha \in (3/4, 1)$, for every $\overline{w} \in H^2\big(\mathbb{R}\backslash\{0\}\big)$ such that $\overline{w}(0-) > \overline{w}(0+)$, the Cauchy problem [(1.1),(1.5)] admits a unique piecewise regular solution u defined for $t \in [0,T]$, for some $T > 0$ sufficiently small. Moreover, the map $t \mapsto u(t, 0\pm)$ is locally Lipchitz and satisfies*

$$|\dot{u}(t, 0\pm)| \leq 2\Gamma_1 t^{\alpha-1} \quad a.e.\ t \in [0,T], \tag{2.1}$$

for some constant $\Gamma_1 > 0$.

Remark 2.1 *The local existence and uniqueness result can be extended to the case of solutions with finitely many non-interacting shocks. Moreover, our result can be applied to both Fornberg-Whitham equation [9] and Burgers-Possion equation [8, 10, 11].*

Remark 2.2 *We point out that the lower bound $3/4$ on the constant α in the definition of $\mathcal{X}_\alpha$ is somehow sharp within our analysis. The best lower bound on α remains an open question.*

The main steps in the proof of our main theorem are introduced below, while the details are provided in the subsequent sections.

2.1 New coordinate system for solutions with one shock

The first step in the proof of our main theorem consists in transferring the equation (1.1) to a new coordinate system so that the location of the shock of the constructed piecewise regular solution always remains at the origin. The details for the change of coordinates are included below.

Assume that u is a piecewise regular solution of the balance law (1.1), with one single shock. By the Rankine-Hugoniot condition in (1.8), the location $y(t)$ of the shock at time t satisfies

$$\dot{y}(t) \;=\; \frac{f(u^-(t)) - f(u^+(t))}{u^-(t) - u^+(t)}, \qquad u^{\pm}(t) \;=\; \lim_{x \to y(t)\pm} u(t,x).$$

As in [4, 5], we shift the space coordinate, by replacing x with $x - y(t)$, so that in the new coordinate system the shock is always located at the origin. In these new coordinates, the Cauchy problem [(1.1), (1.5)] becomes

$$u_t + \left(f'(u) - \frac{f(u^-(t)) - f(u^+(t))}{u^-(t) - u^+(t)} \right) \cdot u_x \;=\; \mathbf{G}[u], \tag{2.2}$$

with initial data of the form

$$u(0,x) \;=\; \overline{w}(x) + \bar{v}(x), \qquad \overline{w} \;\in\; H^2(\mathbb{R}\backslash\{0\}), \bar{v} \in \mathcal{X}_\alpha. \tag{2.3}$$

Let $\eta \in C^\infty(\mathbb{R})$ be an even cut-off function which is nonincreasing on $[0,\infty[$ and satisfies

$$\mathrm{supp}(\eta) \;\subseteq\; [-2,2], \qquad \eta(x) \;=\; 1 \qquad \text{for all } x \in [-1,1]. \tag{2.4}$$

Without loss of generality, we can assume $\bar{v}(0) = 0$ with $\mathrm{supp}(v) \subseteq [-2,2]$. This assumption is justified due to the fact that (2.3) can be rearranged as

$$\overline{w} + \bar{v} = \Big[\overline{w} + \bar{v} - \Big(\bar{v} - \bar{v}(0)\Big) \cdot \eta\Big] + \Big(\bar{v} - \bar{v}(0)\Big) \cdot \eta \;\in\; H^2(\mathbb{R}\backslash\{0\}) + \mathcal{X}_\alpha$$

2.2 Structure of piecewise regular solution

The key idea in proving a local existence result for the Cauchy problem (2.2)-(2.3) is to look for solutions of the form

$$u(t,x) \;=\; w(t,x) + \varphi^{(w)}(t,x), \tag{2.5}$$

where $w(t,\cdot)$ belongs to $H^2(\mathbb{R}\backslash\{0\})$ for $t>0$ and the corrector term $\varphi^{(w)}$ depends explicitly on time t such that $\varphi^{(w)}(t,0)=0$ for all $t\geq 0$, the strength of the jump

$$\sigma^{(w)}(t) \;\doteq\; w^-(t)-w^+(t), \qquad \text{where } \; w^\pm(t)\;\doteq\; w(t,0\pm), \tag{2.6}$$

and also on the difference between the speed of characteristic and the speed of the shock curve

$$b_\pm^{(w)}(t) \;\doteq\; b^{(\omega)}(t,0\pm), \qquad b^{(w)}(t,x) \;\doteq\; f'(w(t,x)) - \frac{f(w^-(t))-f(w^+(t))}{w^-(t)-w^+(t)}\,. \tag{2.7}$$

In order to make an appropriate ansatz for the function $\varphi^{(w)}$, we observe from (1.3) and (1.4) that if $g\in H^1(\mathbb{R}\backslash\{0\})$ with $\mathrm{supp}(g)\subseteq[-2,2]$, then for every $x\neq 0$ we have

$$\mathbf{G}[g](x) \;=\; \int_{-2}^{2} g'(y)\cdot\Lambda(x-y)\;dy + \big[g(0+)-g(0-)\big]\cdot\Lambda(x). \tag{2.8}$$

The function $\Lambda\in C^3(\mathbb{R}\backslash\{0\})$ is the antiderivative of K, which is given by

$$\Lambda(x) = -\left(\int_x^2 K(y)dy\right)\cdot\chi_{]0,2]}(x) + \left(\int_2^x K(y)dy\right)\cdot\chi_{]2,\infty[}(x),\;\; x\in]0,\infty[, \;\text{ and} \tag{2.9}$$

$$\Lambda(x) = -\left(\int_x^{-2} K(y)dy\right)\cdot\chi_{]-\infty,-2[}(x) + \left(\int_{-2}^x K(y)dy\right)\cdot\chi_{]-2,0[}(x),\;\; x\in]-\infty,0[\tag{2.10}$$

where χ_E is the indicator function of E.

At this point we note that equation (2.2) can be approximated by the simpler equation

$$u_t + \left(f'(u) - \frac{f(u^-(t))-f(u^+(t))}{u^-(t)-u^+(t)}\right)\cdot u_x \;=\; \big[u^+(t)-u^-(t)\big]\cdot\Lambda(x). \tag{2.11}$$

Indeed, we expect that the solutions of (2.2) and (2.11) with the same initial data will have the same asymptotic structure near the origin. Their difference will lie in $H^2\big(\mathbb{R}\backslash\{0\}\big)$. With this in mind, let $\Phi:[0,\infty[\to\mathbb{R}$ be the antiderivative of Λ such that

$$\Phi(0) \;=\; 0, \qquad \Phi(x) \;=\; \int_0^x \Lambda(y)dy \qquad \text{for all } x\in\mathbb{R}\backslash\{0\}. \tag{2.12}$$

Consider the function

$$\phi(x,b) \;\doteq\; \eta(x)\cdot[\Phi(b)-\Phi(x+b)] \qquad \text{for all } x,b\in\mathbb{R}. \tag{2.13}$$

We make the ansatz

$$\varphi^{(w)}(t,x) \;=\; \begin{cases} \dfrac{\sigma^{(w)}(t)}{b_-^{(w)}(t)}\cdot\Big[\phi(x,0)-\phi\left(x,-t\cdot b_-^{(w)}(t)\right)\Big] \\ \qquad +\eta(x)\cdot\Big[\overline{v}\left(x-t\cdot b_-^{(w)}(t)\right)-\bar{v}\left(-t\cdot b_-^{(w)}(t)\right)\Big] & \text{if} \quad x<0, \\ \dfrac{\sigma^{(w)}(t)}{b_+^{(w)}(t)}\cdot\Big[\phi(x,0)-\phi\left(x,-t\cdot b_+^{(w)}(t)\right)\Big] \\ \qquad +\eta(x)\cdot\Big[\overline{v}\left(x-t\cdot b_+^{(w)}(t)\right)-\bar{v}\left(-t\cdot b_+^{(w)}(t)\right)\Big] & \text{if} \quad x>0. \end{cases} \tag{2.14}$$

From (2.2), (2.5) and (2.7), the remaining component $w(t,\cdot)$ from (2.5) solves the equation

$$w_t + a(t,x,w)\cdot w_x \;=\; F(t,x,w), \tag{2.15}$$

where a and F are respectively given by

$$a(t,x,w) \;=\; b^{(w)}(t,x) + f'\left(w+\varphi^{(w)}\right) - f'(w), \tag{2.16}$$

and

$$\begin{aligned} F(t,x,w) \;=\; & \mathbf{G}\left[\varphi^{(w)}\right](t,x) - \left[f'\left(w+\varphi^{(w)}\right) - f'(w)\right]\cdot \varphi_x^{(w)}(t,x) \\ & + \left(\mathbf{G}\left[w\right](t,x) - \left[\varphi_t^{(w)}(t,x) + b^{(w)}(t,x)\cdot\varphi_x^{(w)}(t,x)\right]\right). \end{aligned} \tag{2.17}$$

2.3 Construction of solution

In order to finish the proof of Theorem 2.1, we construct solutions to the Cauchy problem (2.15)-(2.17) with initial data satisfying

$$w(0,\cdot) \;=\; \overline{w}(\cdot)\in H^2(\mathbb{R}\backslash\{0\}), \qquad \overline{w}(0-)-\overline{w}(0+) \;>\; 0. \tag{2.18}$$

Following the analysis in [4], the solution will be obtained as the limit of a sequence of approximations. Namely, consider a sequence of linear approximations constructed as follows. As a first step, we set

$$w_1(t,x) \;=\; \overline{w}(x) \qquad \text{for all}\;\; t\geq 0,\;\; x\in\mathbb{R}\,.$$

By induction, let w_n be given. We define w_{n+1} to be the solution of the semilinear, non-homogeneous Cauchy problem

$$w_t + a(t,x,w_n)\cdot w_x \;=\; F(t,x,w), \qquad w(0,\cdot) \;=\; \overline{w}. \tag{2.19}$$

The induction argument requires two main steps:

(i). Existence and uniqueness of solutions to each semilinear problem (2.19) such that

$$\sup_{t\in[0,T]} \left\|w_n(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{0\})} \;<\; \infty \quad \text{for all } n\geq 1.$$

(ii). Convergence in the weaker norm $H^1(\mathbb{R}\backslash\{0\})$, which will follow from the contractive property below

$$\sup_{t\in[0,T]} \left\|w_{n+1}(t,\cdot)-w_n(t,\cdot)\right\|_{H^1(\mathbb{R}\backslash\{0\})} < \frac{1}{2}\sup_{t\in[0,T]} \left\|w_n(t,\cdot)-w_{n-1}(t,\cdot)\right\|_{H^1(\mathbb{R}\backslash\{0\})} \tag{2.20}$$

The details for these three main steps will be provided in the following sections.

3 Key estimates on the source term F

In order to prove existence and uniqueness of solutions to each linear problem (2.19) above, we need a priori estimates on the source term F defined in (2.17) and recalled here:

$$\begin{aligned} F(t,x,w) &= \mathbf{G}\left[\varphi^{(w)}\right](t,x) - \left[f'\left(w+\varphi^{(w)}\right) - f'(w)\right]\cdot\varphi_x^{(w)}(t,x) \\ &\qquad + \left(\mathbf{G}\left[w\right](t,x) - \left[\varphi_t^{(w)}(t,x) + b^{(w)}(t,x)\cdot\varphi_x^{(w)}(t,x)\right]\right). \end{aligned}$$

First, we rewrite the source F in the following way:

$$F(t,x,w) = A^{(w)}(t,x) + B^{(w)}(t,x) - C^{(w)}(t,x), \quad \text{with} \tag{3.1}$$

$$A^{(w)}(t,x) \doteq \mathbf{G}\left[\varphi^{(w)}\right](t,x) - \left[f'\left(w+\varphi^{(w)}\right) - f'(w)\right]\cdot\varphi_x^{(w)}(t,x), \tag{3.2}$$

$$B^{(w)}(t,x) \doteq \mathbf{G}\left[w\right](t,x) - \sigma^{(\omega)}(t)\cdot\Lambda(x)\cdot\eta(x) \ \text{ and} \tag{3.3}$$

$$C^{(\omega)}(t,x) = \varphi_t^{(w)}(t,x) + b^{(w)}(t,x)\cdot\varphi_x^{(w)}(t,x) + \sigma^{(\omega)}(t)\cdot\Lambda(x)\cdot\eta(x). \tag{3.4}$$

3.1 Estimating the corrector term $\varphi^{(w)}(t,x)$

Given two constants $M_0, \delta_0 > 0$, we shall assume that $w(t,\cdot)$ is in $H^2(\mathbb{R}\backslash\{0\})$ and satisfies

$$\|w(t,\cdot)\|_{H^2(\mathbb{R}\backslash\{0\})} \le M_0, \qquad \sigma^{(w)}(t) > \delta_0 \qquad \text{for all } t\in[0,T]. \tag{3.5}$$

In addition, the map $t \mapsto w^{\pm}(t) \doteq w(t,0\pm)$ is locally Lipschitz and

$$\left|\dot{w}^{\pm}(t)\right| \le \ell(t) \qquad a.e.\ t\in]0,T]. \tag{3.6}$$

From (3.5) and (2.6), we have the following estimates for all $x\in\mathbb{R}\backslash\{0\}$ and $t\in[0,T]$

$$|w(t,x)|, |w_x(t,x)| \le 2M_0, \qquad \left|\sigma^{(w)}(t)\right| \le |w^+(t)| + |w^-(t)| \le 4M_0, \tag{3.7}$$

and

$$\begin{cases} \left|b^{(w)}(t,x)\right|, \left|b_x^{(w)}(t,x)\right| \le 4\cdot\|f\|_{\mathcal{C}^2([-2M_0,2M_0])}M_0 \doteq b_1, \\ \left|b_{xx}^{(w)}(t,x)\right| \le \|f\|_{\mathcal{C}^3([-2M_0,2M_0])}\left(|w_{xx}(t,x)| + 4M_0^2\right). \end{cases} \tag{3.8}$$

By the strict convexity of f in (1.2), one has that

$$-b_1 \le b_+^{(w)}(t) \le -b_0 < 0 < b_0 \le b_-^{(w)}(t) \le b_1, \tag{3.9}$$

with b_0 defined by

$$b_0 \doteq \delta_0 \cdot \min_{a,b\in[-2M_0,2M_0],\, b-a\ge\delta_0} \int_0^1\int_0^1 f''(b - rs(b-a))\,s\,dr\,ds > 0. \tag{3.10}$$

Moreover, assumption (3.6) implies that

$$\left|\dot{b}_{\pm}^{(w)}(t)\right| \le 2\|f\|_{\mathcal{C}^2([-2M_0,2M_0])}\cdot\ell(t) \qquad \text{for all } t\in[0,T]. \tag{3.11}$$

Using (3.7)-(3.9), we provide some estimates on the corrector term $\varphi^{(w)}(t,x)$ defined (2.12)-(2.14). In the following, as usual, by the Landau symbol $\mathcal{O}(1)$ we shall denote a uniformly bounded quantity which does not depend on M_0, δ_0 and f.

Lemma 3.1 *For every* $0 < |x| < 1/4$ *and* $0 < t < \frac{1}{4b_1}$ *with* b_1 *defined in (3.8), we have that*

$$\left|\varphi^{(w)}(t,x)\right| \;\leq\; C_0 \cdot |x| \left(\left|\ln|x|\right| + t^{\alpha-1}\right), \tag{3.12}$$

$$\left|\frac{d}{dx}\varphi^{(w)}(t,x)\right| \;\leq\; C_0 \cdot \left(\left|\ln|x|\right| + \frac{1}{(|x|+t)^{1-\alpha}}\right), \tag{3.13}$$

$$\left|\frac{d^i}{dx^i}\varphi^{(w)}(t,x)\right| \leq\; C_0 \cdot \left(\frac{1}{|x|^{i-1}} + \frac{1}{(|x|+t)^{i-\alpha}}\right), \qquad i \in \{2,3\} \tag{3.14}$$

Moreover, for $\delta > 0$ *small, we obtain that*

$$\left\|\varphi^{(\omega)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; C_0 \cdot \delta^{\alpha-3/2},$$

with a constant $C_0 > 0$ *satisfying*

$$\left\|\varphi^{(w)}(t,\cdot)\right\|_{L^\infty} \;\leq\; C_0 \;\doteq\; \mathcal{O}(1)\cdot\left(M_0 b_0^{-1}+1\right)\cdot\left(b_0^{\alpha-3}+1\right).$$

Proof. We rewrite $\varphi^{(w)}$ equivalently as $\varphi^{(w)} = \widetilde{\varphi}^{(w)} + \overline{\varphi}^{(w)}$, with

$$\begin{cases} \widetilde{\varphi}^{(w)}(t,x) \;=\; \dfrac{\sigma^{(w)}(t)}{b_-^{(w)}(t)}\cdot\left[\phi(x,0) - \phi\left(x, -t\cdot b_-^{(w)}(t)\right)\right]\cdot\chi_{]-\infty,0[} \\ \qquad\qquad\qquad + \dfrac{\sigma^{(w)}(t)}{b_+^{(w)}(t)}\cdot\left[\phi(x,0) - \phi\left(x, -t\cdot b_+^{(w)}(t)\right)\right]\cdot\chi_{]0,\infty[}, \\ \overline{\varphi}^{(w)}(t,x) \;=\; \eta(x)\cdot\left[\overline{v}\left(x - t\cdot b_-^{(w)}(t)\right) - \overline{v}\left(-t\cdot b_-^{(w)}(t)\right)\right]\cdot\chi_{]-\infty,0[} \\ \qquad\qquad\qquad + \eta(x)\cdot\left[\overline{v}\left(x - t\cdot b_+^{(w)}(t)\right) - \overline{v}\left(-t\cdot b_+^{(w)}(t)\right)\right]\cdot\chi_{]0,\infty[}. \end{cases} \tag{3.15}$$

From the assumption **(H2)**, for all $x \in [-1/4, 1/4]\setminus\{0\}$, it holds that

$$|\Phi(x)| \;\leq\; \mathcal{O}(1)\cdot\left|x\ln|x|\right|, \quad |\Lambda(x)| \;\leq\; \mathcal{O}(1)\cdot\left|\ln|x|\right|, \quad K(x) \;\leq\; \frac{C}{|x|^{i+1}}, \quad i = 0,1,2.$$

Recalling (3.7) and (3.9), we estimate for every $x \in (0, 1/4]$ and $0 < t < \frac{1}{4b_1}$ that

$$\begin{cases} \left|\widetilde{\varphi}^{(w)}(t,x)\right| \;\leq\; \dfrac{4M_0}{b_0}\cdot\left|\phi(x,0) - \phi\left(x,-t\cdot b_+^{(w)}(t)\right)\right| \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_0}{b_0}\cdot\left|x\ln|x|\right|, \\ \left|\dfrac{d}{dx}\widetilde{\varphi}^{(w)}(t,x)\right| \;\leq\; \dfrac{4M_0}{b_0}\cdot\left|\Lambda\left(x - tb_+^{(w)}(t)\right) - \Lambda(x)\right| \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_0}{b_0}\cdot\left|\ln x\right|, \\ \left|\dfrac{d^i}{dx^i}\widetilde{\varphi}^{(w)}(t,x)\right| \;\leq\; \mathcal{O}(1)\cdot\dfrac{M_0}{b_0}\cdot\dfrac{1}{x^{(i-1)}}, \qquad i \in \{2,3\}, \end{cases}$$

and

$$\left|\overline{\varphi}^{(w)}(t,x)\right| \;=\; \left|\overline{v}\left(x - t\cdot b_+^{(w)}(t)\right) - \overline{v}\left(-t\cdot b_+^{(w)}(t)\right)\right| \;\leq\; \mathcal{O}(1)\cdot\min\left\{1, \frac{x}{(b_0 t)^{1-\alpha}}\right\},$$

$$\left|\frac{d^i}{dx^i}\overline{\varphi}^{(w)}(t,x)\right| \;=\; \left|\frac{d^i}{dx^i}\overline{v}\left(x - t\cdot b_+^{(w)}(t)\right)\right| \;\leq\; \mathcal{O}(1)\cdot\frac{1}{(x+b_0 t)^{i-\alpha}} \qquad i \in \{1,2,3\}.$$

The same estimates hold for $x \in [-1/4, 0)$, $0 < t < 1/(4b_1)$, and this yields (3.12)-(3.14). $\square$

3.2 Estimating source F

The next lemma provides some estimates on the function F in (2.17). These estimates will be used in Lemma 4.3 to establish a priori bounds on the approximations solutions of the linear, non-homogeneous Cauchy problem (2.19).

Lemma 3.2 *Assume that $w : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies assumptions (3.5)-(3.6). Then for $|x| < 1/4$ and a.e. $t \in [0,T]$, we have*

$$\begin{cases} |F(t,x,w)| & \leq \ \Gamma_1 \cdot \left[\ell(t) \cdot \left(\big|x \ln |x|\big| + t^{\alpha}\right) + t^{\alpha-1}\right], \\ |F_x(t,x,w)| & \leq \ \Gamma_1 \cdot \left[t^{\alpha-3/2} + \left(\ell(t) + t^{\alpha-1}\right) \cdot |x|^{\alpha-1}\right]. \end{cases} \tag{3.16}$$

Furthermore, for every $\delta > 0$ sufficiently small, we obtain that

$$\|F(t,\cdot,w)\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \leq \ \Gamma_1 \cdot \left[t^{-3/4} + \left(t^{\alpha-1} + \ell(t)\right) \cdot \delta^{-3/4}\right], \quad \textit{with} \tag{3.17}$$

$$\Gamma_1 \ = \ \mathcal{O}(1) \cdot \Gamma_f^2 C_0^2 (M_0^3 + 1)\left[1 + b_0^{\alpha-3}\right], \ \Gamma_f \ = \ 1 + \|f\|_{\mathcal{C}^4([-2M_0 - C_0, 2M_0 + C_0])}. \tag{3.18}$$

Proof. First, recall that we expressed F in (3.1) in terms of $A^{(w)}$, $B^{(w)}$ and $C^{(w)}$.

1. Estimates on $B^{(w)}$. Applying Lemma A.2 for $w(t,\cdot)$, we get for every $0 < |x| < 1/4$ and $\delta > 0$ that

$$\left|B^{(w)}(t,x)\right| \ \leq \ \mathcal{O}(1) \cdot M_0, \qquad \left|B_x^{(w)}(t,x)\right| \ \leq \ \mathcal{O}(1) \cdot M_0 \cdot \ln^2 |x|,$$

and

$$\left\|B^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \leq \ \mathcal{O}(1) \cdot M_0 \cdot \delta^{-2/3}.$$

2. Estimates on $A^{(w)}$. We are now giving some bounds on $A^{(w)}$ by splitting it into two parts

$$A^{(w)} \ = \ \mathbf{G}\left[\varphi^{(w)}\right] - A_1^{(w)}, \qquad A_1^{(w)} \ = \ d^{(w)} \cdot \varphi_x^{(w)} \tag{3.19}$$

with

$$d^{(w)} \ \doteq \ f'\left(w + \varphi^{(w)}\right) - f'(w).$$

From (3.7)-(3.9) and Lemma 3.1, we estimate

$$\begin{cases} \left|\dfrac{d}{dx} d^{(w)}(t,x)\right| \ \leq \ \mathcal{O}(1) \cdot \Gamma_f C_0 \left(\big|\ln |x|\big| + \dfrac{1}{(|x|+t)^{1-\alpha}}\right), \\ \left|\dfrac{d^2}{dx^2} d^{(w)}(t,x)\right| \ \leq \ \mathcal{O}(1) \cdot \Gamma_f C_0 \left(|w_{xx}| + \dfrac{1}{|x|} + \dfrac{1}{(|x|+t)^{2-\alpha}}\right). \end{cases} \tag{3.20}$$

Notice that $d^{(w)}(t,0) = 0$, we derive from Lemma 3.1 that

$$\begin{cases} \left|A_1^{(w)}(t,x)\right| & \leq \ \Gamma_f \cdot \left|\varphi^{(w)} \varphi_x^{(w)}\right| \ \leq \ \Gamma_f C_0^2 \cdot |x|^{\alpha}\left(|t|^{\alpha-1} + \ln^2 |x|\right), \\ \left|\dfrac{d}{dx} A_1^{(w)}(t,x)\right| & \leq \ \mathcal{O}(1) \cdot \Gamma_f \cdot \left(\left|\varphi^{(w)} \varphi_{xx}^{(w)}\right| + \left|\varphi_x^{(w)}\right|^2 + \left|w_x \varphi^{(w)} \varphi_x^{(w)}\right|\right) \\ & \leq \ \mathcal{O}(1) \cdot \Gamma_f C_0^2 (M_0 + 1) \cdot |tx|^{\alpha-1}. \end{cases}$$

Moreover, keeping the leading order terms, one also gets

$$\begin{aligned}
&\Big|\frac{d^2}{dx^2}A_1^{(w)}(t,x)\Big| \\
&\qquad \le \mathcal{O}(1)\cdot\Gamma_f(M_0^2+1)\cdot\Big[\big|\varphi^{(w)}\varphi^{(w)}_{xxx}\big| + \big|\varphi^{(w)}_x\varphi^{(w)}_{xx}\big| + \big|\varphi^{(w)}_x\big|^3 + \big|\varphi^{(w)}\varphi^{(w)}_x w_{xx}\big|\Big] \\
&\qquad \le \mathcal{O}(1)\cdot\Gamma_f(M_0^2+1)C_0^2\cdot\left[\big(\big|\ln|x|\big| + t^{\alpha-1}\big)\cdot\left(\frac{1}{|x|} + \frac{1}{(|x|+t)^{2-\alpha}}\right) + \frac{|w_{xx}|}{|x|^{1-\alpha}}\right].
\end{aligned}$$

In particular, setting $\Gamma \doteq \Gamma_f^2 C_0^2(M_0^3+1)\left[1+b_0^{\alpha-3}\right]$, we have

$$\left\|A_1^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \le\ \mathcal{O}(1)\Gamma\cdot\big(|\ln\delta| + t^{\alpha-1}\big)\left(\delta^{-1/2} + (\delta+t)^{\alpha-3/2}\right).$$

To estimate the term $\mathbf{G}\left[\varphi^{(w)}\right](t,x)$ in $A^{(w)}$, we first recall Lemma A.2 to obtain that

$$\begin{cases}
\big|\mathbf{G}\big[\widetilde{\varphi}^{(w)}(t,\cdot)\big](x)\big| \le \mathcal{O}(1)\cdot\dfrac{M_0}{b_0}, \qquad \left|\dfrac{d}{dx}\mathbf{G}\big[\widetilde{\varphi}^{(w)}(t,\cdot)\big](x)\right| \le \mathcal{O}(1)\cdot\dfrac{M_0}{b_0}\cdot\ln^2|x|, \\[2mm]
\big\|\mathbf{G}\big[\widetilde{\varphi}^{(w)}(t,\cdot)\big]\big\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \le \mathcal{O}(1)\cdot\dfrac{M_0}{b_0}\cdot\delta^{-2/3}.
\end{cases}$$

We observe that

$$\left|\overline{\varphi}^{(w)}(t,0)\right| \ =\ 0, \quad \text{and} \quad \left|\frac{d^i}{dx^i}\overline{\varphi}^{(w)}(t,x)\right| \ \le\ \mathcal{O}(1)\cdot(|x|+b_0t)^{\alpha-i}, \ i\in\{1,2\},$$

and therefore we obtain

$$\begin{aligned}
\left\|\overline{\varphi}^{(w)}(t,\cdot)\right\|_{H^1(\mathbb{R})} \ &\le\ \mathcal{O}(1)\cdot\big(1+b_0^{\alpha-1/2}\big)\cdot t^{\alpha-1/2}, \\
\left\|\overline{\varphi}^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R})} \ &\le\ \mathcal{O}(1)\cdot\big(1+b_0^{\alpha-3/2}\big)\cdot t^{\alpha-3/2}.
\end{aligned}$$

Using the $\mathbf{L}^2$-continuity of $\mathbf{G}$, we then have that

$$\begin{aligned}
\left\|\mathbf{G}\left[\overline{\varphi}^{(w)}(t,\cdot)\right]\right\|_{L^\infty(\mathbb{R})} &\le 2\cdot\left\|\mathbf{G}\left[\overline{\varphi}^{(w)}(t,\cdot)\right]\right\|_{H^1(\mathbb{R})} \le \mathcal{O}(1)\cdot\big(1+b_0^{\alpha-1/2}\big)\cdot t^{\alpha-1/2}, \\
\left\|\frac{d}{dx}\mathbf{G}\left[\overline{\varphi}^{(w)}(t,\cdot)\right]\right\|_{L^\infty(\mathbb{R})} &\le 2\cdot\left\|\mathbf{G}\left[\overline{\varphi}^{(w)}(t,\cdot)\right]\right\|_{H^2(\mathbb{R})} \le \mathcal{O}(1)\cdot\big(1+b_0^{\alpha-3/2}\big)\cdot t^{\alpha-3/2}.
\end{aligned}$$

Thus, for every $t>0$ and $|x|<1/4$, we have the following estimates:

$$\begin{aligned}
\left|A^{(w)}(t,x)\right| \ &\le\ \mathcal{O}(1)\cdot\Gamma\cdot\big(|x|^\alpha\ln^2|x| + t^{\alpha-1}\big), \\
\left|\frac{d}{dx}A^{(w)}(t,x)\right| \ &\le\ \mathcal{O}(1)\cdot\Gamma\cdot t^{\alpha-1}\cdot\big(|x|^{\alpha-1} + t^{-1/2}\big), \\
\left\|A^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ &\le\ \mathcal{O}(1)\cdot\Gamma\cdot t^{\alpha-1}\cdot\left(t^{-1/2} + \delta^{-2/3} + (\delta+t)^{\alpha-3/2}\right).
\end{aligned}$$

3. Estimates on $C^{(w)}$. From (2.17) and (3.2)-(3.3), we have that

$$C^{(\omega)}(t,x) \ =\ \varphi_t^{(w)}(t,x) + b^{(w)}(t,x)\cdot\varphi_x^{(w)}(t,x) + \sigma^{(\omega)}(t)\cdot\Lambda(x)\cdot\eta(x).$$

Recalling (3.15), we compute for $0 < x < 1/4$ that

$$\widetilde{\varphi}_x^{(w)}(t,x) = \frac{\sigma^{(w)}(t)}{b_+^{(w)}(t)} \cdot \left[\phi_x(x,0) - \phi_x\left(x, -t \cdot b_+^{(w)}(t)\right)\right], \quad \overline{\varphi}_x^{(w)}(t,x) = \overline{v}'\left(x - t \cdot b_+^{(w)}(t)\right)$$

$$\overline{\varphi}_t^{(w)}(t,x) = \left(b_+^{(w)}(t) + t\dot{b}_+^{(w)}(t)\right) \cdot \left[\overline{v}'\left(-t \cdot b_+^{(w)}(t)\right) - \overline{v}'\left(x - t \cdot b_+^{(w)}(t)\right)\right],$$

$$\widetilde{\varphi}_t^{(w)}(t,x) = \left(\frac{\dot{\sigma}^{(w)}(t)}{\sigma^{(w)}(t)} - \frac{\dot{b}_+^{(w)}(t)}{b_+^{(w)}(t)}\right) \cdot \widetilde{\varphi}^{(w)}(x,t) + \frac{\sigma^{(w)}(t)}{b_+^{(w)}(t)} \cdot \left(b_+^{(w)}(t) + t \cdot \dot{b}_+^{(w)}(t)\right) \cdot \frac{d}{db}\phi\left(x, -tb_+^{(w)}(t)\right)$$

Since $\frac{d}{db}\phi(x,b) = \Phi'(b) + \frac{d}{dx}\phi(x,b)$ and $\frac{d}{dx}\phi(x,0) = -\Lambda(x)$, one has

$$\begin{aligned} C^{(w)}(t,x) \;=\; & \left(\frac{\dot{\sigma}^{(w)}(t)}{\sigma^{(w)}(t)} - \frac{\dot{b}_+^{(w)}(t)}{b_+^{(w)}(t)}\right) \cdot \widetilde{\varphi}^{(w)}(t,x) + \left(b^{(w)}(t,x) - b_+^{(w)}(t)\right) \cdot \varphi_x^{(w)}(t,x) \\ & - t\dot{b}_+^{(w)}(t) \cdot \left[\frac{\sigma^{(w)}(t)}{b_+^{(w)}(t)} \cdot \Phi'\left(x - t \cdot b_+^{(w)}(t)\right) + \overline{v}'\left(x - t \cdot b_+^{(w)}(t)\right)\right] + E^{(w)}(t), \end{aligned} \tag{3.21}$$

with $E^{(w)}$ defined by

$$E^{(w)}(t) \;\doteq\; \left(b_+^{(w)}(t) + t \cdot \dot{b}_+^{(w)}(t)\right) \cdot \left[\frac{\sigma^{(w)}(t)}{b_+^{(w)}(t)} \cdot \Phi'\left(-t \cdot b_+^{(w)}(t)\right) + \overline{v}'\left(-t \cdot b_+^{(w)}(t)\right)\right].$$

From (3.5)-(3.11) and Lemma 3.1 we obtain

$$\begin{aligned} \left|C^{(w)}(t,x)\right| \;&\leq\; \mathcal{O}(1)\Gamma \cdot \left[\ell(t) \cdot \left(\left|x \ln|x|\right| + t^{\alpha}\right) + t^{\alpha-1}\right], \\ \left|C_x^{(w)}(t,x)\right| \;&\leq\; \mathcal{O}(1)\Gamma \cdot \left[\ell(t)\left[\frac{t}{(|x|+t)^{2-\alpha}} + \left|\ln|x|\right|\right] + \frac{1}{(|x|+t)^{1-\alpha}}\right], \\ \left|C_{xx}^{(w)}(t,x)\right| \;&\leq\; \mathcal{O}(1)\Gamma \cdot \left(\ell(t)\left[\frac{t}{(|x|+t)^{3-\alpha}} + \frac{1}{|x|}\right] + (|w_{xx}|+1)\left[\left|\ln|x|\right| + \frac{1}{(|x|+t)^{1-\alpha}}\right]\right), \end{aligned}$$

and

$$\left\|C^{(w)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; \mathcal{O}(1)\Gamma \cdot \left(t^{\alpha-1} + t\ell(t)(\delta+t)^{\alpha-5/2} + \ell(t)\delta^{-1/2}\right).$$

From the previous estimates on $A^{(w)}$, $B^{(w)}$, and (3.2), one then achieves (3.16)-(3.17) for $\alpha > 3/4$. □

To complete this section, we study the change in the function $F(t,x,w)$ as w takes different values which plays a key role in the proof of convergence of the approximations. More precisely, for any given two functions $w_1, w_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying (3.5)-(3.6), we provide a priori estimates on the difference

$$\mathbf{F}^{(w_1,w_2)}(t,x) \;\doteq\; F(t,x,w_2) - F(t,x,w_1). \tag{3.22}$$

in terms of $M_i(t)$ for $i \in \{1,2\}$ with

$$\mathbf{w} \;\doteq\; w_2 - w_1, \qquad M_i(t) \;\doteq\; \|\mathbf{w}(t,\cdot)\|_{H^i(\mathbb{R}\setminus\{0\})}. \tag{3.23}$$

For all $x \in \mathbb{R}\backslash\{0\}$ and $t \in [0,T]$, it holds

$$|\mathbf{w}(t,x)| \ \leq\ 2M_1(t), \qquad |\mathbf{w}_x(t,x)| \ \leq\ 2M_2(t), \tag{3.24}$$

For simplicity, recalling (2.7) and (3.15), we set

$$\mathbf{b} \ \doteq\ b^{(w_2)} - b^{(w_1)}, \qquad\qquad \Psi \ \doteq\ \varphi^{(w_2)} - \varphi^{(w_1)}. \tag{3.25}$$

We first estimate $\mathbf{b}^{(w_1,w_2)}$ by direct computations

$$\begin{cases} \big|\mathbf{b}(t,x)\big| \ \leq\ 4\Gamma_f \cdot M_1(t), \qquad \big|\mathbf{b}_x(t,x)\big| \ \leq\ 2\Gamma_f \cdot (M_0 M_1(t) + |\mathbf{w}_x(t,x)|)\,, \\ \big|\mathbf{b}_{xx}(t,x)\big| \ \leq\ 4\Gamma_f \cdot \left(\left[M_0^2 + |w_{1,xx}(t,x)|\right] \cdot M_1(t) + 2M_0M_2(t) + |\mathbf{w}_{xx}(t,x)|\right), \end{cases} \tag{3.26}$$

with Γ_f given in (3.18). In particular, this yields

$$\big|\mathbf{b}_{\pm}(t)\big| \ \doteq\ \big|b^{(w_2)}(t,0\pm) - b^{(w_2)}(t,0\pm)\big| \ \leq\ 4\Gamma_f \cdot M_1(t). \tag{3.27}$$

Secondly, the difference of the corrector term Ψ and its transform $\mathbf{G}\big[\Psi\big]$ is bounded by the following lemma.

Lemma 3.3 *For every $0 < |x| < 1/4$ and $0 < t < \dfrac{1}{4b_1}$, we have*

$$\begin{cases} |\Psi(t,x)| \leq \Gamma_{f,1} M_1(t)|x|^{\alpha}, \qquad \left|\dfrac{d}{dx}\Psi(t,x)\right| \leq \Gamma_{f,1}M_1(t)\left(\big|\ln|x|\big| + (|x|+t)^{\alpha-1}\right), \\ \left|\dfrac{d^i}{dx^i}\Psi(t,x)\right| \leq \Gamma_{f,1}M_1(t)\left(|x|^{1-i} + (|x|+t)^{\alpha-i}\right), \quad i = 2,3, \\ \|\mathbf{G}\left[\Psi(t,\cdot)\right]\|_{H^1(\mathbb{R}\backslash\{0\})}, \|\Psi(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \ \leq\ \Gamma_{f,1}M_1(t)t^{\alpha-1/2}, \end{cases} \tag{3.28}$$

with $\Gamma_{f,1} = \mathcal{O}(1)(4b_1 + 16\Gamma_f M_0)(b_0^{\alpha-3} + b_0^{-2})$. Moreover, for every $\delta > 0$ small, it holds

$$\begin{cases} \|\Psi(t,\cdot)\|_{H^2(\mathbb{R}\backslash[-\delta,\delta])} \ \leq\ \Gamma_{f,1}M_1(t)\left(t^{\alpha-3/2} + \delta^{-1/2}\right), \\ \|\mathbf{G}\left[\Psi(t,\cdot)\right]\|_{H^2(\mathbb{R}\backslash[-\delta,\delta])} \ \leq\ \Gamma_{f,1}M_1(t)\left(t^{\alpha-3/2} + \delta^{-2/3}\right). \end{cases} \tag{3.29}$$

Proof. 1. Recalling (3.15), we first write $\Psi = \overline{\Psi} + \widetilde{\Psi}$ with

$$\overline{\Psi} \ \doteq\ \overline{\varphi}^{(w_2)} - \overline{\varphi}^{(w_1)}, \qquad\qquad \widetilde{\Psi} \ \doteq\ \widetilde{\varphi}^{(w_2)} - \widetilde{\varphi}^{(w_1)}. \tag{3.30}$$

From (3.15) and (1.6), for every $t \in [0, 1/(4b_1)]$ and $|x| > 0$, one has that $\overline{\Psi}(t,0) = 0$, $\mathrm{supp}\left(\overline{\Psi}(t,\cdot)\right) \subseteq [-2,2]$, and

$$\left|\frac{d^i}{dx^i}\overline{\Psi}(t,x)\right| \ \leq\ \mathcal{O}(1)\cdot\frac{M_1(t)}{(|x|+tb_0)^{i-\alpha}}, \quad i = 1,2,3,$$

whence

$$\begin{cases} \big\|\overline{\Psi}(t,\cdot)\big\|_{H^1(\mathbb{R})} \ \leq\ \mathcal{O}(1)\cdot\left(1 + b_0^{\alpha-1/2}\right)\cdot t^{\alpha-1/2}\cdot M_1(t), \\ \big\|\overline{\Psi}(t,\cdot)\big\|_{H^2(\mathbb{R})} \ \leq\ \mathcal{O}(1)\cdot\left(1 + b_0^{\alpha-3/2}\right)\cdot t^{\alpha-3/2}\cdot M_1(t), \end{cases}$$

and so the $\mathbf{L}^2$-continuity of $\mathbf{G}$ yields

$$\left\|\mathbf{G}\left[\overline{\Psi}(t,\cdot)\right]\right\|_{\mathbf{L}^\infty(\mathbb{R})} \leq 2\left\|\mathbf{G}\left[\overline{\Psi}(t,\cdot)\right]\right\|_{H^1(\mathbb{R})} \leq \mathcal{O}(1)\cdot\left(1+b_0^{\alpha-1/2}\right)\cdot t^{\alpha-1/2}\cdot M_1(t)$$

and

$$\left\|\mathbf{G}\left[\overline{\Psi}(t,\cdot)\right]\right\|_{H^2(\mathbb{R})} \leq \mathcal{O}(1)\cdot\left(1+b_0^{\alpha-3/2}\right)\cdot t^{\alpha-3/2}\cdot M_1(t).$$

2. Fix $0<x<1/4$ (similar computations hold for $-1/4<x<0$). To estimate $\widetilde{\Psi}$, we write

$$\widetilde{\Psi}(t,x) = \left(\frac{\sigma^{(w_2)}(t)}{b_+^{(w_2)}(t)} - \frac{\sigma^{(w_1)}(t)}{b_+^{(w_1)}(t)}\right)\left(\phi(x,0)-\phi\Big(x,-t\cdot b_+^{(w_1)}\Big)\right) + \frac{\sigma^{(w_2)}(t)\mathbf{b}_+(t)t}{b_+^{(w_2)}(t)}\cdot\int_0^1 \frac{d}{db}\phi\left(x,-t\cdot b_+^{(w_2)}(t)+\tau\mathbf{b}_+(t)t\right)d\tau.$$

From (3.9) and (3.27), it holds

$$\begin{aligned}\left|\frac{\sigma^{(w_2)}(t)}{b_+^{(w_2)}(t)} - \frac{\sigma^{(w_1)}(t)}{b_+^{(w_1)}(t)}\right| &\leq \frac{b_1\cdot|\mathbf{w}(t,0+)-\mathbf{w}(t,0-)|+4M_0\cdot\left|\mathbf{b}_+(t)\right|}{b_0^2}\\ &\leq \frac{4b_1+16\Gamma_f M_0}{b_0^2}\cdot M_1(t) \doteq \Gamma_{f,0}M_1(t).\end{aligned} \tag{3.31}$$

Recalling (1.4), we estimate

$$\begin{aligned}\left|\phi_x\left(x,-tb_+^{(w_1)}(t)\right)-\phi_x(x,0)\right| &= \left|\int_0^{-tb_+^{(w_1)}(t)} K(x+\tau)\,d\tau\right| \leq \mathcal{O}(1)\cdot\int_0^{b_1 t}\frac{d\tau}{x+\tau}\\ &= \mathcal{O}(1)\cdot\ln\left(1+\frac{b_1 t}{x}\right) \leq \mathcal{O}(1)\cdot\min\left\{|\ln x|,\left(\frac{b_1 t}{x}\right)^{\alpha-1/2}\right\},\end{aligned}$$

and

$$\left|\frac{d}{dx}\int_0^1\frac{d}{db}\phi\left(x,-t\cdot b_+^{(w_2)}(t)+\tau\mathbf{b}_+(t)t\right)d\tau\right| = \left|\int_0^1 K\left(x-t\cdot b_+^{(w_2)}(t)+\tau\mathbf{b}_+(t)t\right)d\tau\right|$$
$$\leq \mathcal{O}(1)\cdot\int_0^1\frac{d\tau}{x-(1-\tau)tb_+^{(w_2)}(t)-\tau tb_+^{(w_1)}} \leq \mathcal{O}(1)\cdot\int_0^1\frac{d\tau}{x+b_0 t} = \mathcal{O}(1)\cdot\frac{1}{x+b_0 t}.$$

Thus, keeping into account that $\widetilde{\Psi}(t,0)=0$, we then have

$$\begin{aligned}\left|\widetilde{\Psi}(t,x)\right| &\leq \mathcal{O}(1)\cdot\Gamma_{f,0}b_1^{\alpha-1/2}M_1(t)\cdot\min\left\{t^{\alpha-1/2}x^{3/2-\alpha},\left|x\ln x\right|\right\},\\ \left|\frac{d}{dx}\widetilde{\Psi}(t,x)\right| &\leq \mathcal{O}(1)\cdot\Gamma_{f,0}b_1^{\alpha-1/2}b_0^{-1}M_1(t)\cdot\min\left\{\left(\frac{t}{x}\right)^{\alpha-1/2},\left|\ln x\right|\right\}.\end{aligned}$$

This, together with the $\mathbf{L}^2$-continuity of $\mathbf{G}$, yields

$$\begin{aligned}\left\|\mathbf{G}\left[\widetilde{\Psi}(t,\cdot)\right]\right\|_{\mathbf{L}^\infty(\mathbb{R})} &\leq 2\cdot\left\|\mathbf{G}\left[\widetilde{\Psi}(t,\cdot)\right]\right\|_{H^1(\mathbb{R}\setminus\{0\})} \leq \mathcal{O}(1)\cdot\left\|\widetilde{\Psi}(t,\cdot)\right\|_{H^1(\mathbb{R}\setminus\{0\})}\\ &\leq \mathcal{O}(1)\cdot\Gamma_{f,0}b_1^{\alpha-1/2}M_1(t)t^{\alpha-1/2}.\end{aligned}$$

Concerning the other derivatives of $\widetilde{\Psi}$, direct computations, together with (3.9), (3.27), and (3.31), yield

$$\left|\frac{d^i}{dx^i}\widetilde{\Psi}(t,x)\right| \;\le\; O(1)\cdot\Gamma_{f,1}M_1(t)\frac{1}{x^{i-1}}, \qquad i\in\{2,3\},$$

and, in particular,

$$\left\|\widetilde{\Psi}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\le\; \mathcal{O}(1)\cdot\Gamma_{f,1}M_1(t)\delta^{-1/2}.$$

3. In order to estimate $\left\|\mathbf{G}\left[\widetilde{\Psi}(t,\cdot)\right]\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])}$, we write $\widetilde{\Psi}=\widetilde{\Psi}_1+\widetilde{\Psi}_2$, with

$$\widetilde{\Psi}_1(t,x)=\left[\left(\frac{\sigma^{(w_2)}(t)}{b_-^{(w_2)}(t)}-\frac{\sigma^{(w_1)}(t)}{b_-^{(w_1)}(t)}\right)\chi_{]-\infty,0[}+\left(\frac{\sigma^{(w_2)}(t)}{b_+^{(w_2)}(t)}-\frac{\sigma^{(w_1)}(t)}{b_+^{(w_1)}(t)}\right)\cdot\chi_{]0,\infty[}\right]\phi(x,0)$$

From (2.13), (3.31), and Lemma A.1, we obtain

$$\begin{aligned}\left\|\frac{d^2}{dx^2}\mathbf{G}[\widetilde{\Psi}_1(t,\cdot)]\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} &\;\le\; \mathcal{O}(1)\cdot\Gamma_{f,0}M_1(t)\left\|\frac{d^2}{dx^2}\mathbf{G}[\phi(x,0)]\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])}\\ &\;\le\; \mathcal{O}(1)\cdot\Gamma_{f,0}M_1(t)\delta^{-2/3}.\end{aligned}$$

On the other hand, from (3.15) and (3.30), for every $x>0$, we can write

$$\begin{aligned}\widetilde{\Psi}_2(t,x) \;=\; &\left(\frac{\sigma^{(w_1)}(t)}{b_+^{(w_1)}(t)}-\frac{\sigma^{(w_2)}(t)}{b_+^{(w_2)}(t)}\right)\phi\left(x,-t\cdot b_+^{(w_1)}(t)\right)\\ &+\frac{\sigma^{(w_2)}(t)\mathbf{b}_+(t)t}{b_+^{(w_2)}(t)}\cdot\int_0^1\frac{d}{db}\phi\left(x,-t\cdot b_+^{(w_2)}(t)+\tau\mathbf{b}_+(t)t\right)d\tau.\end{aligned}$$

Combining (3.9), (3.27), and (3.31), we obtain

$$\left|\frac{d^2}{dx^2}\widetilde{\Psi}_2(t,x)\right| \;\le\; \mathcal{O}(1)\cdot\frac{\Gamma_{f,0}M_1(t)}{(|x|+b_0t)}.$$

Similarly, the above estimates also hold for $x<0$. Thus, noticing that $\widetilde{\Psi}_2(t,0)=0$ and using again the $\mathbf{L}^2$-continuity of $\mathbf{G}$, we get

$$\left\|\mathbf{G}\left[\widetilde{\Psi}_2(t,\cdot)\right]\right\|_{H^2(\mathbb{R})} \;\le\; \mathcal{O}(1)\cdot\left\|\widetilde{\Psi}_2(t,\cdot)\right\|_{H^2(\mathbb{R})} \;\le\; \mathcal{O}(1)\cdot\Gamma_{f,0}b_0^{-1/2}M_1(t)t^{-1/2}.$$

Combining all the above estimates and using the fact that $3/4<\alpha<1$, we finally obtain (3.28) and (3.29). □

Using the estimates in Lemma 3.3, we provide an H^2 bound on $\mathbf{F}^{(w_1,w_2)}$ which allows us to obtain the convergence of a sequence of approximate solutions $w^{(k)}$ to (2.19) in Lemma 4.4.

Lemma 3.4 *There exists a constant* $\Gamma_2\doteq\mathcal{O}(1)\cdot\Gamma_f\left(\Gamma_{f,1}^2M_0^2C_0^3b_1+1\right)$ *and* $T>0$ *sufficiently small such that for every* $|x|<1/4$, $0<t<T$, *and* $\delta>0$, *we have*

$$\left|\mathbf{F}^{(w_1,w_2)}(t,x)\right|\le\Gamma_2\cdot\left(M_2(t)\cdot\left[t^{\alpha-1}+\ell(t)(t^\alpha+|x|^\alpha)\right]+\Sigma(t)|x|^\alpha\right) \tag{3.32}$$

$$\left\|\mathbf{F}^{(w_1,w_2)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])}\le\Gamma_2\cdot\left(M_2(t)\cdot\left[\ell(t)\delta^{\alpha-3/2}+\delta^{-2/3}t^{2\alpha-11/6}\right]+\Sigma(t)\delta^{\alpha-3/2}\right), \tag{3.33}$$

with $\Sigma(t)\doteq\max\left\{\left|\dot{w}_2^+(t)-\dot{w}_1^+(t)\right|,\left|\dot{w}_2^-(t)-\dot{w}_1^-(t)\right|\right\}$.

Proof. 1. Recalling (3.2), (3.19), and (3.23)-(3.25), we have

$$\mathbf{F}^{(w_1,w_2)}(t,x) \;=\; \mathbf{A}_2(t,x) - \mathbf{A}_1(t,x) + B^{(\mathbf{w})}(t,x) - \mathbf{C}(t,x),$$

with

$$\mathbf{A}_2 \;\doteq\; \mathbf{G}\big[\Psi\big], \qquad \mathbf{A}_1 \;\doteq\; A_1^{(w_2)} - A_1^{(w_1)}, \qquad \mathbf{C} \;\doteq\; C^{(w_2)} - C^{(w_1)}.$$

From Lemma A.2 and Lemma 3.3, for every $0 < |x| < 1/4$ and $\delta > 0$, it holds

$$\begin{cases} \big|B^{(\mathbf{w})}(t,x)\big| \;\leq\; \mathcal{O}(1)\cdot M_1(t), \qquad \big\|B^{(\mathbf{w})}(t,\cdot)\big\|_{H^1(\mathbb{R}\setminus\{0\})} \;\leq\; \mathcal{O}(1)\cdot M_1(t), \\ \big\|B^{(\mathbf{w})}(t,\cdot)\big\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; \mathcal{O}(1)\cdot M_2(t)\cdot\delta^{-2/3}, \end{cases}$$

and

$$\begin{cases} |\mathbf{A}_2(t,x)| \;\leq\; \mathcal{O}(1)\cdot\Gamma_{f,0}M_1(t), \qquad \|\mathbf{A}_2(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})} \;\leq\; \mathcal{O}(1)\cdot\Gamma_{f,1}M_1(t), \\ \|\mathbf{A}_2(t,\cdot)\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; \mathcal{O}(1)\cdot\Gamma_{f,1}M_1(t)\left(t^{\alpha-3/2}+\delta^{-2/3}\right). \end{cases}$$

2. To bound the term $\mathbf{A_1}$, we first recall $d^{(w)} \doteq f'\left(w+\varphi^{(w)}\right) - f'(w)$ and write

$$\mathbf{A}_1 \;=\; d^{(w_1)}\Psi_x + \mathbf{d}\cdot\varphi_x^{(w_2)} \;\doteq\; \mathbf{A}_{1,1} + \mathbf{A}_{1,2}, \qquad \mathbf{d} \;\doteq\; d^{(w_2)} - d^{(w_1)} \;=\; \mathbf{d}_1 + \mathbf{d}_2\,,$$

with

$$\begin{cases} \mathbf{d}_1 \;=\; \left(\displaystyle\int_0^1 f''\left(w_1+\tau\cdot\varphi^{(w_1)}\right)d\tau\right)\cdot\Psi, \\ \mathbf{d}_2 \;=\; \varphi^{(w_2)}\cdot\displaystyle\int_0^1\left[\int_0^1 f^{(3)}\left(w_1+\tau\cdot\varphi^{(w_1)}+[\mathbf{w}+\tau\Psi]\cdot s\right)ds\cdot(\mathbf{w}+\tau\Psi)\right]d\tau. \end{cases}$$

Since $\|w_i\|_{H^2(\mathbb{R}\setminus\{0\})} \leq M_0$ and $f\in C^4$, we directly estimate

$$\left|\frac{d}{dx}\mathbf{d}^{(\mathbf{w})}(t,x)\right| \leq \mathcal{C}\cdot\min\left\{M_1(t)|x|^{\alpha-1}, M_2(t)\left(\big|\ln|x|\big| + \frac{1}{(|x|+t)^{1-\alpha}}\right)\right\} \tag{3.34}$$

$$\left|\frac{d^2}{dx^2}\mathbf{d}^{(\mathbf{w})}(t,x)\right| \leq \mathcal{C}\cdot\left[|\mathbf{w}_{xx}| + M_2(t)\left(\big|w_{2,xx}\big| + \frac{1}{|x|} + \frac{1}{(|x|+t)^{2-\alpha}}\right)\right], \tag{3.35}$$

where $\mathcal{C} = \mathcal{O}(1)\cdot\Gamma_f\Gamma_{f,1}C_0^2$.

Notice that $d^{(w_1)}(t,0) = \mathbf{d}^{(\mathbf{w})}(t,0) = 0$. By keeping the leading order terms, we derive from (3.20), Lemma 3.1 and Lemma 3.3 that for every $|x| < 1/4$ and $0 < t < T$, it holds

$$\left|\mathbf{A}_1(t,x)\right| \;\leq\; |\mathbf{A}_{1,1}(t,x)| + |\mathbf{A}_{1,2}(t,x)| \;\leq\; \Gamma_2 M_2(t)\cdot|x|^\alpha t^{\alpha-1},$$

$$\left|\frac{d}{dx}\mathbf{A}_1(t,x)\right| \;\leq\; |\mathbf{A}_{1,1}(t,x)| + |\mathbf{A}_{1,2}(t,x)| \;\leq\; \Gamma_2 M_2(t)\cdot\left(\ln^2|x| + \frac{t^{\alpha-1}}{(|x|+t)^{1-\alpha}}\right),$$

and

$$\begin{aligned}\left|\frac{d^2}{dx^2}\mathbf{A}_1(t,x)\right| \;\leq\; \Gamma_2\cdot\Bigg[&M_2(t)\left(\big|\ln|x|\big| + t^{\alpha-1}\right)\left(\frac{1}{|x|} + \frac{1}{(|x|+t)^{2-\alpha}}\right) \\ &+\Big(|\mathbf{w}_{xx}| + M_2(t)(|w_{1,xx}| + |w_{2,xx}|)\Big)\left(\big|\ln|x|\big| + \frac{1}{(|x|+t)^{1-\alpha}}\right)\Bigg],\end{aligned}$$

and thus

$$\|\mathbf{A}_1(t,\cdot)\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; \Gamma_2 M_2(t)\Big[\big(\ln\delta + t^{\alpha-1}\big)\big(\delta^{-1/2} + (\delta+t)^{\alpha-3/2}\big)\Big].$$

3. It remains to estimate $\mathbf{C}$. From (3.21), (3.23), (3.24), (3.27), we have

$$\begin{aligned}\big|\mathbf{C}(t,0+)\big| \;\leq\; & \Big|\sigma^{(w_2)}\cdot\Phi'\big(-tb_+^{(w_2)}\big) - \sigma^{(w_1)}\cdot\Phi'\big(-tb_+^{(w_1)}\big)\Big| \\ & + \Big|b_+^{(w_2)}\cdot\bar{v}'\big(-tb_+^{(w_2)}\big) - b_+^{(w_1)}\cdot\bar{v}'\big(-tb_+^{(w_1)}\big)\Big| \;\leq\; \Gamma_2 M_1(t)t^{\alpha-1}.\end{aligned}$$

Moreover, for $0<x<1/4$ (similar estimates hold when $-1/4<x<0$), it holds

$$\frac{d}{dx}\mathbf{C}(t,x) \;=\; \frac{d}{dx}\mathbf{C}_1(t,x) + \frac{d}{dx}\mathbf{C}_2(t,x) + \frac{d}{dx}\mathbf{C}_3(t,x),$$

with

$$\mathbf{C}_1 = \left[\frac{\dot\sigma^{(w_2)}(t)}{\sigma^{(w_2)}(t)} - \frac{\dot\sigma^{(w_1)}(t)}{\sigma^{(w_1)}(t)} - \frac{\dot b_+^{(w_2)}(t)}{b_+^{(w_2)}(t)} + \frac{\dot b_+^{(w_1)}(t)}{b_+^{(w_1)}(t)}\right]\widetilde{\varphi}^{(w_2)} + \left(\frac{\dot\sigma^{(w_1)}(t)}{\sigma^{(w_1)}(t)} - \frac{\dot b_+^{(w_1)}(t)}{b_+^{(w_1)}(t)}\right)\widetilde{\Psi},$$

$$\mathbf{C}_2 = (\mathbf{b}(t,x) - \mathbf{b}_+(t))\cdot\varphi_x^{(w_2)} + \Big(b^{(w_1)}(t,x) - b_+^{(w_1)}(t)\Big)\cdot\Psi_x,$$

$$\mathbf{C_3} = -t\dot{\mathbf{b}}_+(t)\cdot D^{(w_2)} + t\dot b_+^{(w_1)}(t)\cdot\Big[D^{(w_1)} - D^{(w_2)}\Big],$$

and

$$D^{(w)}(t,x) \;\doteq\; \frac{\sigma^{(w)}(t)}{b_+^{(w)}(t)}\Phi'\Big(x - tb_+^{(w)}(t)\Big) + \bar{v}'\Big(x - tb_+^{(w)}(t)\Big).$$

Recalling (3.5), (3.9), (3.24), and (3.27), and taking $\tilde{C} = \mathcal{O}(1)\frac{\Gamma_f(M_0+b_1)+1}{b_0^2}$, we have

$$\left|\frac{\dot\sigma^{(w_2)}(t)}{\sigma^{(w_2)}(t)} - \frac{\dot\sigma^{(w_1)}(t)}{\sigma^{(w_1)}(t)}\right| + \left|\frac{\dot b_+^{(w_2)}(t)}{b_+^{(w_2)}(t)} - \frac{\dot b_+^{(w_1)}(t)}{b_+^{(w_1)}(t)}\right| \;\leq\; \tilde{C}\Big(\ell(t)M_1(t) + \Sigma(t)\Big),$$

and this implies

$$\left|\frac{d^i}{dx^i}\mathbf{C}_1\right| \;\leq\; \Gamma_2\cdot\left[\Big(\ell(t)M_1(t)+\Sigma(t)\Big)\cdot\left|\frac{d^i}{dx^i}\widetilde{\varphi}^{(w_2)}\right| + \Gamma\cdot\ell(t)\cdot\left|\frac{d^i}{dx^i}\widetilde{\Psi}\right|\right].$$

Moreover, from (3.8) and (3.26), taking $\hat{C} = \mathcal{O}(1)(\Gamma_f M_0^2 + \Gamma_f + b_1 + 1)$, we obtain

$$\begin{cases}\left|\dfrac{d}{dx}\mathbf{C}_2\right| \;\leq\; \hat{C}\displaystyle\sum_{i=1}^{2} x^{i-1}\left(M_2(t)\left|\frac{d^i}{dx^i}\varphi^{(w_2)}\right| + \left|\frac{d^i}{dx^i}\Psi\right|\right), \\ \left|\dfrac{d^2}{dx^2}\mathbf{C}_2\right| \;\leq\; \hat{C}\Bigg[\displaystyle\sum_{i=2}^{3} x^{i-2}\left(M_2(t)\left|\frac{d^i}{dx^i}\varphi^{(w_2)}\right| + \left|\frac{d^i}{dx^i}\Psi\right|\right) \\ \qquad + \big[|\mathbf{w}_{xx}| + M_2(t)(|w_{2,xx}|+1)\big]\left|\varphi_x^{(w_2)}(t,x)\right| + \big(|w_{2,xx}|+1\big)|\Psi_x|\Bigg].\end{cases}$$

In particular, Lemma 3.1 and Lemma 3.3 imply

$$\begin{cases} \left|\dfrac{d}{dx}\mathbf{C}_1\right| \le \Gamma_2 \cdot \left[\ell(t)M_1(t)+\Sigma(t)\right]|\ln x|, \ \left|\dfrac{d^2}{dx^2}\mathbf{C}_1\right| \le \Gamma_2 \cdot \dfrac{\ell(t)M_1(t)+\Sigma(t)}{x}, \\ \left|\dfrac{d}{dx}\mathbf{C}_2\right| \le \Gamma_2 M_2(t)\cdot\left(|\ln x| + (x+t)^{\alpha-1}\right), \\ \left|\dfrac{d^2}{dx^2}\mathbf{C}_2\right| \le \Gamma_2 \cdot \Big(M_2(t)\left(x^{-1}+(x+t)^{\alpha-2}\right) + \left(|\mathbf{w}_{xx}| + M_2(t)\,|w_{2,xx}|\right)\tilde{C}(x,t)\Big), \end{cases} \tag{3.36}$$

with $\tilde{C}(x,t) = |\ln x| + (x+t)^{\alpha-1}$. Noticing that

$$\left|\dot{\mathbf{b}}_+(t)\right| \ \le \ \mathcal{O}(1)\Gamma_f(M_0+1)\big[\ell(t)M_1(t)+\Sigma(t)\big], \qquad \left|\dot{b}_+^{(w_1)}(t)\right| \ \le \ \mathcal{O}(1)\Gamma_f(M_0+1)\cdot\ell(t),$$

we have

$$\left|\frac{d^i}{dx^i}\mathbf{C}_3\right| \le \tilde{C}t\left(\left[\ell(t)M_1(t)+\Sigma(t)\right]\left|\frac{d^i}{dx^i}D^{(w_2)}\right| + \ell(t)\left|\frac{d^i}{dx^i}\left(D^{(w_2)}-D^{(w_1)}\right)\right|\right).$$

with $\tilde{C} = \mathcal{O}(1)\Gamma_f(M_0+1)$. Thus, a direct computation yields

$$\begin{cases} \left|\dfrac{d^i}{dx^i}D^{(w_2)}(t,x)\right| \ \le \ \mathcal{O}(1)\cdot\left(1+b_0^{\alpha-1-i}\right)\dfrac{1}{(x+t)^{1-\alpha+i}}, \\ \left|\dfrac{d^i}{dx^i}\left(D^{(w_2)}(t,x)-D^{(w_1)}(t,x)\right)\right| \ \le \ \mathcal{O}(1)\cdot\Gamma_f\left(1+b_0^{\alpha-2-i}\right)\dfrac{M_1(t)}{(x+t)^{1-\alpha+i}}, \end{cases}$$

and

$$\left|\frac{d^i}{dx^i}\mathbf{C}_3(t,x)\right| \ \le \ \Gamma_2\cdot\frac{\ell(t)M_1(t)+\Sigma(t)}{(x+t)^{i-\alpha}}.$$

Finally, combining the above estimates and (3.36), we obtain

$$\begin{cases} \left|\dfrac{d}{dx}\mathbf{C}(t,x)\right| \ \le \ \Gamma_2\cdot\Big[\ell(t)M_1(t)+M_2(t)+\Sigma(t)\Big]\left(|\ln x| + \dfrac{1}{(x+t)^{1-\alpha}}\right), \\ \left|\dfrac{d^2}{dx^2}\mathbf{C}(t,x)\right| \ \le \ \Gamma_2\cdot\Big\{\Big[\ell(t)M_1(t)+M_2(t)+\Sigma(t)\Big]\left(\dfrac{1}{x}+\dfrac{1}{(x+t)^{2-\alpha}}\right) \\ \qquad\qquad + \Big(|\mathbf{w}_{xx} + M_2(t)|w_{2,xx}|\Big)\left(|\ln x| + \dfrac{1}{(x+t)^{1-\alpha}}\right)\Big\}, \\ \|\mathbf{C}(t,\cdot)\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \le \ \Gamma_2\cdot\Big[\ell(t)M_1(t)+M_2(t)+\Sigma(t)\Big]\left(\delta^{-1/2}+(\delta+t)^{\alpha-3/2}\right), \end{cases}$$

and then we obtain the desired estimates. □

4 Local existence and uniqueness of an entropic solution

Throughout this section, we give a proof of Theorem 2.1 by constructing a local solution to the Cauchy problem (2.2) with general initial data of the form $\overline{u} = \overline{w} + \overline{v}$ as in (2.3) where $\overline{v}$ is in $\mathcal{X}_\alpha$ defined in (1.6) and $\overline{w} \in H^2(\mathbb{R}\backslash\{0\})$ satisfies

$$\delta_0 \ \doteq \ \overline{w}(0-) - \overline{w}(0+) \ > \ 0, \qquad \frac{M_0}{2} \ \doteq \ \|\overline{w}\|_{H^2(\mathbb{R}\backslash\{0\})} \ < \ \infty. \tag{4.1}$$

The solution will be obtained as limit of a Cauchy sequence of approximate solutions $w_n(t,x)$ in $L^\infty([0,T], H^2(\mathbb{R}\backslash\{0\}))$ for some $T>0$ sufficiently small, following the two steps (i)-(ii) outlined at the end of Section 2. Indeed, we first establish the existence and uniqueness of solutions to the Cauchy problem (2.19).

4.1 Construction of approximate solutions

For fixed $n \geq 1$, let $w_n : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a given function such that

$$\|w_n(t,\cdot)\|_{H^2(\mathbb{R}\backslash\{0\})} \ \leq\ M_0, \qquad |w_n(t,0\pm)-\overline{w}(0\pm)| \ \leq\ \frac{\delta_0}{3}. \tag{4.2}$$

For simplicity, recalling the definition of the corrector function $\varphi^{(w)}(t,x)$ in (2.14), (2.16), and (2.17), we set

$$\begin{cases} \sigma_n(t) \ \doteq\ w_n^-(t)-w_n^+(t), \qquad w_n^\pm(t) \ \doteq\ w_n(t,0\pm), \\ \varphi_n(t,x) \ \doteq\ \varphi^{(w_n)}(t,x), \qquad a_n(t,x) \ \doteq\ a(t,x,w_n). \end{cases} \tag{4.3}$$

The Cauchy problem (2.19) can be rewritten as

$$w_t + a_n(t,x)\cdot w_x \ =\ F(t,x,w), \qquad w(0,x) \ =\ \overline{w}(x). \tag{4.4}$$

Hence, to complete step (i), we shall prove that (4.4) admits a unique solution w_{n+1} which satisfies the bounds listed in (4.2). The construction of w_{n+1} is divided into three steps:

Step 1. Let $t\mapsto x_n(t;\tau,x_\tau)$ be the solution to the Cauchy problem

$$\dot{x}(t) \ =\ a_n(t,x(t)), \qquad x(\tau) \ =\ x_\tau,$$

with $a_n(t,x)$ being the characteristic speed of (4.4), i.e.

$$a_n(t,x) \ =\ b^{(w_n)}(t,x) + f'\left(w_n+\varphi_n\right) - f'(w_n).$$

The solution w_{n+1} will be constructed by considering a sequence of approximate solutions $w^{(k)}$ to (2.19), inductively defined as follows.

1. $w^{(1)}(t,\cdot) \ \doteq\ \overline{w}(\cdot)$ for all $t\geq 0$.
2. For every $k\geq 1$, $w^{(k+1)}(t,\cdot)$ solves the linear equation

$$w_t + a_n(t,x)\cdot w_x \ =\ F^{(k)}(t,x) \ \doteq\ F\left(t,x,w^{(k)}\right), \qquad w(0,\cdot) \ =\ \overline{w}(\cdot). \tag{4.5}$$

   Equivalently, $w^{(k+1)}$ satisfies the integral identity

$$w^{(k+1)}(t_0,x_0) \ =\ \overline{w}(x_n(0;t_0,x_0)) + \int_0^{t_0} F^{(k)}(t,x_n(t;t_0,x_0))dt. \tag{4.6}$$

For every $0\leq t\leq\tau<\infty$, we denote by

$$I_t^\tau \ \doteq\ \left[-b_0(\tau-t)/2, b_0(\tau-t)/2\right]. \tag{4.7}$$

The next lemma provides some properties including the Lipschitz continuous dependence of the characteristic curves $t\mapsto x_n(t;\tau,x_\tau)$.

Lemma 4.1 *Assume that w_n and φ_n satisfy (4.2)-(4.3). Then there exist constants $\delta_1, T, K > 0$ depending only on M_0, δ_0 and f such that for every $0 \le t \le \tau$, and for all $(x_\tau, \tau) \in ([-\delta_1, \delta_1]\backslash\{0\}) \times (0, T]$, we have*

$$x_n(t;\tau,x_\tau) \notin I_t^\tau, \quad \text{and} \quad \frac{b_0(\tau-t)}{2} \le \left|x_n(t;\tau,x_\tau)-x_\tau\right| \le 2b_1(\tau-t). \tag{4.8}$$

Moreover, for any $0 < x_1 < x_2 \le \delta_1$ or $-\delta_1 \le x_1 < x_2 < 0$, one has

$$\left|x_n(t;\tau,x_1)-x_n(t;\tau,x_2)\right| \le K \cdot |x_1 - x_2|. \tag{4.9}$$

Proof. From lemma 3.1, it holds for every $0 < |x| < 1/4$ and $0 < t < \dfrac{1}{4b_1}$ that

$$\begin{aligned} \left|\frac{d}{dx}a_n(t,x)\right| &\le \left|b_x^{(w_n)}(t,x)\right| + \Gamma_f \cdot \left(\left|\varphi_{n,x}(t,x)\right| + 2\cdot\left|w_{n,x}(t,x)\right|\right) \\ &\le b_1 + \Gamma_f \cdot \left(C_0 \cdot |x|^{\alpha-1} + 4M_0\right). \end{aligned} \tag{4.10}$$

Recalling (3.9), we have that

$$b_0 \le a_n(t,0-) = b_-^{(w_n)}(t) \le b_1, \qquad -b_1 \le a_n(t,0+) = b_+^{(w_n)}(t) \le -b_0,$$

there exists a constant $\delta_1 > 0$ depending on b_1, b_0, C_0, M_0, and Γ_f such that

$$\begin{cases} -2b_1 \le a_n(t,x) \le -\dfrac{b_0}{2}, & (t,x) \in [0,1/(4b_1)]\times\,]0,2\delta_1], \\ \dfrac{b_0}{2} \le a_n(t,x) \le 2b_1, & (t,x) \in [0,1/(4b_1)] \times [-2\delta_1, 0[\,. \end{cases} \tag{4.11}$$

In particular, set $T \doteq \min\{1/(4b_1), \delta_1/(2b_1)\}$. For $(x_\tau, \tau) \in ([-\delta_1,\delta_1]\backslash\{0\}) \times (0,T]$, one has

$$\left|x_n(t;\tau,x_\tau)\right| \le 2b_1|\tau| + |x_\tau| \le 2\delta_1, \qquad 0 \le t \le \tau,$$

and (4.11) implies (4.8).

To complete the proof, we shall establish (4.9) for $0 < x_1 < x_2 \le \delta_1$, the other case being entirely similar. Set $z(t) \doteq \left|x_n(t;\tau,x_1) - x_n(t;\tau,x_2)\right|$. From (4.10), one has

$$\begin{aligned} \left|\frac{d}{dt}z(t)\right| &\le \left(b_1 + \Gamma_f \cdot \left[C_0 \cdot |x_n(t;\tau,x_2)|^{\alpha-1} + 4M_0\right]\right) \cdot z(t) \\ &\le \left(b_1 + \Gamma_f \cdot \left[C_0 \cdot \left|\frac{b_0(\tau-t)}{2}\right|^{\alpha-1} + 4M_0\right]\right) \cdot z(t), \end{aligned}$$

and this yields (4.9). □

As a consequence, for every $\tau \in [0,T]$, all characteristics starting at time $t = 0$ inside the interval I_0^τ hit the origin before time τ. On the other hand, since $a_n = b^{(w_n)} + d^{(w_n)}$, from (3.8), (3.20), and (4.7), there exists a constant $\widetilde{C}_0 > 0$ depending only on M_0, δ_0, α and f such that for every $x \in \mathbb{R}\backslash I_t^\tau$ and $t \in [0,\tau]$, it holds

$$\left|a_{n,x}(t,x)\right| \le \widetilde{C}_0 \cdot (\tau-t)^{\alpha-1}, \qquad \left|a_{n,xx}(t,x)\right| \le \widetilde{C}_0 \cdot \left(|w_{n,xx}(t,x)| + |x|^{\alpha-2}\right). \tag{4.12}$$

Hence, one can use the same arguments as in [4, Lemma 4.1] to prove the following Lemma:

Lemma 4.2 *Under the same assumptions as in Lemma 4.1, there exists a constant $T > 0$ depending only on M_0, δ_0 and f such that for every $\tau \in [0, T]$ and any solution v of the linear equation*

$$v_t + a_n(t,x) \cdot v_x \;=\; 0, \qquad v(0,\cdot) \;=\; \bar{v} \;\in\; H^2\big(\mathbb{R}\backslash I_0^\tau\big), \tag{4.13}$$

one has

$$\|v(\tau,\cdot)\|_{H^2\big(\mathbb{R}\backslash\{0\}\big)} \;\leq\; \frac{3}{2} \cdot \|\bar{v}\|_{H^2(\mathbb{R}\backslash I_0^\tau)}. \tag{4.14}$$

Proof. To establish (4.14) for a fixed $\tau \in [0, T]$, we set

$$Z(t) \;\doteq\; \|v(t,\cdot)\|^2_{H^2(\mathbb{R}\backslash I_t^\tau)} \qquad \text{for all } t \in [0,\tau].$$

Multiplying the linear equation (4.13) by $2v$, we have

$$\big(v^2\big)_t + \big(a_n v^2\big)_x \;=\; a_{n,x} v^2, \qquad v(0,x) \;=\; \bar{v}(x).$$

Integrating the above equation over the domain

$$\Omega \;\doteq\; \bigcup_{t\in[0,\tau]} \{t\} \times (\mathbb{R}\backslash I_t^\tau) \;=\; \left\{(t,x) \in [0,\tau] \times \mathbb{R} : x \in \mathbb{R}\backslash I_t^\tau\right\},$$

and using the the first inequality in (4.12), we get

$$\begin{aligned} \int_{-\infty}^{\infty} v^2(\tau,x)dx \;&\leq\; \int_{x\in\mathbb{R}\backslash I_0^\tau} \bar{v}^2(x)dx + \int_0^\tau \int_{x\in\mathbb{R}\backslash I_t^\tau} a_{n,x}(t,x) v^2(t,x)\; dxdt \\ &\leq\; \|\bar{v}(\cdot)\|^2_{L^2(\mathbb{R}\backslash I_0^\tau)} + \widetilde{C}_0 \int_0^\tau (\tau - t)^{\alpha-1} \cdot Z(t)\; dt. \end{aligned} \tag{4.15}$$

Similarly, differentiating equation (4.13) with respect to x and multiplying by $2v_x$ (and by $2v_{xx}$), we have

$$\begin{cases} \big(v_x^2\big)_t + \big(a_n\, v_x^2\big)_x \;=\; -\, a_{n,x}\, v_x^2, \qquad v_x(0,x) \;=\; \bar{v}'(x)\,, \\ \big(v_{xx}^2\big)_t + \big(a_n\, v_{xx}^2\big)_x = -3a_{n,x}\, v_{xx}^2 - 2a_{n,xx}\, v_x\, v_{xx}, \qquad v_{xx}(0,x) \;=\; \bar{v}''(x). \end{cases}$$

Integrating the above equations over Ω, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} v_x^2(\tau,x)dx \;&\leq\; \int_{x\in\mathbb{R}\backslash I_0^\tau} [\bar{v}']^2(x)\,dx - \int_0^\tau \int_{x\in\mathbb{R}\backslash I_t^\tau} a_{n,x}(t,x) v_x^2(t,x)\; dxdt \\ &\leq\; \|\bar{v}'(\cdot)\|^2_{L^2(\mathbb{R}\backslash I_0^\tau)} + \widetilde{C}_0 \int_0^\tau (\tau - t)^{\alpha-1} Z(t)\; dt, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} v_{xx}^2(\tau,x)dx \;&\leq\; \int_{x\in\mathbb{R}\backslash I_0^\tau} [\bar{v}'']^2(x)\,dx - \int_0^\tau \int_{x\in\mathbb{R}\backslash I_t^\tau} 3a_{n,x} v_{xx}^2 + 2a_{n,xx} v_x v_{xx}\; dxdt \\ &\leq \|\bar{v}''(\cdot)\|^2_{L^2(\mathbb{R}\backslash I_0^\tau)} + 3\widetilde{C}_0 \int_0^\tau (\tau - t)^{\alpha-1} Z(t)dt + 2\widetilde{C}_0 \int_0^\tau \int_{x\in\mathbb{R}\backslash I_t^\tau} a_{n,xx} v_x v_{xx} dxdt. \end{aligned} \tag{4.17}$$

Using the second inequality in (4.12), (4.7), and Hölder's inequality, we estimate

$$\begin{aligned}\int_{x\in\mathbb{R}\setminus I_t^\tau} a_{n,xx} v_x v_{xx} dx \;&\leq\; \int_{x\in\mathbb{R}\setminus I_t^\tau} \left(1+|w_{xx}(t,x)|+|x|^{\alpha-2}\right)\left|v_x(t,x)v_{xx}(t,x)\right|\,dx \\ &\leq\; \|v_x(t,\cdot)\|_{L^2(\mathbb{R}\setminus I_t^\tau)}\|v_{xx}(t,\cdot)\|_{L^2(\mathbb{R}\setminus I_t^\tau)} \\ &+\|v_x(t,\cdot)\|_{L^\infty(\mathbb{R}\setminus I_t^\tau)}\cdot\int_{x\in\mathbb{R}\setminus I_t^\tau}\left[|w_{n,xx}(t,x)|+|x|^{\alpha-2}\right]\cdot\left|v_{xx}(t,x)\right|\,dx \\ &\leq\; Z(t)+2Z^{1/2}(t)\cdot\int_{x\in\mathbb{R}\setminus I_t^\tau}\left[|w_{n,xx}(t,x)|+|x|^{\alpha-2}\right]\cdot\left|v_{xx}(t,x)\right|\,dx \\ &\leq\; Z(t)\cdot\left(1+2M_0+(2-\alpha)^{-1/2}\cdot\left[\frac{b_0(\tau-t)}{2}\right]^{\alpha-3/2}\right).\end{aligned}$$

Thus, summing up (4.15), (4.16), and (4.17), we get

$$Z(t)\leq Z(0)+\widetilde{C}_0\int_0^\tau\left(2+4M_0+5(\tau-t)^{\alpha-1}+\frac{2}{(2-\alpha)^{1/2}}\left[\frac{b_0(\tau-t)}{2}\right]^{\alpha-3/2}\right)Z(t)dt,$$

and by Gronwall's lemma, we derive for $\tau>0$ sufficiently small that

$$\|v(\tau,\cdot)\|^2_{H^2\left(\mathbb{R}\setminus\{0\}\right)} \;=\; Z(\tau)\;\leq\;\frac{9}{4}\cdot Z(0)\;=\;\frac{9}{4}\cdot\|\bar{v}\|^2_{H^2(\mathbb{R}\setminus I_0^\tau)},$$

proving (4.14). □

As a consequence of Lemma 4.1, by Duhamel's formula, we get from (4.6) that

$$\left\|w^{(k+1)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\setminus\{0\})}\;\leq\;\frac{3}{2}\cdot\|\overline{w}\|_{H^2(\mathbb{R}\setminus I_0^\tau)}+\frac{3}{2}\cdot\int_0^\tau\left\|F^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus I_t^\tau)}dt. \tag{4.18}$$

for all $k\geq 1$.

Step 2. Using Lemma 3.2, Lemma 4.1, and (4.18), we are now going to establish a priori estimates on the sequence of approximations $\left(w^{(k)}\right)_{k\geq 1}$.

Lemma 4.3 *Assume that w_n and φ_n satisfy (4.2)-(4.3). There exists $T>0$ depending only on M_0,δ_0, and f such that*

$$\left|w^{(k)}(\tau,0\pm)-\overline{w}(0\pm)\right|\;\leq\;\frac{\delta_0}{3},\qquad\left\|w^{(k)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\setminus\{0\})}\;\leq\;M_0. \tag{4.19}$$

In addition, the map $\tau\mapsto w^{(k)}(\tau,0\pm)$ is locally Lipchitz and

$$\left|\dot{w}^{(k)}(\tau,0\pm)\right|\;\leq\;2\Gamma_1\tau^{\alpha-1}\quad a.e.\ \tau\in[0,T]. \tag{4.20}$$

Proof. 1. Since $w^{(1)}(\tau,\cdot)=\overline{w}(\cdot)$ for all $\tau\in[0,T]$, the estimates (4.19)-(4.20) hold for $k=1$. Assume that (4.19)-(4.20) hold for a given $k\geq 1$. Applying Lemma 3.2 for $w=w^{(k)}$, $\ell(t)=2\Gamma_1 t^{\alpha-1}$ and $\delta=\dfrac{b_0(\tau-t)}{2}$, we get

$$\left\|F^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus I_t^\tau)}\;\leq\;\Gamma_1\cdot\left[t^{-3/4}+(2\Gamma_1+1)t^{\alpha-1}\cdot\left(\frac{b_0(\tau-t)}{2}\right)^{-3/4}\right],$$

and (4.18), (4.1) yield

$$\begin{aligned}\left\|w^{(k+1)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\setminus\{0\})} &\leq \frac{3M_0}{4}+\frac{3}{2}\cdot\int_0^\tau \left\|F^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\setminus I_t^\tau)}dt \\ &\leq \frac{3M_0}{4}+\Gamma_{1,1}\cdot\left(\tau^{1/4}+\tau^{\alpha-3/4}\right) \leq \frac{3M_0}{4}+2\Gamma_{1,1}\cdot\tau^{\alpha-3/4}\end{aligned} \tag{4.21}$$

for some constant $\Gamma_{1,1}>0$ that depends on Γ_1, b_0 and M_0.

2. For any $\tau\in[0,T]$ and $-\delta_1<\bar{x}_2<\bar{x}_1<0$, consider the characteristics $t\mapsto x_i(t)\doteq x(t;\tau,\bar{x}_i)$ for $i=1,2$. Recalling Lemma 3.2, Lemma 4.1, and (4.6), we have

$$\begin{aligned}&\left|w^{(k+1)}(\tau,\bar{x}_2)-w^{(k+1)}(\tau,\bar{x}_1)\right| \\ &\leq \left|\overline{w}(x_2(0))-\overline{w}(x_1(0))\right|+\int_0^\tau \left|F^{(k)}(t,x_2(t))-F^{(k)}(t,x_1(t))\right|dt \\ &\leq 2M_0K|\bar{x}_2-\bar{x}_1|+(2\Gamma_1^2+\Gamma_1)\int_0^\tau \left[\left(\frac{b_0t|\tau-t|}{2}\right)^{\alpha-1}+t^{\alpha-3/2}\right]|x_2(t)-x_1(t)|dt \\ &\leq 2M_0K|\bar{x}_2-\bar{x}_1|+\Gamma_{1,2}\cdot\left(\tau^{2\alpha-1}+\tau^{\alpha-1/2}\right)\cdot|\bar{x}_2-\bar{x}_1| \\ &\leq \left(2M_0K+2\Gamma_{1,2}\cdot\tau^{1/4}\right)\cdot|\bar{x}_2-\bar{x}_1|\end{aligned} \tag{4.22}$$

for some constant $\Gamma_{1,2}>0$ that depends on Γ_1, b_0. An entirely similar estimate holds for $\tau\in[0,T]$ and $0<\bar{x}_2<\bar{x}_1<\delta_1$.

3. Given $0\leq\tau_1<\tau_2\leq T$, let $x_2^{\pm}(t)\doteq x(t;\tau_2,0\pm)$ be the backward characteristic starting from negative side and positive side of the origin at time τ_2 . From Lemma 3.2, Lemma 4.1, (4.6), and (4.22), one has

$$\begin{aligned}&\left|w^{(k+1)}(\tau_2,0\pm)-w^{(k+1)}(\tau_1,0\pm)\right| \\ &\leq \left|w^{(k+1)}(\tau_1,x_2^{\pm}(\tau_1))-w^{(k+1)}(\tau_1,0\pm)\right|+\int_{\tau_1}^{\tau_2}\left|F^{(k)}(t,x_2^{\pm}(t))\right|dt \\ &\leq 6M_0Kb_1(\tau_2-\tau_1)+\Gamma_1\int_{\tau_1}^{\tau_2}t^{\alpha-1}\left(1+2\Gamma_1\cdot\left[\left|\frac{b_0(\tau-t)}{2}\ln\left(\frac{b_0(\tau-t)}{2}\right)\right|+t^{\alpha}\right]\right)dt \\ &\leq 6M_0Kb_1(\tau_2-\tau_1)+\left[\Gamma_1+2\Gamma_1^2\left(T^\alpha+\left|\frac{b_0T}{2}\ln\left(\frac{b_0T}{2}\right)\right|\right)\right]\cdot\min\left\{\frac{\tau_2^\alpha}{\alpha},\tau_1^{\alpha-1}(\tau_2-\tau_1)\right\}.\end{aligned}$$

In particular, for almost every $\tau\in[0,T]$, it holds

$$\left|w^{(k+1)}(\tau,0\pm)-\overline{w}(0\pm)\right| \leq 6M_0Kb_1T+\left[\Gamma_1+2\Gamma_1^2\left(T^\alpha+\left|\frac{b_0T}{2}\ln\left(\frac{b_0T}{2}\right)\right|\right)\right]\cdot\frac{T^\alpha}{\alpha}, \tag{4.23}$$

and

$$\left|\dot{w}^{(k+1)}(\tau,0\pm)\right| \leq 6M_0Kb_1+\left[\Gamma_1+2\Gamma_1^2\left(T^\alpha+\left|\frac{b_0T}{2}\ln\left(\frac{b_0T}{2}\right)\right|\right)\right]\cdot\tau^{\alpha-1}. \tag{4.24}$$

Thus, from (4.21), (4.23), and (4.24), there exists a sufficiently small time $T>0$ depending only on M_0,δ_0 and f so that (4.19)-(4.20) holds. □

Step 3. Thanks to the above estimates, we now complete step (i), which is a key step toward the proof of Theorem 2.1, by proving that the sequence of approximations $w^{(k)}$ is Cauchy and converges to a solution w of the linear problem (2.19).

Lemma 4.4 *Under the same settings in Lemma 4.3, the sequence of approximations $\left(w^{(k)}\right)_{k\geq 1}$ converges to a limit function w in $\mathbf{L}^\infty([0,T],H^2(\mathbb{R}\backslash\{0\}))$ for sufficiently small $T>0$ depending only on M_0,δ_0, and f. Moreover, $w(\tau,\cdot)$ satisfies (4.19)-(4.20) in $[0,T]$.*

Proof. 1. For every $k\geq 1$, we set $\mathbf{w}^{(k)}\doteq w^{(k+1)}-w^{(k)}$ and

$$\beta^{(k)}(\tau) \ \doteq \ \sup_{t\in[0,\tau]}\left\|\mathbf{w}^{(k)}(t,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{0\})} \qquad \text{for all } \tau\in[0,T]. \tag{4.25}$$

By (4.20), for almost every $\tau\in[0,T]$, we define

$$\Sigma^{(k)}(\tau) \ \doteq \ \max\left\{\left|\dot{\mathbf{w}}^{(k)}(\tau,0+)\right|,\left|\dot{\mathbf{w}}^{(k)}(\tau,0-)\right|\right\} \ \leq \ 4\Gamma_1\tau^{\alpha-1}. \tag{4.26}$$

From (3.22) and (4.5), $\mathbf{w}^{(k)}$ solves the semilinear equation

$$v_t+a_n(t,x)\cdot v_x \ = \ \mathbf{F}^{\left(w^{(k)},w^{(k+1)}\right)}(t,x), \qquad v(0,\cdot) \ = \ 0\,.$$

In particular, Lemma 4.2 and the Duhamel formula yield

$$\left\|\mathbf{w}^{(k+1)}(\tau,\cdot)\right\|_{H^2(\mathbb{R}\backslash\{0\})} \ \leq \ \frac{3}{2}\cdot\int_0^\tau\left\|\mathbf{F}^{\left(w^{(k)},w^{(k+1)}\right)}(s,\cdot)\right\|_{H^2(\mathbb{R}\backslash I_s^\tau)}ds$$

Thus, by (4.7), (4.20), and the second inequality in (3.32), we derive

$$\begin{aligned}\beta^{(k+1)}(\tau)\leq \ & C_1\cdot\left(\beta^{(k)}(\tau)\int_0^\tau s^{\alpha-1}(\tau-s)^{\alpha-\frac{3}{2}}+(\tau-s)^{-\frac{2}{3}}s^{2\alpha-\frac{11}{6}}+\Sigma^{(k)}(s)(\tau-s)^{\alpha-\frac{3}{2}}\,ds\right)\\ \leq \ & C_2\beta^{(k)}(\tau)\tau^{2\alpha-\frac{3}{2}}+C_1\cdot\int_0^\tau\Sigma^{(k)}(s)(\tau-s)^{\alpha-\frac{3}{2}}\,ds.\end{aligned} \tag{4.27}$$

for some constants C_1,C_2 depend only on Γ_2, and b_0

2. Next, we are going to provide a bound on $\Sigma^{(k)}$ in terms of $\beta^{(k)}$ and $\beta^{(k+1)}$. Given any $0<\tau_1<\tau_2\leq T$, let $\tilde{x}_2^\pm(t)=x_n(t;\tau_2,0\pm)$ be the characteristics, which reach the origin at time τ_i, from the positive or negative side, respectively. From (4.6), it holds

$$\mathbf{w}^{(k+1)}(\tau_2,0\pm) \ = \ \mathbf{w}^{(k+1)}\left(\tau_1,\tilde{x}_2^\pm(\tau_1)\right)+\int_{\tau_1}^{\tau_2}\mathbf{F}^{\left(w^{(k)},w^{(k+1)}\right)}(t,\tilde{x}_2^\pm(t))dt.$$

Using (4.8), (4.25), (4.26), and the first inequality in (3.32), we estimate

$$\begin{aligned}&\left|\mathbf{w}^{(k+1)}(\tau_2,0\pm)-\mathbf{w}^{(k+1)}(\tau_1,0\pm)\right|\\ &\quad\leq \ \left|\mathbf{w}^{(k+1)}\left(\tau_1,\tilde{x}_2^\pm(\tau_1)\right)-\mathbf{w}^{(k+1)}(\tau_1,0\pm)\right|+\int_{\tau_1}^{\tau_2}\left|\mathbf{F}^{\left(w^{(k)},w^{(k+1)}\right)}\left(t,\tilde{x}_2^\pm(t)\right)\right|dt\\ &\quad\leq \ \beta^{(k+1)}(\tau_1)\left|\tilde{x}_2^\pm(\tau_1)\right|+4\Gamma_2\Gamma_1\tau_1^{\alpha-1}\int_{\tau_1}^{\tau_2}\left|\tilde{x}_2^\pm(t)\right|^\alpha dt\\ &\qquad\qquad+\Gamma_2\beta^{(k)}(\tau_2)\int_{\tau_1}^{\tau_2}t^{\alpha-1}+2\Gamma_1t^{\alpha-1}\left(t^\alpha+\left|\tilde{x}_2^\pm(t)\right|^\alpha\right)dt\\ &\quad\leq \ C_3\cdot\left(\beta^{(k+1)}(\tau_1)+\tau_1^{\alpha-1}(\tau_2-\tau_1)^\alpha+\beta^{(k)}(\tau_2)\tau_1^{\alpha-1}\right)\cdot(\tau_2-\tau_1),\end{aligned}$$

and the increasing property of the map $\tau \mapsto \beta^{(k+1)}(\tau)$ implies that

$$\left|\dot{\mathbf{w}}^{(k+1)}(s,0\pm)\right| \;\leq\; C_3 \cdot \Big(\beta^{(k+1)}(s) + \beta^{(k)}(s)s^{\alpha-1}\Big), \qquad a.e.\ s \in]0,T], \tag{4.28}$$

where $C_3 > 0$ only depends on Γ_1, Γ_2, b_1, and α.

3. To complete the proof, we introduce the sequence of maps $\tau \mapsto \gamma^{(k)}(\tau)$ defined by

$$\gamma^k(\tau) \;=\; \frac{\beta^{(k)}(\tau)}{4C_1} + \int_0^\tau \Sigma^{(k)}(s)(\tau-s)^{\alpha-3/2}\,ds \qquad \text{for all } \tau \in [0,T].$$

with C_1 being the same as in (4.27). Then, from (4.26), (4.27), (4.28), and the increasing property of $\beta^{(k)}$, it holds

$$\begin{aligned}\gamma^{k+1}(\tau) \;&\leq\; \frac{C_2\tau^{2\alpha-1/2}}{4C_1}\beta^{(k)}(\tau) + \frac{1}{4}\int_0^\tau \Sigma^{(k)}(s)(\tau-s)^{\alpha-3/2}\,ds \\ &\qquad + C_3\int_0^\tau \Big(\beta^{(k+1)}(\tau) + \beta^{(k)}(\tau)s^{\alpha-1}\Big)(\tau-s)^{\alpha-3/2}\,ds \\ &\leq\; C_4\tau^{\alpha-1/2}\beta^{(k+1)}(\tau) + C_4\tau^{2\alpha-3/2}\beta^{(k)}(\tau) + \frac{1}{4}\int_0^\tau \Sigma^{(k)}(s)(\tau-s)^{\alpha-3/2}\,ds\end{aligned}$$

for some constant $C_4 > 0$ that only depends on C_1, C_2 and C_3. In particular, by choosing $T > 0$ sufficiently small such that

$$C_4T^{\alpha-1/2} \;\leq\; \frac{1}{2}, \qquad C_4T^{2\alpha-3/2} \;\leq\; \frac{1}{16C_1},$$

one obtains a contractive property

$$\sup_{\tau\in[0,T]} \gamma^{(k+1)}(\tau) \;\leq\; \frac{1}{2}\cdot \sup_{\tau\in[0,T]} \gamma^{(k)}(\tau) \qquad \text{for all } k \geq 1.$$

It follows that $\big(w^{(k)}\big)_{k\geq 1}$ is a Cauchy sequence in $\mathbf{L}^\infty([0,T], H^2(\mathbb{R}\backslash\{0\}))$ and converges to a function $w_{n+1} \in \mathbf{L}^\infty([0,T], H^2(\mathbb{R}\backslash\{0\}))$. This implies that

$$\|w_{n+1}(\tau,\cdot)\|_{H^2(\mathbb{R}\backslash\{0\})} \;\leq\; M_0, \qquad \lim_{k\to\infty} w^{(k)}(\tau,0\pm) \;=\; w_{n+1}(\tau,0\pm) \qquad \text{for all } \tau \in [0,T],$$

and w_{n+1} satisfies (4.19)–(4.20). Furthermore, from (4.28) and (4.20), the limit $\lim\limits_{k\to\infty} \dot{w}^{(k)}(\tau,0\pm)$ exist and bounded by $2\Gamma_1\tau^{\alpha-1}$ for almost every $\tau \in (0,T)$. Thus, for every $0 < \tau_1 < \tau_2 < T$, one has

$$\begin{aligned} w_{n+1}(\tau_2,0\pm) - w_{n+1}(\tau_1,0\pm) =&\; \lim_{k\to\infty}\Big(w^{(k)}(\tau_2,0\pm) - w^{(k)}(\tau_1,0\pm)\Big) \\ =&\; \lim_{k\to\infty}\int_{\tau_1}^{\tau_2} \dot{w}^{(k)}(\tau,0\pm)d\tau \;=\; \int_{\tau_1}^{\tau_2} \lim_{k\to\infty} \dot{w}^{(k)}(\tau,0\pm)d\tau\,, \end{aligned}$$

and this yields

$$\lim_{k\to\infty} \dot{w}^{(k)}(\tau,0\pm) \;=\; \dot{w}_{n+1}(\tau,0\pm) \quad a.e.\ \tau \in [0,T].$$

Thus, recalling the first estimate in (3.32), we have

$$\lim_{k\to\infty} F\big(\tau,x,w^{(k)}(\tau,x)\big) \;=\; F\big(\tau,x,w_{n+1}(\tau,x)\big).$$

Finally, taking $k \to +\infty$ in (4.6), we obtain that for all $t_0 \in [0,T]$,

$$w_{n+1}(t_0,x_0) \;=\; \overline{w}(x_n(0;t_0,x_0)) + \int_0^{t_0} F\Big(t, x_n(t;t_0,x_0), w_{n+1}\Big)\,dt,$$

and w_{n+1} is the solution to the semilinear equation (4.4). □

4.2 Proof of Theorem 2.1

1. By Lemma 4.4, we inductively construct the sequence of approximate solutions $(w_n)_{n\geq 1}$ where each w_n solves (2.19) and satisfies (4.2), on a suitably small time interval $[0,T]$. Moreover, the map $\tau \mapsto w_n(\tau,0\pm)$ is locally Lipschitz and

$$|\dot{w}_n(\tau,0\pm)| \;\leq\; 2\Gamma_1\cdot\tau^{\alpha-1} \quad a.e.\ \tau\in[0,T].$$

As outlined at the end of Section 2, we show that the sequence $(w_n)_{n\geq 1}$ is Cauchy w.r.t. the norm in $H^1(\mathbb{R}\backslash\{0\})$, hence it converges to a unique limit w providing an entropic solution to the Cauchy problem (2.15). In order to do so, for a fixed $n\geq 2$, we define

$$\begin{cases} \mathbf{w}_n \;\doteq\; w_n - w_{n-1}, & \Psi_n \;\doteq\; \varphi^{(w_n)} - \varphi^{(w_{n-1})}, \qquad \mathbf{u}_n \;\doteq\; \mathbf{w}_n + \Psi_n, \\[2ex] A_n \;\doteq\; a_n - a_{n-1}, & \beta_n(\tau) \;\doteq\; \sup\limits_{t\in[0,\tau]} \|\mathbf{w}_n(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})}\,. \end{cases}$$

Recalling (2.19), we have

$$\mathbf{u}_{n+1,t} + a_n\cdot\mathbf{u}_{n+1,x} \;=\; \mathbf{G}[\Psi_{n+1}+\mathbf{w}_{n+1}] - A_n w_{n,x} - A_{n+1}\varphi_x^{(w_{n+1})}.$$

From Lemma 4.2 and Duhamel's formula, we obtain that for all $\tau\in[0,T]$, we have

$$\begin{aligned} &\|\mathbf{u}_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \leq \\ &\qquad \leq \frac{3}{2}\int_0^\tau \Big\| \Big(\mathbf{G}[\Psi_{n+1}+\mathbf{w}_{n+1}] - A_n w_{n,x} - A_{n+1}\varphi_x^{(w_{n+1})}\Big)(t,\cdot)\Big\|_{H^1(\mathbb{R}\backslash I_t^\tau)}\,dt \end{aligned} \tag{4.29}$$

2. To bound the right-hand side of (4.29), we first recall the last inequality in (3.28) and Lemma A.2 to get

$$\begin{cases} \|\mathbf{G}[\Psi_{n+1}\|_{H^1(\mathbb{R}\backslash\{0\})} \;\leq\; \Gamma_{f,1}\cdot\|\mathbf{w}_{n+1}(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \;\leq\; \Gamma_{f,1}\cdot\beta_{n+1}(t), \\[2ex] \|\mathbf{G}[\mathbf{w}_{n+1}\|_{H^1(\mathbb{R}\backslash[-\delta,\delta])} \;\leq\; \dfrac{\|\mathbf{w}_{n+1}(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})}}{\delta^{1/2}} \;\leq\; \dfrac{\beta_{n+1}(t)}{\delta^{1/2}}, \end{cases}$$

for $\delta>0$ sufficiently small. On the other hand, since $a_n = b^{(w_n)} + d^{(w_n)}$, from (3.26) and (3.34), it holds

$$\begin{cases} |A_n(t,x)| \leq \widetilde{C}_1\|\mathbf{w}_n(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})}, \\[2ex] |A_{n,x}(t,x)| \leq \widetilde{C}_1\left(\frac{\|\mathbf{w}_n(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})}}{|x|^{1-\alpha}} + \mathbf{w}_{n,x}(t,x)\right) \end{cases} \tag{4.30}$$

and this implies that

$$\|A_n(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \;\leq\; \widetilde{C}_2\cdot\|\mathbf{w}_n(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})},$$

for some constant $\widetilde{C}_1, \widetilde{C}_2 > 0$ depending on f, M_0, and C_0. Thus, using (3.5) and (3.12)-(3.14), we derive

$$\begin{aligned}
\|A_n(t,\cdot)w_{n,x}(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})} &\leq 4\|A_n(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})}\cdot\|w_{n,x}\|_{H^1(\mathbb{R}\setminus\{0\})}\\
&\leq 4M_0\cdot\|A_n(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})} \leq 4\widetilde{C}_2 M_0\beta_n(t),\\
\left\|A_{n+1}(t,\cdot)\varphi_x^{(w_{n+1})}(t,\cdot)\right\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} &\leq 4\|A_{n+1}(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})}\|\varphi_x^{(w_{n+1})}(t,\cdot)\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])}\\
&\leq \frac{4C_0}{\delta^{3/2-\alpha}}\|A_{n+1}(t,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})} \leq \frac{4C_0\widetilde{C}_2\beta_{n+1}(t)}{\delta^{3/2-\alpha}}
\end{aligned}$$

Thus, from (4.29) and (4.7), we obtain

$$\begin{aligned}
&\|\mathbf{u}_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})}\\
&\leq \frac{3}{2}\int_0^\tau 4\widetilde{C}_2M_0\beta_n(t)+\left(\Gamma_{f,1}+\left|\frac{2}{b_0(\tau-t)}\right|^{1/2}+4C_0\widetilde{C}_2\left|\frac{2}{b_0(\tau-t)}\right|^{3/2-\alpha}\right)\beta_{n+1}(t)dt\\
&\leq 6\widetilde{C}_2M_0\tau\beta_n(\tau)+\widetilde{C}_3\tau^{\alpha-1/2}\beta_{n+1}(\tau)
\end{aligned}\tag{4.31}$$

for some constant $\widetilde{C}_3 > 0$ depending on $C_0, \widetilde{C}_2, \Gamma_{f,1}, b_0$ and α. From the last inequality in (3.28), one has

$$\begin{aligned}
\|\mathbf{u}_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})} &\geq \|\mathbf{w}_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})}-\|\Psi_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})}\\
&\geq \left(1-\Gamma_{f,1}\tau^{\alpha-1/2}\right)\cdot\|\mathbf{w}_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})},
\end{aligned}$$

and (4.31) yields

$$\begin{aligned}
\|\mathbf{w}_{n+1}(\tau,\cdot)\|_{H^1(\mathbb{R}\setminus\{0\})} &\leq \frac{\tau^{\alpha-1/2}}{1-\Gamma_{f,1}\tau^{\alpha-1/2}}\cdot\left(\frac{3}{2}\widetilde{C}_2M_0\tau^{3/2-\alpha}\cdot\beta_n(\tau)+\widetilde{C}_3\cdot\beta_{n+1}(\tau)\right)\\
&\leq \frac{1}{4}\cdot\beta_n(\tau)+\frac{1}{2}\cdot\beta_{n+1}(\tau)
\end{aligned}$$

for all $\tau\in[0,T]$ with $T>0$ sufficiently small. In particular,

$$\beta_{n+1}(\tau) \leq \frac{1}{2}\cdot\beta_n(\tau)\qquad\text{for all }\tau\in[0,T],$$

and $(w_n)_{n\geq1}$ is a Cauchy sequence in $\mathbf{L}^\infty([0,T],H^1(\mathbb{R}\setminus\{0\}))$ which converges to the unique limit w such that

$$w(t,0-)-w(t,0+) \geq \frac{\delta_0}{3},\qquad \|w(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \leq M_0,\quad\text{for all } t\in[0,T].$$

Moreover, the map $t\mapsto w(t,0\pm)$ is locally Lipschitz and

$$\left|\dot{w}(t,0\pm)\right| \leq 2\Gamma_1\cdot t^{\alpha-1}\quad a.e.\ t\in[0,T].$$

Hence, $u \doteq w+\varphi^{(w)}$ satisfies (i)-(ii) in Definition 1.1 and (2.1). To verify that u a piecewise regular solution to (2.2)-(2.3), we notice that w_{n+1} is the solution to (2.19) and

$$\frac{d}{dt}\left[w_{n+1}+\varphi^{(w_{n+1})}\right]+a_n\cdot\frac{d}{dx}\left[w_{n+1}+\varphi^{(w_{n+1})}\right] = \mathbf{G}\left[w_{n+1}+\varphi^{(w_{n+1})}\right]-A_{n+1}\cdot\varphi_x^{(w_{n+1})}.$$

Denoting by $x_n(\cdot;t_0,x_0)$ the solution to

$$\dot{x}(t) \;=\; a_n(t,x(t)), \qquad x(t_0) \;=\; x_0,$$

the formula above implies

$$\begin{aligned}\Big[w_{n+1}+\varphi^{(w_{n+1})}\Big](t_0,x_0) \;&=\; (\overline{w}+\overline{v})(x_n(0;t_0,x_0))+\\ +\int_0^{t_0}\mathbf{G}\left[w_{n+1}+\varphi^{(w_{n+1})}\right](t,x_n(t;t_0,x_0))dt&-\int_0^t\Big[A_{n+1}\varphi_x^{(w_{n+1})}\Big](t,x_n(t;t_0,x_0))dt\end{aligned} \tag{4.32}$$

From the first inequality in (4.30), it holds

$$\begin{aligned}\lim_{n\to\infty}\int_0^t\Big[A_{n+1}\cdot\varphi_x^{(w_{n+1})}\Big](t,x(t;t_0,x_0))\;dt \;\le\\ \le \widetilde{C}_1\cdot\beta_{n+1}(t_0)\int_0^{t_0}\Big|\varphi^{(w_{n+1})}(t,x(t;t_0,x_0))\Big|\;dt \;=\; 0.\end{aligned}$$

Taking $n\to\infty$ in (4.32), we obtain

$$u(t_0,x_0) \;=\; (\overline{w}+\overline{v})(x(0;t_0,x_0))+\int_0^{t_0}\mathbf{G}\left[u(t,\cdot)\right](x(t;t_0,x_0))\;dt$$

with $t\mapsto x(t;t_0,x_0)$ being the characteristics curve, obtained by solving

$$\dot{x} \;=\; \tilde{a}(t,x,u) \;\doteq\; \left(f'(u(t,x))-\frac{f(u^-(t))-f(u^+(t))}{u^-(t)-u^+(t)}\right), \qquad x(t_0) \;=\; x_0.$$

3. It remains to prove the uniqueness of (2.2)-(2.3). Assume that $\tilde{u}$ is a piecewise regular solution to (2.2)-(2.3). Then we have

$$\sup_{t\in[0,T]}\|\tilde{u}(t,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \;\doteq\; M_1 \;<\; \infty,$$

and for every $\delta>0$ there exists a constant $M_\delta>0$ such that

$$|\tilde{u}_x(t,x)| \;\le\; M_\delta \qquad \text{for all } (t,x)\in[0,T]\times\mathbb{R}\backslash(-\delta,\delta).$$

By (1.7), (1.9), and the continuity $\mathbf{G}[\tilde{u}(t,\cdot)](\cdot)$ outside the origin, $\tilde{u}$ is continuously differentiable with respect to both variables t,x for $x\neq 0$. In particular, the map $t\mapsto\tilde{u}_x(t,0\pm)$ is continuous and

$$\inf_{t\in[0,T]}\tilde{u}(t,0-)-\tilde{u}(t,0+) \;>\; 0.$$

Following the same argument as in the proof of Lemma 4.1, there exist $\tilde{\delta}_1,\tilde{b}_0>0$ and $0<T_1\le T$ small such that for all $\tau\in[0,T_1]$ and $t\in[0,\tau]$, it holds

$$\tilde{a}(t,x,\tilde{u})\cdot\text{sign}(x) \;<\; 0 \qquad \text{for all } x\in[-\tilde{\delta}_1,\tilde{\delta}_1]\backslash\{0\},$$

and

$$\begin{cases}\tilde{x}(t;\tau,x_\tau) \;\ge\; x_\tau+\tilde{b}_0\cdot(\tau-t) & \text{if } \;x_\tau\in]0,\tilde{\delta}_1]\,,\\[2mm] \tilde{x}(t;\tau,x_\tau) \;\le\; -x_\tau-\tilde{b}_0\cdot(\tau-t) & \text{if } \;x_\tau\in[-\tilde{\delta}_1,0[\,,\end{cases} \tag{4.33}$$

where $\tilde{x}(\cdot;\tau,x_\tau)$ is the solution to

$$\dot{x}(t) \;=\; \tilde{a}(t,x(t),\tilde{u}), \qquad \tilde{x}(\tau) \;=\; x_\tau.$$

For every $0<\delta<\tilde{\delta}_1$ small, we shall provide a better upper bound for

$$\tilde{\gamma}_\tau(\delta,t) \;=\; \sup_{x\in\mathbb{R}\setminus[\tilde{x}(t;\tau,-\delta),\tilde{x}(t;\tau,\delta)]} |u_x(t,x)| \qquad \text{for all } t\in[0,\tau].$$

For every $x_1<x_2<\tilde{x}(t;\tau,-\delta)$ or $\tilde{x}(t;\tau,\delta)<x_1<x_2$, set $\tilde{x}_i(\cdot)\doteq\tilde{x}(\cdot;t,x_i)$ for $i=1,2$, one has

$$\begin{aligned}\left|\dot{\tilde{x}}_1(s)-\dot{\tilde{x}}_2(s)\right| \;&\le\; |\tilde{a}(s,x_1(s),\tilde{u})-\tilde{a}(s,x_2(s),\tilde{u})|\\ &\le\; \widetilde{C}_2\cdot|\tilde{u}(s,x_1(s))-\tilde{u}(s,x_2(s))| \;\le\; \widetilde{C}_2\cdot\tilde{\gamma}_\tau(\delta,s)\cdot|\tilde{x}_1(s)-\tilde{x}_2(s)|\end{aligned}$$

with $\widetilde{C}_2=\max_{\omega\in[-2M_1,2M_1]}\left|f''(\omega)\right|$. Applying Gronwall's inequality, we get

$$|\tilde{x}_1(s)-\tilde{x}_2(s)| \;\le\; \exp\left(\widetilde{C}_2\cdot\int_s^t\tilde{\gamma}_\tau(\delta,r)dr\right)\cdot|x_1-x_2| \quad \text{for all } s\in[0,t].$$

By Lemma A.2, (1.6), and (4.33), we estimate

$$\begin{aligned}|\tilde{u}(t,x_1)-\tilde{u}(t,x_2)| \;&\le\; \left|(\bar{v}+\overline{w})(\tilde{x}_1(0))-(\bar{v}+\overline{w})(\tilde{x}_2(0))\right|\\ &\qquad+\int_0^t\left|\mathbf{G}[\tilde{u}(t,\cdot)](\tilde{x}_1(s))-\mathbf{G}[\tilde{u}(s,\cdot)](\tilde{x}_2(s))\right|ds\\ &\le\; \left(2M_0+\widetilde{C}_3\cdot(\delta+\tau)^{\alpha-1}\right)\cdot|\tilde{x}_1(0)-\tilde{x}_2(0)|\\ &\qquad+\widetilde{C}_3\cdot M_1\int_0^t\left(\ln^2\left[\delta+\tilde{b}_0(\tau-s)\right]+\frac{1}{\delta+\tilde{b}_0(\tau-s)}\right)\cdot|\tilde{x}_2(s)-\tilde{x}_1(s)|ds\\ &\le\; \widetilde{C}_4\cdot\left((\delta+\tau)^{\alpha-1}+\left|\ln(\delta+\tilde{b}_0(\tau-t))\right|\right)\cdot\exp\left(\widetilde{C}_2\cdot\int_0^t\tilde{\gamma}_\tau(\delta,s)ds\right)\cdot|x_1-x_2|.\end{aligned}$$

Thus, for $t\in[0,\tau]$, we have

$$\tilde{\gamma}_\tau(\delta,t) \;\le\; \widetilde{C}_4\cdot\left((\delta+\tau)^{\alpha-1}+\left|\ln(\delta+\tilde{b}_0(\tau-t))\right|\right)\cdot\exp\left(\widetilde{C}_2\cdot\int_0^t\tilde{\gamma}_\tau(\delta,s)ds\right) \tag{4.34}$$

Equivalently,

$$-\frac{d}{dt}\exp\left(-\widetilde{C}_2\cdot\int_0^t\tilde{\gamma}_\tau(\delta,s)ds\right) \;\le\; \widetilde{C}_2\widetilde{C}_4\cdot\left((\delta+\tau)^{\alpha-1}+\left|\ln(\delta+\tilde{b}_0(\tau-t))\right|\right),$$

and thus there exists a small time $0<T_2<T_1$ and a constant $\widetilde{C}_5>0$ such that

$$\exp\left(\widetilde{C}_2\cdot\int_0^\tau\tilde{\gamma}_\tau(\delta,s)ds\right) \;\le\; \widetilde{C}_5 \qquad \text{for all } \tau\in[0,T_2].$$

Recalling (4.34), we finally get

$$\sup_{x\in\mathbb{R}\setminus[-\delta,\delta]}|\tilde{u}_x(\tau,x)| \;=\; \tilde{\gamma}_\tau(\delta,\tau) \;\le\; \widetilde{C}_6\cdot\delta^{\alpha-1} \quad \text{for all } \tau\in[0,T_2] \tag{4.35}$$

for some constant $\widetilde{C}_6>0$ which does not depend on δ.

4. Finally, to show that $\tilde{u}(t,\cdot)$ coincides with $u(t,\cdot)$ for all $t \in [0,T]$, defining

$$\mathbf{u}(t,x) \doteq \tilde{u}(t,x) - u(t,x), \qquad A(t,x) \ \doteq \ \tilde{a}(t,x,\tilde{u}) - \tilde{a}(t,x,u),$$

we have

$$\mathbf{u}_t + \tilde{a}(t,x,\tilde{u}) \cdot \mathbf{u}_x \ = \ \mathbf{G}[\mathbf{u}] - A(t,x) \cdot u_x.$$

Multiplying the above equation by $2\mathbf{u}$, we derive

$$\left(\mathbf{u}^2\right)_t + \left(\tilde{a}(t,x,\tilde{u}) \cdot \mathbf{u}^2\right)_x \ = \ \mathbf{u} \cdot \left[2\mathbf{G}[\mathbf{u}] - 2A(t,x)u_x + \tilde{a}_x(t,x,\tilde{u})\mathbf{u}\right]. \tag{4.36}$$

For every $\tau \in [0,T_2]$, integrating (4.36) over the domain

$$\Omega_\tau \ \doteq \ \left\{(t,x) \in [0,\tau] \times \mathbb{R} : x \in \mathbb{R}\backslash[\tilde{x}_\tau^-(t), \tilde{x}_\tau^+(t)]\right\},$$

where $\tilde{x}_\tau^\pm(t) \doteq \tilde{x}(t;\tau,0\pm)$, we get

$$\|\mathbf{u}(\tau,\cdot)\|^2_{L^2(\mathbb{R})} \ \le \ \int_0^\tau \int_{\mathbb{R}\backslash[\tilde{x}_\tau^-(t),\tilde{x}_\tau^+(t)]} \left|\mathbf{u} \cdot \left[2\mathbf{G}[\mathbf{u}] - 2A(t,x)u_x + \tilde{a}_x(t,x,\tilde{u})\mathbf{u}\right]\right| dxdt.$$

Similarly, arguing as in the proof of Lemma 4.2, we obtain

$$\begin{aligned}&\|\mathbf{u}_x^2(\tau,x)\|^2_{L^2(\mathbb{R})} \\ &\quad \le \ \int_0^\tau \int_{\mathbb{R}\backslash[\tilde{x}_\tau^-(t),\tilde{x}_\tau^+(t)]} \mathbf{u}_x \left[2\frac{d}{dx}\mathbf{G}[\mathbf{u}] - \tilde{a}_x(t,x,\widetilde{u})\mathbf{u}_x - 2A_x(t,x)u_x - 2A(t,x)u_{xx}\right] dxdt.\end{aligned}$$

From Lemma A.2, (4.35), the two inequalities above, and the fact that $\alpha > 3/4$, there exists $T_3 > 0$ so small that for every $\tau \in [0,T_3]$

$$\|\mathbf{u}(\tau,\cdot)\|^2_{H^1(\mathbb{R}\backslash\{0\})} \ \le \ \widetilde{C}_7 \int_0^\tau [\chi_\tau(t)]^{-1/2} \|\mathbf{u}(t,\cdot)\|^2_{H^1(\mathbb{R}\backslash\{0\})} dt,$$

where $\widetilde{C}_7 > 0$ does not depend on τ and $\chi_\tau \doteq \max\{\tilde{x}_\tau^+, -\tilde{x}_\tau^-\}$. From (4.33) and Gronwall's inequality, we conclude that

$$\|\mathbf{u}(\tau,\cdot)\|_{H^1(\mathbb{R}\backslash\{0\})} \ = \ 0 \qquad \text{for all } \tau \in [0,T_3].$$

Finally, we set

$$\widetilde{T} \ \doteq \ \sup \ \{\tau \in [0,T] : \tilde{u}(t,\cdot) = u(t,\cdot) \qquad \text{for all } t \in [0,\tau]\}.$$

By the continuity of $\tilde{u}, u$ outside the origin, $\tilde{u}(\widetilde{T},\cdot) = u(\widetilde{T},\cdot)$ has the same regularity as $\overline{w} + \overline{v}$. Consequently, if $\widetilde{T} < T$, then arguing as above we can find $\overline{T} \in]\widetilde{T},T]$ such that $\tilde{u}(\tau,\cdot) = u(\tau,\cdot)$ for every $\tau \in [0,\overline{T}]$, which contradicts the definition of $\widetilde{T}$. □

A Estimates on the nonlocal source

In this section, we shall establish basic estimates which are used in the proof of Lemma 3.2 and Lemma 3.4. Assume that K satisfies **(H1)-(H2)**. Recalling the definition of Λ, Φ, η and ϕ in (2.9)-(2.10) and (2.12)-(2.13), we set

$$\phi_b(x) \ \doteq \ \phi(x,b) \ = \ \eta(x) \cdot [\Phi(b) - \Phi(x+b)] \ = \ \eta(x) \cdot \left[\int_0^b \Lambda(y)\,dy - \int_0^{x+b} \Lambda(y)\,dy\right].$$

We first provide some bounds on $\mathbf{G}[\chi_{[0,\infty[}\phi_b]$ with $\chi_{[0,\infty[}$ being the indicator function on $[0,\infty[$ for $b > 0$ small. As usual, by the Landau symbol $\mathcal{O}(1)$ we shall denote a uniformly bounded quantity which does not depends on b.

Lemma A.1 *Assume that* $0<b<1/4$*. For every* $0<|x|<1/4$ *and* $\delta>0$*, we have*

$$\begin{cases} \left|\mathbf{G}[\chi_{[0,\infty[}\phi_b](x)\right| \ \leq\ \mathcal{O}(1), & \left|\dfrac{d}{dx}\mathbf{G}[\chi_{[0,\infty[}\phi_b](x)\right| \ \leq\ \mathcal{O}(1)\cdot\ln^2|x|, \\ \left|\dfrac{d^2}{dx^2}\mathbf{G}[\chi_{[0,\infty[}\phi_b](x)\right| \ \leq\ \mathcal{O}(1)\cdot\left|\dfrac{\ln|x|}{x}\right|, & \left\|\mathbf{G}[\chi_{[0,\infty[}\phi_b]\right\|_{H^2(\mathbb{R}\backslash[-\delta,\delta])} \ \leq\ \mathcal{O}(1)\cdot\delta^{-2/3}. \end{cases} \tag{A.1}$$

Proof. From (2.8) and $\phi_b(0)=0$, it holds

$$\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x) \ =\ -\int_0^2\Phi'(y+b)\cdot\Lambda(x-y)\ dy \qquad \text{for all } |x|<1/2.$$

Case 1. Assume that $-1/4<x<0$. Observe that for all $y\in[-2,2]\backslash\{0\}$,

$$|\Lambda(y)|,|\Phi'(y)| \ \leq\ \mathcal{O}(1)\cdot\big(\big|\ln|y|\big|+1\big), \qquad |\Phi(y)| \ \leq\ \mathcal{O}(1)\cdot|y|\cdot\big(\big|\ln|y|\big|+1\big),$$

we have

$$\begin{aligned}\left|\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)\right| \ &=\ \left|\int_0^2\Phi'(y+b)\cdot\Lambda(x-y)\ dy\right| \\ &\leq\ \mathcal{O}(1)\cdot\int_0^2\big(1+\big|\ln|y+b|\big)\big(1+\big|\ln|x-y|\big|\big)\ dy \ \leq\ \mathcal{O}(1).\end{aligned}$$

To estimate derivatives of $\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)$, we consider two cases:

- If $x+b>0$ then

$$\left|\frac{d}{dx}\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)\right| \ =\ \left|\int_{|x|}^{2+|x|}\Phi'(x+b+z)\cdot K(-z)\ dz\right|$$

$$\leq\ \mathcal{O}(1)\cdot\int_{|x|}^{2+|x|}\frac{1+\big|\ln|x+b+z|\big|}{z}\ dz \ \leq\ \mathcal{O}(1)\cdot\big(1+\big|\ln|x|\big|\big)\cdot\int_{|x|}^{2+|x|}\frac{1}{z}\ dz \ \leq\ \mathcal{O}(1)\cdot\ln^2|x|,$$

and

$$\begin{aligned}\left|\frac{d^2}{dx^2}\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)\right| \ &=\ \left|-\Phi'(x+b+z)\cdot K(-z)\Big|_{|x|}^{2+|x|}+\int_{|x|}^{2+|x|}\Phi''(x+b+z)\cdot K(-z)\ dz\right| \\ &\leq\ \mathcal{O}(1)\cdot\frac{1}{|x|}\cdot\left(|\ln b|+\int_{|x|}^{2+|x|}\frac{1}{z}\ dz\right) \ \leq\ \mathcal{O}(1)\cdot\left|\frac{\ln|x|}{x}\right|.\end{aligned}$$

- Otherwise, if $x+b\leq 0$ then

$$\begin{aligned}\left|\frac{d}{dx}\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)\right| \ &=\ \left|\int_{|x|}^{2+|x|}\Phi'(x+b+z)\cdot K(-z)\ dz\right| \\ &\leq\ \mathcal{O}(1)\cdot\left[\big|\ln|x|\big|+\int_{|x|}^{2+|x|}\big|\Phi(x+b+z)\cdot K'(-z)\big|\ dz\right] \ \leq\ \mathcal{O}(1)\cdot\ln^2|x|,\end{aligned}$$

and

$$\left|\frac{d^2}{dx^2}\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)\right| \leq \mathcal{O}(1)\cdot\left|\frac{\ln|x|}{x}\right| + \left|\int_{|x|}^{2+|x|}\Phi'(x+b+z)\cdot K'(-z)\,dz\right|$$
$$\leq \mathcal{O}(1)\cdot\left[\left|\frac{\ln|x|}{x}\right| + \int_{|x|}^{2+|x|}\big|\Phi(x+b+z)\cdot K''(-z)\big|\,dz\right] \leq \mathcal{O}(1)\cdot\left|\frac{\ln|x|}{x}\right|.$$

Case 2. Assume that $0<x<1/4$. We write

$$\mathbf{G}[\chi_{[0,\infty[}\phi_b](x) = -\int_0^{1-b}\Lambda(y+b)\Lambda(x-y)\,dy - \int_{1-b}^{2}\Phi'(y+b)\Lambda(x-y)dy \doteq -I_1 - I_2.$$

Since Λ is C^3 in $[1/2,2]$, it holds

$$\left|I_2^{(i)}(x)\right| \leq \mathcal{O}(1) \qquad \text{for all } i\in\{0,1,2\}.$$

On the other hand, we split I_1 into three parts as follows

$$I_1 = \int_0^{x/2}\Lambda(y+b)\cdot\Lambda(x-y)dy + \int_{x/2}^{3x/2}\Lambda(y+b)\cdot\Lambda(x-y)dy$$
$$+\int_{3x/2}^{1-b}\Lambda(y+b)\cdot\Lambda(x-y)dy = I_{11}+I_{12}+I_{13}.$$

We estimate

$$\begin{cases} \big|I_{11}(x)\big| \leq \mathcal{O}(1)\cdot\displaystyle\int_0^{\frac{x}{2}}|\ln(y+b)||\ln(x-y)|\,dy \leq \mathcal{O}(1)\cdot\big|\ln x\big|\int_0^{\frac{x}{2}}\big|\ln y\big|dy \leq \mathcal{O}(1)\cdot x\ln^2 x,\\ \big|I_{11}'(x)\big| = \left|\displaystyle\int_0^{\frac{x}{2}}\Lambda(y+b)K(x-y)\,dy + \frac{\Lambda(\frac{x}{2}+b)\Lambda(\frac{x}{2})}{2}\right| \leq \mathcal{O}(1)\cdot\ln^2 x,\\ \big|I_{11}''(x)\big| = \left|\displaystyle\int_0^{\frac{x}{2}}\Lambda(y+b)K'(x-y)dy + \frac{3\Lambda(\frac{x}{2}+b)K(\frac{x}{2})}{4} + \frac{\Lambda(\frac{x}{2})K(\frac{x}{2}+b)}{4}\right| \leq \mathcal{O}(1)\cdot\left|\frac{\ln x}{x}\right|, \end{cases}$$

and

$$\begin{cases} \big|I_{13}(x)\big| \leq \mathcal{O}(1)\cdot\displaystyle\int_{\frac{3x}{2}}^{1-b}(1+|\ln(y+b)|)\cdot(1+|\ln(y-x)|)\,dy \leq \mathcal{O}(1)\cdot x\ln^2 x,\\ \big|I_{13}'(x)\big| = \left|\displaystyle\int_{\frac{3x}{2}}^{1-b}\Lambda(y+b)K(x-y)\,dy - \frac{3\Lambda(\frac{3x}{2}+b)\Lambda(-\frac{x}{2})}{2}\right| \leq \mathcal{O}(1)\cdot\ln^2 x,\\ \big|I_{13}''(x)\big| = \left|\displaystyle\int_{\frac{3x}{2}}^{1-b}\Lambda(y+b)K'(x-y)\,dy - \frac{9K(\frac{3x}{2}+b)\Lambda(-\frac{x}{2})}{4} - \frac{3\Lambda(\frac{3x}{2}+b)K(-\frac{x}{2})}{4}\right|\\ \qquad\qquad \leq \mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x}\right|. \end{cases}$$

Concerning I_{12}, first of all, by a change of variable, it holds

$$I_{12}(x) = \lim_{\varepsilon\to 0+}\left(\int_{x/2}^{x-\varepsilon} + \int_{x+\varepsilon}^{\frac{3x}{2}}\right)\Lambda(y+b)\cdot\Lambda(x-y)\,dy = \int_{-\frac{x}{2}}^{\frac{x}{2}}\Lambda(x+b-z)\cdot\Lambda(z)\,dz,$$

and one directly computes that

$$\begin{cases} \big|I_{12}(x)\big| & \le \mathcal{O}(1)\cdot\big|\ln x\big|\cdot\displaystyle\int_0^{\frac{x}{2}}|\ln z|\,dz \le \mathcal{O}(1)\cdot x\ln^2 x, \\ \big|I'_{12}(x)\big| & = \left|\displaystyle\int_{-\frac{x}{2}}^{\frac{x}{2}} K(x+b-z)\cdot\Lambda(z)\,dz + \tfrac{1}{2}\cdot\left[\Lambda(\tfrac{3x}{2}+b)\Lambda(-\tfrac{x}{2})+\Lambda(\tfrac{x}{2}+b)\Lambda(\tfrac{x}{2})\right]\right| \\ & \le \mathcal{O}(1)\cdot\left(\dfrac{1}{x}\cdot\displaystyle\int_0^{\frac{x}{2}}|\ln z|\,dz+\ln^2 x\right)\le\mathcal{O}(1)\cdot\ln^2 x, \\ \big|I''_{12}(x)\big| & \le \mathcal{O}(1)\cdot\left(\left|\displaystyle\int_{-\frac{x}{2}}^{\frac{x}{2}} K'(x+b-z)\cdot\Lambda(z)\,dz\right|+\left|\dfrac{\ln x}{x}\right|\right) \\ & \le \mathcal{O}(1)\cdot\left(\dfrac{1}{x^2}\cdot\displaystyle\int_0^{\frac{x}{2}}|\ln z|\,dz+\left|\dfrac{\ln x}{x}\right|\right)\le\mathcal{O}(1)\cdot\left|\dfrac{\ln x}{x}\right|. \end{cases}$$

From the previous step, we obtain the first three estimates in (A.1) for $0<|x|<1/4$.

Finally, observe that $\chi_{[0,\infty[}\phi_b$ is continuous with compact support and smooth outside the origin. Hence, $\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big]$ is smooth outside the origin. As $|x|\to+\infty$, for $i\in\{1,2\}$ we have

$$\big|\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)\big| \ \le\ \mathcal{O}(1)\cdot x^{-1}, \qquad \left|\frac{d^i}{dx^i}\mathbf{G}\big[\chi_{[0,\infty[}\phi_b\big](x)\right| \ \le\ \mathcal{O}(1)\cdot x^{-(1+i)},$$

and using the first three estimates in (A.1) we obtain the last estimate in (A.1). □

Following the same argument in Lemma A.1, one can show that

Remark A.1 *Given $\lambda_1,\lambda_2\in\mathbb{R}$, the function*

$$v(x) = \big(\lambda_1\cdot\chi_{]-\infty,0[}+\lambda_2\cdot\chi_{]0,\infty[}\big)\cdot\eta(x)x$$

is more regular than Φ. Thus, one can follow the same argument as in Lemma A.1 to obtain for all $|x|<1/2$ that

$$|\mathbf{G}[v](x)| \ \le\ \mathcal{O}(1)\cdot\big(|\lambda_1|+|\lambda_2|\big), \qquad \left|\frac{d}{dx}\mathbf{G}[v](x)\right| \ \le\ \mathcal{O}(1)\cdot\big(|\lambda_1|+|\lambda_2|\big)\cdot\ln^2|x|,$$

and

$$\big\|\mathbf{G}[v]\big\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \le\ \mathcal{O}(1)\cdot\big(|\lambda_1|+|\lambda_2|\big)\cdot\delta^{-2/3} \qquad \text{for all } \delta>0.$$

Lemma A.2 *Let $v\in H^2(\mathbb{R}\setminus\{0\})$ be such that*

$$\|v\|_{H^i(\mathbb{R}\setminus\{0\})} \ \le\ M_i, \qquad i\in\{1,2\}.$$

Set $D^{(v)}(x)\doteq\mathbf{G}[v](x)-\big[v(0+)-v(0-)\big]\cdot\eta(x)\cdot\Lambda(x)$. Then for every $0<|x|<1/2$ and $\delta>0$ small, we have that

$$\begin{cases} \big|D^{(v)}(x)\big| \ \le\ \mathcal{O}(1)\cdot M_1, & \left|D^{(v)}_x(x)\right| \ \le\ \mathcal{O}(1)\cdot M_1\cdot\ln^2|x|, \\ \big\|D^{(v)}\big\|_{H^1(\mathbb{R}\setminus\{0\})} \ \le\ \mathcal{O}(1)\cdot M_1, & \big\|D^{(v)}\big\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \ \le\ \mathcal{O}(1)\cdot M_2\cdot\delta^{-2/3}. \end{cases} \tag{A.2}$$

Consequently, $\mathbf{G}[v]$ is in $H^1_{loc}(\mathbb{R}\setminus\{0\})$ and

$$\|\mathbf{G}[v]\|_{H^1(\mathbb{R}\setminus[-\delta,\delta])} \ \le\ M_1\cdot\delta^{-1/2}.$$

Proof. We first split v into two parts

$$v = v_1 + v_2, \qquad \text{where} \qquad v_1(x) \;=\; \begin{cases} v(0-)\cdot\eta(x) & \text{if} \;\; x<0,\\[2mm] v(0+)\cdot\eta(x) & \text{if} \;\; x>0. \end{cases}$$

For $i\in\{1,2\}$, one has that

$$\|v_1\|_{H^{(i)}(\mathbb{R}\backslash\{0\})} \;\leq\; \big(|v(0-)|+|v(0+)|\big)\cdot\|\eta\|_{H^{(i)}(\mathbb{R}\backslash\{0\})} \;\leq\; 4M_1\cdot\|\eta\|_{H^{(i)}(\mathbb{R}\backslash\{0\})},$$

and the function $v_2\in H^1(\mathbb{R})$ satisfies

$$\|v_2\|_{H^1(\mathbb{R})} \;\leq\; \big(1+4\|\eta\|_{H^1(\mathbb{R})}\big)\cdot M_1, \qquad \|v_2\|_{H^2(\mathbb{R}\backslash\{0\})} \;\leq\; \big(1+4\|\eta\|_{H^2(\mathbb{R})}\big)\cdot M_2.$$

Step 1. For every $0<|x|<1/2$, we have

$$\mathbf{G}[v_1](x) = \int_{-2}^{2} v_1'(y)\cdot\Lambda(x-y)\,dy + \big[v(0+)-v(0-)\big]\cdot\Lambda(x) = I_1 + \big[v(0+)-v(0-)\big]\cdot\Lambda(x). \quad \text{(A.3)}$$

Recalling (2.4), we estimate

$$\begin{aligned} \big|I_1(x)\big| \;&=\; \left|v(0-)\cdot\int_{-2}^{-1}\eta'(y)\cdot\Lambda(x-y)\,dy + v(0+)\cdot\int_1^2\eta'(y)\cdot\Lambda(x-y)\,dy\right| \\ &\leq\; \mathcal{O}(1)\cdot M_1\cdot\left(\int_{-2}^{-1}|\Lambda(x-y)|\,dy+\int_1^2|\Lambda(x-y)|\,dy\right) \;\leq\; \mathcal{O}(1)\cdot M_1, \end{aligned}$$

and

$$\left|I_1^{(i)}(x)\right| \;\leq\; \mathcal{O}(1)\cdot M_1\cdot\left(\int_{-2}^{-1}\left|\Lambda^{(i)}(x-y)\right|\,dy+\int_1^2\left|\Lambda^{(i)}(x-y)\right|\,dy\right) \;\leq\; \mathcal{O}(1)\cdot M_1$$

for $i\in\{1,2\}$. Hence, (A.3) and (1.4) yield

$$\begin{cases} \mathbf{G}[v_1](x) \;=\; \mathcal{O}(1)\cdot M_1 + \big[v(0+)-v(0-)\big]\cdot\Lambda(x) \\[2mm] \mathbf{G}'[v_1](x) \;=\; \mathcal{O}(1)\cdot M_1 + \big[v(0+)-v(0-)\big]\cdot K(x), \\[2mm] \mathbf{G}''[v_1](x) \;=\; \mathcal{O}(1)\cdot M_1 + \big[v(0+)-v(0-)\big]\cdot K'(x). \end{cases} \tag{A.4}$$

Moreover, since v_1 is bounded and compactly supported, as $|x|\to+\infty$

$$|\mathbf{G}[v_1](x)| \;\leq\; \mathcal{O}(1)\cdot M_1\cdot x^{-1}, \qquad \left|\frac{d^i}{dx^i}\mathbf{G}[v_1](x)\right| \;\leq\; \mathcal{O}(1)\cdot M_1\cdot x^{-(1+i)} \quad i\in\{1,2\},$$

and we have

$$\begin{cases} \|\mathbf{G}[v_1]-\big[v(0+)-v(0-)\big]\cdot\Lambda\cdot\eta\|_{H^1(\mathbb{R}\backslash\{0\})} \;\leq\; \mathcal{O}(1)\cdot M_1, \\[2mm] \|\mathbf{G}[v_1]-\big[v(0+)-v(0-)\big]\cdot\Lambda\cdot\eta\|_{H^2(\mathbb{R}\backslash\{0\})} \;\leq\; \mathcal{O}(1)\cdot M_1. \end{cases} \tag{A.5}$$

Step 2. To estimate $\mathbf{G}[v_2]$, we first recall that $v_2\in H^1(\mathbb{R})$. By the continuity of the linear operator $\mathbf{G}:\mathbf{L}^2(\mathbb{R})\to\mathbf{L}^2(\mathbb{R})$, we have

$$\|\mathbf{G}[v_2]\|_{H^1(\mathbb{R})} \;\leq\; \mathcal{O}(1)\cdot\|v_2\|_{H^1(\mathbb{R})} \;\leq\; \mathcal{O}(1)\cdot M_1,$$

and this particularly yields

$$\left|\mathbf{G}[v_2](x)\right| \;\leq\; 2\cdot\left\|\mathbf{G}[v_2]\right\|_{H^1(\mathbb{R})} \;\leq\; \mathcal{O}(1)\cdot M_1 \qquad \text{for all } x\in\mathbb{R}.$$

Thus, recalling (A.4) and (A.5), we get the first and the third estimate in (A.2).

Step 3. To achieve the second and the fourth estimate in (A.2), we split v_2 into two parts

$$v_2 = v_{21}+v_{22}, \qquad\qquad v_{21}(x) \;=\; \begin{cases} v_{2,x}(0-)\cdot x\eta(x) & \text{if } \;x<0,\\[2mm] v_{2,x}(0+)\cdot x\eta(x) & \text{if } \;x>0.\end{cases}$$

Since $|v_{2,x}(0\pm)|\leq 2\cdot\|v_{2,x}\|_{H^1(\mathbb{R}\setminus\{0\})}\leq 2M_2$, one has that $\|v_{22}(\cdot)\|_{H^2(\mathbb{R})}\leq\mathcal{O}(1)\cdot M_2$. Thus, by the continuity of the linear operator $\mathbf{G}:\mathbf{L}^2(\mathbb{R})\to\mathbf{L}^2(\mathbb{R})$, we get

$$\left\|\mathbf{G}[v_{22}](\cdot)\right\|_{H^2(\mathbb{R})} \;\leq\; \mathcal{O}(1)\cdot M_2,\;\; |\mathbf{G}[v_{22}](x)|, \left|\frac{d}{dx}\mathbf{G}[v_{22}](x)\right| \;\leq\; \mathcal{O}(1)\cdot M_2,\;\; \text{for all } x\in\mathbb{R}.$$

Finally, by Remark A.1 we have

$$|\mathbf{G}[v_{21}](x)| \;\leq\; \mathcal{O}(1)\cdot M_2, \qquad \left|\frac{d}{dx}\mathbf{G}[v_{21}](x)\right| \;\leq\; \mathcal{O}(1)\cdot M_2\cdot\ln^2|x| \qquad \text{for all } |x|<1/2,$$

and

$$\left\|\mathbf{G}[v_{21}]\right\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \;\leq\; \mathcal{O}(1)\cdot M_2\cdot\delta^{-2/3} \qquad \text{for all } \delta>0.$$

Thus, (A.4) and (A.5) yield the second and the fourth estimates in (A.2). □

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