ON THE BURGERS-POISSON EQUATION

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ABSTRACT. In this paper, we prove the existence and uniqueness of weak entropy solutions to the Burgers-Poisson equation for initial data in $\mathbf{L}^1(\mathbb{R})$. Additional an Oleinik type estimate is established and some criteria on local smoothness and wave breaking for weak entropy solutions are provided.

1. INTRODUCTION

Consider the balance law obtained from Burgers' equation by adding a nonlocal source term

$$u_t + \left(\frac{u^2}{2}\right)_x = [G * u]_x \,, \tag{1.1}$$

where

$$G(x) = -\frac{1}{2}e^{-|x|} \quad \text{and} \quad [G*u](x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}\cdot u(z)dz$$

solves the Poisson equation

$$\varphi_{xx} - \varphi \; = u \,. \tag{1.2}$$

Equation (1.1) has been derived in [11] as a simplified model of shallow water waves and admits conservation of both momentum and energy. For sufficiently smooth initial data

$$u(0,x) \; = \overline{u}(x) \,, \tag{1.3}$$

the local existence and uniqueness of solutions of (1.1) has been established in [6]. Additional the analysis of traveling waves showed that the equation features wave breaking in finite time. Hence it is natural to study existence and uniqueness of weak entropy solutions which are global in time.

Definition 1.1. *A function* $u \in \mathbf{L}^1_{loc}([0,\infty[\times\mathbb{R}) \cap \mathbf{L}^\infty_{loc}(]0,\infty[,\mathbf{L}^\infty(\mathbb{R}))$ *is an* **entropy weak solution** *of* (1.1)-(1.3) *if* u *satisfies the following properties:*

(i) *the map* $t \mapsto u(t,\cdot)$ *is continuous with values in* $\mathbf{L}^1(\mathbb{R})$ *and satisfies the initial condition* (1.3).

(ii) *For any* $k \in \mathbb{R}$ *and any non-negative test function* $\phi \in C^1_c(]0,\infty[\times\mathbb{R},\mathbb{R})$ *one has*

$$\int\int \Big[|u-k|\phi_t + \text{sign}(u-k)\Big(\frac{u^2}{2}-\frac{k^2}{2}\Big)\phi_x + \text{sign}(u-k)[G_x * u(t,\cdot)](x)\phi\Big]\, dxdt \;\geq\; 0\,. \tag{1.4}$$

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For any initial data $\bar{u} \in BV(\mathbb{R})$, the existence of a global weak entropy solution to (1.1)-(1.3) has been studied in [6]. The proof is based on the vanishing viscosity method yielding a sequence of approximating smooth solutions. Due to the BV bound of $\bar{u}$, one obtains that the approximating solutions also satisfy an a priori uniform BV bound for all positive times, yielding the compactness of the approximating sequence of solutions. However this method cannot be applied in the more general case with initial data in $\mathbf{L}^1(\mathbb{R})$. Additional there are no uniqueness or continuity results for global weak entropy solutions of (1.1) established in [6]. Thus our main goal is to study the existence and uniqueness for global weak entropy solutions for initial data $\bar{u} \in \mathbf{L}^1(\mathbb{R})$. To be more explicit we are going to show the following theorem.

Theorem 1.2. *Given any initial data $u(0,\cdot) = \bar{u}(\cdot) \in \mathbf{L}^1(\mathbb{R})$, the Cauchy problem* (1.1)-(1.3) *has a unique weak entropy solution $u(t,x)$ in $[0,\infty)\times\mathbb{R}$. Furthermore, for any $t > 0$*

$$\|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq e^t \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \,, \tag{1.5}$$

and

$$u(t,y) - u(t,x) \leq \left[\frac{1}{t} + 2 + 2t + 4te^t \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\right](y-x), \qquad x < y\,. \tag{1.6}$$

Moreover, let $v(t,x)$ be the weak entropy solution of (1.1) *with initial data $v(0,\cdot) = \bar{v}(\cdot) \in \mathbf{L}^1(\mathbb{R})$. Then, for every $t > 0$, it holds*

$$\|u(t,\cdot) - v(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq e^t \|\bar{u} - \bar{v}\|_{\mathbf{L}^1(\mathbb{R})} \,. \tag{1.7}$$

The above solutions will be constructed by a flux-splitting method. Relying on the decay properties of the semigroup generated by Burgers' equation [4, 10] and the Lipschitz continuity of solutions to the Poisson equation (see Lemma 2.2), we prove that approximating solutions satisfy an Oleinik type inequality. As a consequence the sequence of approximating solutions is precompact and converges in $\mathbf{L}^1_{\text{loc}}(\mathbb{R})$. Moreover we show using an energy estimate that the characteristics are Hölder continuous. This allows us to derive a *Tightness Property* for the sequence of approximating solutions, which implies the continuity property of the solutions.

It is well-known that the nonlocal Possion source term in the Burgers–Poisson equation cannot prevent the breaking induced from the Burgers term uu_x. This means it is possible that the velocity slope u_x blows up in finite time even if the initial data is very smooth and has small C^1-norm. In [9], a criteria on wave breaking has been established in the class of spatially periodic solutions. By a careful study of the derivative of the solution u_x along characteristics, we extend the result in [9] to the general case (see Theorem 4.1). Furthermore, we provide some criteria on local smoothness. In particular, we prove that if the $\mathbf{L}^\infty$-norm of the derivative $u_{0,x}$ of the initial data u_0 is small then the corresponding weak entropy solution of (1.1) will remain smooth for a large time.

The paper is organized as follows. In Section 2 we construct approximating solutions and provide some a priori estimates. In Section 3 we will prove the existence and uniqueness for weak entropy solutions. Finally, we are going to study local smoothness and wave breaking criteria for weak entropy solutions.

2. Approximating solutions and some a priori estimates

The proof of the existence and uniqueness of weak entropy solutions to the Burgers-Poisson equation with initial data in $\mathbf{L}^1(\mathbb{R})$ is based on a limiting process for approximating solutions, which are constructed by the flux-splitting method. Thus this section is concerned with the construction of the approximating solutions as well as the derivation of some a priori estimates for them.

1. Approximating solutions. For some fixed integer $\nu \geq 1$, we define the time steps

$$t_i = i \cdot 2^{-\nu}, \qquad i = 0, 1, 2, \dots$$

The approximating solution of (2.2) is then defined inductively as

(2.1)
$$\begin{cases} u^\nu(0) = \bar{u}, & u^\nu(t_i) = u^\nu(t_i-) + 2^{-\nu} \cdot [G_x * u^\nu(t_i-)], \qquad i = 1, 2, \dots \\ u^\nu(t) = S^B_{t-t_i}(u^\nu(t_i)), & t \in [t_i,\, t_{i+1}[\,, \quad i = 0, 1, 2, \dots \end{cases}$$

Here S^B denotes the semigroup generated by Burgers' equation. More precisely, $t \mapsto S^B_t(\bar{u})$ denotes the Kruzkov entropy solution to

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad u(0,x) = \bar{u}(x) \in \mathbf{L}^1(\mathbb{R})\,. \tag{2.2}$$

For every $u \in \mathbf{L}^1(\mathbb{R})$, we have

$$\left\|S^B_t(u)\right\|_{\mathbf{L}^1(\mathbb{R})} \leq \|u\|_{\mathbf{L}^1(\mathbb{R})}\,, \qquad \text{for all } t \geq 0\,, \tag{2.3}$$

and

$$\|G_x * u\|_{\mathbf{L}^1(\mathbb{R})} \leq \|u\|_{\mathbf{L}^1(\mathbb{R})}\,, \tag{2.4}$$

which implies

$$\begin{aligned} \|u^\nu(t_i)\|_{\mathbf{L}^1(\mathbb{R})} &\leq \left\|S^B_{2^{-\nu}}(u^\nu(t_{i-1}))\right\|_{\mathbf{L}^1(\mathbb{R})} + 2^{-\nu}\left\|[G_x * S^B_{2^{-\nu}}(u^\nu(t_{i-1}))]\right\|_{\mathbf{L}^1(\mathbb{R})} \\ &\leq (1+2^{-\nu})\,\|u^\nu(t_{i-1})\|_{\mathbf{L}^1(\mathbb{R})} \leq (1+2^{-\nu})^i\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \\ &= (1+2^{-\nu})^{2^\nu t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \leq e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\,. \end{aligned}$$

By (2.3), we obtain that

$$\|u^\nu(t)\|_{\mathbf{L}^1(\mathbb{R})} \leq e^t\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\,, \qquad \text{for all } t \geq 0\,. \tag{2.5}$$

2. Oleinik type inequality. We claim that for any $i \geq 1$ and $t \in [t_i, t_{i+1}[$ it holds that

(2.6)
$$u^\nu(t,x_2) - u^\nu(t,x_1) \leq \left[\frac{1}{t_i} + 2 + 2t_i + 4e^{t_i}t_i\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\right] \cdot (x_2 - x_1), \quad \text{for all } x_1 < x_2\,.$$

The proof relies heavily on the positive decay of Burgers' semigroup and the Lipschitz continuity of solutions to the Poisson equation.

Lemma 2.1. *Let $u_0 \in \mathbf{L}^1(\mathbb{R})$ be such that*

$$u_0(x_2) - u_0(x_1) \leq K \cdot (x_2 - x_1), \qquad \text{for all } x_1 < x_2\,, \tag{2.7}$$

for some constant $K > 0$. Then

$$S_t^B(u_0)(x_2) - S_t^B(u_0)(x_1) \leq \frac{K}{1+Kt} \cdot (x_2 - x_1), \qquad \text{for all } x_1 < x_2\,. \tag{2.8}$$

Proof. It is sufficient to prove (2.8) for any point of continuity x_i of $S_t^B(u_0)$. Let $\xi_{x_i}(\cdot)$ be the characteristic through the point (t, x_i), then we have

$$x_i = \xi_{x_i}(0) + tu_0(\xi_{x_i}(0)) \quad \text{and} \quad S_t^B(u_0)(x_i) \ = \ u_0(\xi_{x_i}(0))\,.$$

From the assumption (2.7), we get

$$\begin{aligned} x_2 - x_1 \quad &= \quad \xi_{x_2}(0) - \xi_{x_1}(0) + t \cdot (u_0(\xi_{x_2}(0)) - u_0(\xi_{x_1}(0))) \\ &\leq \quad (1+Kt) \cdot \left[\xi_{x_2}(0) - \xi_{x_1}(0)\right], \end{aligned}$$

which implies that

$$\begin{aligned} S_t^B(u_0)(x_2) - S_t^B(u_0)(x_1) \quad &= \quad \frac{1}{t} \cdot \left[(x_2 - x_1) - (\xi_{x_2}(0) - \xi_{x_1}(0))\right] \\ &\leq \quad \frac{K}{1+Kt} \cdot (x_2 - x_1)\,, \end{aligned}$$

and the proof is complete. □

Lemma 2.2. *Let $u_0 \in \mathbf{L}^1(\mathbb{R})$ be such that*

$$u_0(x_2) - u_0(x_1) \leq K \cdot (x_2 - x_1), \qquad \text{for all } x_1 < x_2\,, \tag{2.9}$$

for some constant $K > 0$. Then

$$\left|[G_x * u_0](x_2) - [G_x * u_0](x_1)\right| \leq \left[\, \|u_0\|_{\mathbf{L}^1(\mathbb{R})} + \sqrt{2K\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\,\right] \cdot |x_2 - x_1|\,. \tag{2.10}$$

Proof. For any $x_1 < x_2$, we compute

$$\begin{aligned} \left|[G_x * u_0](x_2) - [G_x * u_0](x_1)\right| \ &\leq \ \frac{1}{2}\int_{-\infty}^{x_1} |e^{z-x_1} - e^{z-x_2}| \cdot |u_0(z)|\; dz \\ + \frac{1}{2}\int_{x_2}^{\infty} |e^{x_1-z} - e^{x_2-z}| \cdot |u_0(z)|\; dz &+ \frac{1}{2}\int_{x_1}^{x_2} |e^{x_1-z} + e^{z-x_2}| \cdot |u_0(z)|\; dz \\ &\leq \ \left[\, \|u_0\|_{\mathbf{L}^1} + \|u_0\|_{\mathbf{L}^\infty}\,\right] \cdot |x_2 - x_1|\,. \end{aligned}$$

Concluding as in the proof of [1, Lemma 4.2], (2.9) implies that

$$\|u_0\|_{\mathbf{L}^\infty} \leq \sqrt{2K\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\ , \tag{2.11}$$

and hence (2.10). □

Using Lemma 2.1 and Lemma 2.2, we now show by induction that for any $i = 1, 2, \ldots$, one has

$$u^\nu(t_i-, x_2) - u^\nu(t_i-, x_1) \leq a_i \cdot (x_2 - x_1), \qquad \text{for all } x_1 < x_2\,, \tag{2.12}$$

where

$$a_1 = 2^{\nu} \quad \text{and} \quad a_{i+1} = \frac{(1+2^{-\nu})\cdot a_i + 2^{-\nu+1}e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}{1+\Big[(1+2^{-\nu})\cdot a_i + 2^{-\nu+1}e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big]\cdot 2^{-\nu}}\,. \tag{2.13}$$

Indeed, since $u^{\nu}(t_1-,\cdot) = S^B_{2^{-\nu}}(\bar{u})(\cdot)$, (2.12) holds for $i=1$ by Oleinik's inequality, see e.g. [5, Chapter 3.4]. Assume that (2.12) holds for all indices up to i. Then it follows from Lemma 2.2 and (2.5), that

$$\begin{aligned} u^{\nu}(t_i,x_2)-u^{\nu}(t_i,x_1) &= u^{\nu}(t_i-,x_2)-u^{\nu}(t_i-,x_1) \\ &\quad + 2^{-\nu}\cdot\big([G_x * u(t_i-)](x_2) - [G_x * u(t_i-)](x_1)\big) \\ &\le \Big(a_i + 2^{-\nu}\Big(\|u(t_i-)\|_{\mathbf{L}^1(\mathbb{R})} + \sqrt{2a_i\,\|u(t_i-)\|_{\mathbf{L}^1(\mathbb{R})}}\Big)\Big)\cdot(x_2-x_1) \\ &\le \Big(a_i + 2^{-\nu}\Big(e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} + \sqrt{2e^{t_i}a_i\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}\Big)\Big)\cdot(x_2-x_1) \\ &\le \Big((1+2^{-\nu})a_i + 2^{-\nu+1}e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big)\cdot(x_2-x_1) \end{aligned} \tag{2.14}$$

for any $x_1 < x_2$.

Applying Lemma 2.1 to $u_0(\cdot) = u^{\nu}(t_i,\cdot)$ and $t = 2^{-\nu}$, we obtain

$$\begin{aligned} u^{\nu}(t_{i+1}-,x_2)-u^{\nu}(t_{i+1}-,x_1) &= S^B_{2^{-\nu}}(u^{\nu}(t_i))(x_2) - S^B_{2^{-\nu}}(u^{\nu}(t_i))(x_1) \\ &\le \frac{(1+2^{-\nu})a_i + 2^{-\nu+1}e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}{1+\Big[(1+2^{-\nu})a_i + 2^{-\nu+1}e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big]\cdot 2^{-\nu}}\cdot(x_2-x_1)\,, \end{aligned}$$

for any $x_1 < x_2$, which is (2.12) for $i+1$.

Note that (2.14) together with Lemma 2.1 implies, for all $t \in [t_i, t_{i+1}[$, that

$$u^{\nu}(t,x_2)-u^{\nu}(t,x_1) \le \Big[(1+2^{-\nu})a_i + 2^{-\nu+1}e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big]\cdot(x_2-x_1)$$

for all $x_1 < x_2$. Hence (2.6) follows if we can show that

$$(1+2^{-\nu})a_i + 2^{-\nu+1}e^{t_i}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \le \frac{1}{t_i} + 2 + 2t_i + 4e^{t_i}t_i\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\,. \tag{2.15}$$

We therefore establish an upper bound for $\{a_i\}$. Observe first that

$$\frac{1}{a_{i+1}} = 2^{-\nu} + \frac{1}{1+\Big(1+\frac{2e^{t_i}\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}{a_i}\Big)2^{-\nu}}\cdot\frac{1}{a_i}\,.$$

Fix any $T > 0$, set

$$K_T = 1 + 2e^{T}\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\,, \tag{2.16}$$

and define the sequence $\{z_i\}$ by

$$z_1 = 2^{-\nu} \quad \text{and} \quad z_{i+1} = 2^{-\nu} + \frac{1}{1+(1+K_T z_i)2^{-\nu}}\cdot z_i \quad \text{for all } 1 \le i+1 \le T\cdot 2^{\nu}\,. \tag{2.17}$$

By a comparison argument, one has

$$z_{i+1} \le \frac{1}{a_{i+1}}, \qquad \text{for all } 1 \le i+1 \le T\cdot 2^{\nu}\,. \tag{2.18}$$

On the other hand, since

$$z_{i+1} \leq 2^{-\nu} + z_i \qquad \text{for all } 1 \leq i+1 \leq T \cdot 2^{\nu}$$

it holds that

$$z_{i+1} \ \leq \ i \cdot 2^{-\nu} + z_1 \ = \ (i+1) \cdot 2^{-\nu} \ \leq \ T, \quad \text{for all } 1 \leq i+1 \leq T \cdot 2^{\nu}.$$

Recalling (2.17) we get

$$z_{i+1} \ \geq \ 2^{-\nu} + \frac{1}{1 + \big(1 + K_T T\big) \cdot 2^{-\nu}} \cdot z_i \ , \quad \text{for all } 1 \leq i+1 \leq T \cdot 2^{\nu}.$$

Equivalently,

$$z_{i+1} - \alpha \ \geq \ \frac{1}{1 + \big(1 + K_T T\big) \cdot 2^{-\nu}} \cdot (z_i - \alpha), \quad \text{for all } 1 \leq i+1 \leq T \cdot 2^{\nu}$$

where $\alpha := 2^{-\nu} + \frac{1}{1+K_T T}$. This implies that

$$\begin{aligned} z_{i+1} - \alpha \ &\geq \ \frac{1 + \big(1 + K_T T\big) \cdot 2^{-\nu}}{[1 + \big(1 + K_T T\big) \cdot 2^{-\nu}]^{i+1}} \cdot (z_1 - \alpha) \\ &= \ -\frac{1 + \big(1 + K_T T\big) \cdot 2^{-\nu}}{[1 + \big(1 + K_T T\big) \cdot 2^{-\nu}]^{i+1}} \cdot \frac{1}{1 + K_T T} \\ &\geq \ -\frac{1 + \big(1 + K_T T\big) \cdot 2^{-\nu}}{1 + \big(1 + K_T T\big) \cdot t_{i+1}} \cdot \frac{1}{1 + K_T T} \end{aligned}$$

for all $1 \leq i+1 \leq T \cdot 2^{\nu}$. Thus,

$$\begin{aligned} z_{i+1} \ &\geq \ \frac{1}{1 + K_T T} \cdot \Big[1 - \frac{1}{1 + \big(1 + K_T T\big) \cdot t_{i+1}}\Big] \\ &\qquad\qquad + 2^{-\nu} \cdot \Big[1 - \frac{1}{1 + \big(1 + K_T T\big) \cdot t_{i+1}}\Big] \\ &\geq \ \frac{t_{i+1}}{1 + \big(1 + K_T T\big) \cdot t_{i+1}}, \qquad \text{for all } 1 \leq i+1 \leq T \cdot 2^{\nu}. \end{aligned}$$

Recalling (2.18), we have

$$a_{i+1} \ \leq \ \frac{1}{z_{i+1}} \ \leq \ (1 + K_T T) + \frac{1}{t_{i+1}}, \qquad \text{for all } 1 \leq i+1 \leq T \cdot 2^{\nu}, \tag{2.19}$$

and in particular,

$$a_{\lfloor T \cdot 2^{\nu} \rfloor + 1} \ \leq \ (1 + K_T T) + \frac{1}{\lfloor T \cdot 2^{\nu} \rfloor + 1}.$$

Since the above inequality holds for any $T > 0$, we obtain

$$a_i \ \leq \ \frac{1}{t_i} + 1 + t_i + 2e^{t_i} t_i \, \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}, \qquad \text{for all } i \geq 1.$$

which implies (2.15) and thus (2.6).

3. Minimal and maximal backward characteristics. Given some initial data $\bar{u}(x)$, we can split it into a positive and a negative part

$$\bar{u}(x) = \max\{\bar{u}(x), 0\} + \min\{\bar{u}(x), 0\} = \bar{u}^+(x) + \bar{u}^-(x). \tag{2.20}$$

Similarly we can split the source term for each $x \in \mathbb{R}$ into a positive and a negative part,

$$Q^\nu(t_i,x) = [G_x * u^\nu(t_i)](x) = Q^{\nu,+}(t_i,x) + Q^{\nu,-}(t_i,x). \tag{2.21}$$

We then define the function $u^{\nu,+}(t)$ as follows

$$u^{\nu,+}(t) = S^B_{t-t_i}(u^{\nu,+}(t_i)), \quad t \in [t_i, t_{i+1}[, \tag{2.22}$$

$$u^{\nu,+}(0) = \bar{u}^+, \quad u^{\nu,+}(t_i) = u^{\nu,+}(t_i-) + 2^{-\nu} Q^{\nu,+}(t_i). \tag{2.23}$$

Similarly one defines the function $u^{\nu,-}(t)$ as follows

$$u^{\nu,-}(t) = S^B_{t-t_i}(u^{\nu,-}(t_i)), \quad t \in [t_i, t_{i+1}[, \tag{2.24}$$

$$u^{\nu,-}(0) = \bar{u}^-, \quad u^{\nu,-}(t_i) = u^{\nu,-}(t_i-) + 2^{-\nu} Q^{\nu,-}(t_i). \tag{2.25}$$

Here it should be noted that one has in general

$$u^\nu(t,x) \neq u^{\nu,+}(t,x) + u^{\nu,-}(t,x). \tag{2.26}$$

However, one has

$$u^{\nu,-}(t,x) \ \leq \ 0 \ \leq \ u^{\nu,+}(t,x), \qquad \left\| u^{\nu,\pm}(t,\cdot) \right\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ e^t \left\| \bar{u} \right\|_{\mathbf{L}^1(\mathbb{R})} \tag{2.27}$$

$$u^{\nu,-}(t,x) \ \leq \ u^\nu(t,x) \ \leq \ u^{\nu,+}(t,x), \tag{2.28}$$

$$\left\| Q^{\nu,\pm}(t_i,\cdot) \right\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ \left\| u^\nu(t_i,\cdot) \right\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ e^{t_i} \left\| \bar{u} \right\|_{\mathbf{L}^1(\mathbb{R})}. \tag{2.29}$$

Denote by $t \mapsto x(t)$ the generalized characteristic to the approximating solution $u^\nu(t,x)$ through the point $(\tau, x(\tau))$. Additional let $t \mapsto y(t)$ be the minimal backward characteristic, i.e the characteristic for the positive solution $u^{\nu,+}(t,x)$ through the point $(\tau, x(\tau))$. Then $u^\nu(t,x) \leq u^{\nu,+}(t,x)$ implies that $y(t) \leq x(t)$ for all $t \in [0,\tau]$ and in particular $x(\tau) - x(t) \leq y(\tau) - y(t)$ for all $t \in [0,\tau]$. To estimate $y(\tau) - y(t)$, we compute

$$\begin{aligned}
0 \ &\leq \ \int_{y(\tau)}^{\infty} u^{\nu,+}(\tau,x)\ dx \\
&\leq \ \int_{y(t)}^{\infty} u^{\nu,+}(t,x) dx + \sum_{t<t_i\leq\tau} \int_{y(t_i)}^{\infty} \left(u^{\nu,+}(t_i,x) - u^{\nu,+}(t_i-,x)\right)\ dx \\
&\qquad + \int_t^\tau \frac{1}{2} u^{\nu,+,2}(s,y(s)) - u^{\nu,+}(s,y(s))\dot{y}(s)\ ds \\
&\leq \left\| u^{\nu,+}(t,.) \right\|_{\mathbf{L}^1(\mathbb{R})} + \sum_{t<t_i\leq\tau} \int_{y(t_i)}^{\infty} 2^{-\nu} Q^{\nu,+}(t_i,x) dx - \frac{1}{2}\int_t^\tau \dot{y}(s)^2\ ds \\
&\leq \ e^\tau \left\| \bar{u} \right\|_{\mathbf{L}^1(\mathbb{R})} + \sum_{t<t_i\leq\tau} 2^{-\nu} \left\| Q^{\nu,+}(t_i,\cdot) \right\|_{\mathbf{L}^1(\mathbb{R})} - \frac{1}{2}\int_t^\tau \dot{y}(s)^2\ ds \\
&\leq \ e^\tau \left\| \bar{u} \right\|_{\mathbf{L}^1(\mathbb{R})} + \sum_{t<t_i\leq\tau} 2^{-\nu} e^{t_i} \left\| \bar{u} \right\|_{\mathbf{L}^1(\mathbb{R})} - \frac{1}{2}\int_t^\tau \dot{y}(s)^2\ ds \\
&\leq \ (1+\tau) e^\tau \left\| \bar{u} \right\|_{\mathbf{L}^1(\mathbb{R})} - \frac{1}{2}\int_t^\tau \dot{y}(s)^2 ds\,.
\end{aligned}$$

Applying the Cauchy Schwarz inequality then yields

$$\begin{aligned} 0 \ \leq \ y(\tau) - y(t) \ &= \ \int_t^\tau \dot{y}(s) ds \\ &\leq \ (\tau - t)^{1/2} \left(\int_t^\tau \dot{y}(s)^2 ds \right)^{1/2} \leq \sqrt{2(1+\tau)e^\tau \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}} \cdot (\tau - t)^{1/2}. \end{aligned}$$

Denote by $t \mapsto \tilde{y}(t)$ the maximal backward characteristic, i.e. the characteristic for the negative solution $u^{\nu,-}(t,x)$ through the point $(\tau, x(\tau))$. Then $u^{\nu,-}(t,x) \leq u^\nu(t,x)$ implies that $x(t) \leq \tilde{y}(t)$ for all $t \in [0,\tau]$, and in particular $\tilde{y}(\tau) - \tilde{y}(t) \leq x(\tau) - x(t)$ for all $t \in [0,\tau]$. A similar argument as before shows that

$$0 \leq \tilde{y}(t) - \tilde{y}(\tau) \leq \sqrt{2(1+\tau)e^\tau \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}} \cdot (\tau - t)^{1/2}. \tag{2.30}$$

Since

$$\tilde{y}(\tau) - \tilde{y}(t) \ \leq \ x(\tau) - x(t) \leq y(\tau) - y(t),$$

we have shown the following lemma.

Lemma 2.3. *For any $\nu \geq 1$, let $t \mapsto x(t)$ be any characteristic for the approximate solution $u^\nu(t,x)$. Then*

$$|x(\tau) - x(t)| \ \leq \ C_1 \cdot (\tau - t)^{1/2} \qquad \text{for all } 0 \leq t < \tau \leq T, \tag{2.31}$$

where $C_1 = \sqrt{2(1+T)e^T \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}$.

4. Lipschitz type estimate with respect to time. We claim that for any fixed $\delta, R, T > 0$ there exist constants $C_{1,\delta}, C_{2,\delta} > 0$ such that

$$\|u^\nu(t,\cdot) - u^\nu(s,\cdot)\|_{\mathbf{L}^1([-R,R])} \ \leq \ C_{1,\delta} \cdot |t-s| + C_{2,\delta} \cdot 2^{-\nu} \tag{2.32}$$

for all $t_1 \leq \delta \leq s \leq t \leq T$.

Due to (2.1) and (2.4), we have

$$\|u^\nu(t_i,\cdot) - u^\nu(t_i-,\cdot)\|_{\mathbf{L}^1([-R,R])} \ \leq \ 2^{-\nu} e^{t_i} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \ . \tag{2.33}$$

On the other hand, for any $s,\, t \in [t_i, t_{i+1}[$, we have following [8, Theorem 7.10],

$$\begin{aligned} \|u^\nu(t,\cdot) - u^\nu(s,\cdot)\|_{\mathbf{L}^1([-R,R])} \ &= \ \left\| S^B_{t-s}(u^\nu(s))(\cdot) - u^\nu(s,\cdot) \right\|_{\mathbf{L}^1([-R,R])} \\ &\leq \ \max_{t \in [t_i, t_{i+1}]} \left\{ \text{Tot.Var.}\{u^\nu(t,\cdot); [-R,R]\} \right\} \max_{t \in [t_i, t_{i+1}]} \|u^\nu(t,\cdot)\|_{\mathbf{L}^\infty([-R,R])} |t-s|. \end{aligned}$$

We are going to establish an upper bound for $\|u^\nu(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})}$ and Tot.Var.$\{u^\nu(t,.); [-R,R]\}$ for $t \in [t_i, t_{i+1}[$. Let $b_i = \frac{1}{t_i} + 2 + 2t_i + 4t_i e^{t_i} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}$. Combining (2.5) (2.6), and (2.11) then yields

$$\|u^\nu(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ \sqrt{2b_i \|u^\nu(t,.)\|_{\mathbf{L}^1(\mathbb{R})}} \ \leq \ \sqrt{2b_i e^t \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}} \qquad \text{for all } t \in [t_i, t_{i+1}[.$$

Thus it is left to establish an upper bound for the total variation. Observe first that (2.6) implies that the function $u^\nu(t,x) - b_i x$ is decreasing. Then we have, for $t \in [t_i, t_{i+1}[$,

$$\text{Tot.Var.}\{u^\nu(t,\cdot); [-R,R]\} \ \leq \ \text{Tot.Var.}\{u^\nu(t,\cdot) - b_i \cdot; [-R,R]\} + \text{Tot.Var.}\{b_i \cdot : [-R,R]\}$$

$$
\begin{aligned}
&= u^\nu(t,-R) + b_i R - u^\nu(t,R) + b_i R + 2b_i R \\
&= u^\nu(t,-R) - u^\nu(t,R) + 4b_i R \\
&\le 2\sqrt{2b_i e^t \|\bar u\|_{\mathbf{L}^1(\mathbb{R})}} + 4b_i R.
\end{aligned}
$$

Thus for all $s,t \in [t_i, t_{i+1}[$,

$$
\|u^\nu(t,\cdot) - u^\nu(s,\cdot)\|_{\mathbf{L}^1([-R,R])} \le (6b_i e^t \|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + 4b_i^2 R^2)|t-s|. \tag{2.34}
$$

Combining (2.33) and (2.34) finally yields for $t_1 \le \delta \le s \le t \le T$,

$$
\begin{aligned}
&\|u^\nu(t,\cdot) - u^\nu(s,\cdot)\|_{\mathbf{L}^1([-R,R])} \\
&\quad \le \Bigg[6\left(\frac{1}{\lfloor \delta\cdot 2^\nu \rfloor \cdot 2^{-\nu}} + 2 + 2T + 4Te^T \|\bar u\|_{\mathbf{L}^1(\mathbb{R})} \right) e^T \|\bar u\|_{\mathbf{L}^1(\mathbb{R})} \\
&\qquad + 4\left(\frac{1}{\lfloor \delta\cdot 2^\nu \rfloor \cdot 2^{-\nu}} + 2 + 2T + 4Te^T \|\bar u\|_{\mathbf{L}^1(\mathbb{R})} \right)^2 R^2 \\
&\qquad\qquad + e^T \|\bar u\|_{\mathbf{L}^1(\mathbb{R})} \Bigg] \cdot |t-s| + 2^{-\nu} e^T \|\bar u\|_{\mathbf{L}^1(\mathbb{R})} .
\end{aligned}
$$

5. Tightness property. We are establishing a *Tightness Property* for the sequence $u^\nu(t,x)$. Namely, given $\varepsilon > 0$ and $T > 0$, there exists $L(T) > 0$ such that

$$
\int_{|x|>L(T)} |u^\nu(t,z)|\, dz \ \le\ \varepsilon \qquad \text{for all } t \in [0,T[\ , \nu \ge 1\,. \tag{2.35}
$$

To prove (2.35) we are going to use a comparison argument. Given $\bar u \in L^1(\mathbb{R})$, let $C_T = \sqrt{T}\cdot\sqrt{2(1+T)e^T \|\bar u\|_{L^1(\mathbb{R})}}$ and consider any approximating solution $u^\nu(t,x)$ constructed by the flux splitting. By induction we define the sequence of radii $(R_i)_{i\ge 1}$ as follows.

(i) The radius R_1 is chosen so that

$$
\int_{|x|\ge R_1 - C_T} |\bar u(x)| dx \ \le\ \frac{1}{2}. \tag{2.36}
$$

(ii) If R_{i-1} is given, we choose R_i in such a way that

$$
\int_{|x|\ge R_i - C_T} |\bar u(x)| dx \ \le\ 2^{-i} \tag{2.37}
$$

and

$$
R_i - R_{i-1} \ \ge\ (i+1)\ln(2) + 2\sqrt{T}\sqrt{2(1+T)e^T \|\bar u\|_{\mathbf{L}^1(\mathbb{R})}} \tag{2.38}
$$

Given the approximating solution $u^\nu(t,x)$, we denote by $R_i^\pm(t)$ the minimal and maximal backward characteristics through the point $(t,x) = (T, \pm R_i)$. For each $t \in [0,T]$, we define

$$
H_0(t) \ =\ \{u \in \mathbf{L}^1(\mathbb{R}) \mid Supp(u) \subset [R_1^-(t), R_1^+(t)]\} \tag{2.39}
$$

and

$$
H_i(t) \ =\ \{u \in \mathbf{L}^1(\mathbb{R}) \mid Supp(u) \subset [R_{i+1}^-(t), R_i^-(t)] \cup [R_i^+(t), R_{i+1}^+(t)]\} \qquad \text{for } i \ge 1. \tag{2.40}
$$

Furthermore for each $i \geq 1$, let

$$K_i^-(t) \doteq H_0(t) \oplus H_1(t) \oplus H_2(t) \oplus \cdots \oplus H_{i-1}(t), \qquad K_i^+(t) \doteq H_i(t) \oplus H_{i+1}(t) \oplus \cdots$$

with orthogonal projections

$$\pi_i^-(t) : \mathbf{L}^1(\mathbb{R}) \mapsto K_i^-(t), \qquad \pi_i^-(t)(u) = \begin{cases} u(z), & z \in [R_i^-(t), R_i^+(t)], \\ 0, & \text{else}, \end{cases}$$

and

$$\pi_i^+(t) : \mathbf{L}^1(\mathbb{R}) \mapsto K_i^+(t), \qquad \pi_i^+(t)(u) = \begin{cases} u(z), & z \notin [R_i^-(t), R_i^+(t)], \\ 0, & \text{else}. \end{cases}$$

Then $K_i^-(t) \oplus K_i^+(t) = \mathbf{L}^1(\mathbb{R})$ for all $i \geq 1$.

Let $a_1(t) = \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} e^t$ and define $a_i(t)$ for $i \geq 2$ inductively as the solution to

$$\frac{d}{dt} a_i(t) = a_{i-1}(t) + 2^{-i} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} e^t \quad \text{and} \quad a_i(0) = 2^{-i}. \tag{2.41}$$

Then $a_i(t)$ is non-decreasing. Moreover, $A(t) = \sum_{i \geq 1} a_i(t)$ solves

$$\frac{d}{dt} A(t) = A(t) + \frac{3}{2} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} e^t \quad \text{and} \quad A(0) = \frac{1}{2} + \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}. \tag{2.42}$$

Thus $A(t) = (\frac{1}{2} + \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} + \frac{3}{2} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} t) e^t$, and hence to each $\varepsilon, T > 0$ there exists $I \geq 1$ such that $a_i(t) \leq \varepsilon$ for all $t \in [0, T]$ and $i \geq I$.

Hence if we can show that $p_i(t) = \left\|\pi_i^+ u^\nu(t)\right\|_{\mathbf{L}^1(\mathbb{R})}$ $(i \geq 1)$ satisfies

$$p_1(t) \leq \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} e^t \tag{2.43}$$

and

$$p_i(0) \leq 2^{-i} \quad \text{and} \quad p_i(t) \leq a_i(t) \text{ for all } i \geq 2, t \in [0, T], \tag{2.44}$$

the claim follows. As far as $p_1(t)$ is concerned we have

$$p_1(t) = \|u^\nu(t)\|_{\mathbf{L}^1(\mathbb{R} \setminus [R_1^-(t), R_1^+(t)])} \leq \|u^\nu(t)\|_{\mathbf{L}^1(\mathbb{R})} \leq \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} e^t$$

which proves (2.43). By construction we have for $i \geq 2$,

$$p_i(0) = \int_{|x| \geq R_i(0)} |\bar{u}(x)|\, dx \leq \int_{|x| \,\geq\, R_i - C_T} |\bar{u}(x)|\, dx \leq 2^{-i} \tag{2.45}$$

due to Lemma 2.3. Since the curves $R_i^-(t)$ and $R_i^+(t)$ are characteristics, we have

$$\frac{d}{dt}\, p_i(t) \leq 0 \quad \text{for a.e. } t \in [t_{j-1}, t_j[. \tag{2.46}$$

On the other hand, for $t_j = j \cdot 2^\nu$, we have

$$\begin{aligned} |p_i(t_j) - p_i(t_j-)| &\leq 2^{-\nu} \|[G_x * u^\nu(t_j-, \cdot)]\|_{\mathbf{L}^1(\mathbb{R} \setminus [R_i^-(t), R_i^+(t)])} \\ &= 2^{-\nu} \left\|[G_x * (\pi_{i-1}^- u^\nu(t_j-) + \pi_{i-1}^+ u^\nu(t_j-))](\cdot)\right\|_{\mathbf{L}^1(\mathbb{R} \setminus [R_i^-(t), R_i^+(t)])} \\ &\leq 2^{-\nu} \left\|\pi_{i-1}^+ u^\nu(t_j-, .)\right\|_{\mathbf{L}^1(\mathbb{R})} + 2^{-\nu} \left\|[G_x * \pi_{i-1}^- u^\nu(t_j-)]\right\|_{\mathbf{L}^1(\mathbb{R} \setminus [R_i^-(t), R_i^+(t)])}. \end{aligned}$$

As far as the first term on the right hand side is concerned we can apply (2.46). The second one on the other hand is a bit more challenging,

$$\|[G_x * \pi_{i-1}^- u^\nu(t_j-)]\|_{L^1(\mathbb{R} \setminus [R_i^-(t), R_i^+(t)])}$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{R_i^-(t)} \int_{R_{i-1}^-(t)}^{R_{i-1}^+(t)} e^{-|x-y|} |u^\nu(t_j-, y)| \; dy \; dx \\
&\qquad + \int_{R_i^+(t)}^{\infty} \int_{R_{i-1}^-(t)}^{R_{i-1}^+(t)} e^{-|x-y|} |u^\nu(t_j-, y)| \; dy \; dx \\
&\leq \int_{R_{i-1}^-(t)}^{R_{i-1}^+(t)} \left(\int_{-\infty}^{R_i^-(t)} e^{-(y-x)} dx + \int_{R_i^+(t)}^{\infty} e^{-(x-y)} dx \right) |u^\nu(t_j-, y)| \; dx \; dy \\
&\leq \int_{R_{i-1}^-(t)}^{R_{i-1}^+(t)} \left(e^{-(R_{i-1}^-(t) - R_i^-(t))} + e^{-(R_i^+(t) - R_{i-1}^+(t))} \right) |u^\nu(t_j-, y)| \; dy \\
&\leq \|u^\nu(t_j-, \cdot)\|_{\mathbf{L}^1(\mathbb{R})} \left(e^{-(R_{i-1}^-(t) - R_i-(t))} + e^{-(R_i^+(t) - R_{i-1}^+(t))} \right).
\end{aligned}
$$

Now we can use the estimates for the minimal and maximal backward characteristics to conclude the proof. We have

$$
-\big(R_i^+(t) - R_{i-1}^+(t)\big) \;\leq\; 2\sqrt{T}\sqrt{2(1+T)e^T \|\bar{u}\|_{L^1(\mathbb{R})}} - \big(R_i - R_{i-1}\big) \tag{2.47}
$$

and

$$
-\big(R_{i-1}^-(t) - R_i^-(t)\big) \;\leq\; 2\sqrt{T}\sqrt{2(1+T)e^T \|\bar{u}\|_{L^1(\mathbb{R})}} - \big(R_i - R_{i-1}\big). \tag{2.48}
$$

Thus

$$
\begin{aligned}
\big\|[G_x * \pi_{i-1}^- u^\nu(t_j-)]\big\|_{\mathbf{L}^1(\mathbb{R}\setminus[R_i^-(t), R_i^+(t)])} &\leq 2^{-i} \cdot \|u^\nu(t_j-, .)\|_{\mathbf{L}^1(\mathbb{R})} && (2.49)\\
&\leq 2^{-i} \cdot \|u^\nu(t_{j-1}, .)\|_{\mathbf{L}^1(\mathbb{R})} && (2.50)\\
&\leq 2^{-i} \cdot e^{t_{j-1}} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} && (2.51)
\end{aligned}
$$

according to (2.1) and (2.3), and

$$
\begin{aligned}
|p_i(t_j) - p_i(t_j-)| &\leq 2^{-\nu} \cdot [p_{i-1}(t_{j-1}) + 2^{-i} e^{t_{j-1}} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}] && (2.52)\\
&\leq 2^{-\nu} \cdot [a_{i-1}(t_{j-1}) + 2^{-i} e^{t_{j-1}} \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}] && (2.53)\\
&\leq \int_{t_{j-1}}^{t_j} a_{i-1}(t) + 2^{-i} e^t \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} \, dt. && (2.54)
\end{aligned}
$$

This means in particular, together with (2.46), that

$$
\int_{\{x<R_i^-(t)\}\cup\{x>R_i^+(t)\}} |u^\nu(t,x)| dx \;\leq\; a_i(t).
$$

However, Lemma 2.3 yields

$$
|R_i^+(t) - R_i| \;\leq\; C_T \quad \text{and} \quad |R_i^-(t) + R_i| \;\leq\; C_T \qquad \text{for all } t \in [0,T], i \geq 1. \tag{2.55}
$$

Given $\varepsilon > 0$ and $T > 0$, we choose i such that $a_i(T) \leq \varepsilon$ and hence $a_i(t) \leq \varepsilon$ for all $t \in [0,T]$. Choosing $L(T) = R_i + C_T$ finishes the proof of (2.35).

3. Existence and uniqueness of weak entropy solutions

After introducing the approximating sequence $\{u^\nu(t,x)\}_{\nu\in\mathbb{N}}$ and deriving some a priori estimates in the last section, we are going to establish the existence and uniqueness of weak entropy solutions, i.e. Theorem 1.2.

Proof of Theorem 1.2.

1. Existence of a limiting function. Let $\{u^\nu(t,x)\}_{\nu\in\mathbb{N}}$ be the sequence of approximating solutions constructed in Section 2. Additional, we introduce a new sequence $\{\tilde{u}^\nu(t,x)\}_{\nu\in\mathbb{N}}$, by defining

$$\tilde{u}^\nu(t,\cdot) := (1-\theta_t)\cdot u^\nu(t_i,\cdot) + \theta_t \cdot u^\nu(t_{i+1},\cdot) \quad \text{for all } t\in[t_i,t_{i+1}[, \tag{3.1}$$

where $\theta_t \in [0,1[$ such that $t = (1-\theta_t)\cdot t_i + \theta_t \cdot t_{i+1}$. In contrast to $u^\nu(t,x)$ the function $\tilde{u}^\nu(t,x)$ satisfies

$$\tilde{u}^\nu(t_i,x) \;=\; \tilde{u}^\nu(t_i-,x), \qquad \text{for all } i = 1,2,\dots,$$

a property which plays a crucial role in establishing the existence of a convergent subsequence. To this end we are going to apply [8, Theorem A.8], which we state here, in a slightly modified version, for the sake of completeness.

Theorem 3.1. *Let $\tilde{u}^\nu : [\delta,\infty)\times\mathbb{R}\to\mathbb{R}$ be a familiy of functions such that for each $T>\delta$,*

$$|\tilde{u}^\nu(t,x)| \;\le\; C_T, \qquad (t,x)\in[\delta,T]\times\mathbb{R}\,, \tag{3.2}$$

for a constant C_T independent of ν. Assume in addition for all compact $B\subset\mathbb{R}$ and for $t\in[\delta,T]$ that

$$\sup_{|\xi|\le|\rho|}\int_B |\tilde{u}^\nu(t,x+\xi)-\tilde{u}^\nu(t,x)|dx \;\le\; v_{B,T}(|\rho|), \tag{3.3}$$

for a modulus of continuity v. Furthermore, assume for s and t in $[\delta,T]$ that

$$\int_B |\tilde{u}^\nu(t,x)-\tilde{u}^\nu(s,x)|dx \;\le\; w_{B,T}(|t-s|) \quad \text{as } \nu\to\infty, \tag{3.4}$$

for some modulus of continuity $w_{B,T}$. Then there exists a sequence $\nu_j\to\infty$ such that for each $t\in[\delta,T]$ the sequence $\{\tilde{u}^{\nu_j}(t,\cdot)\}$ converges to a function $u(t,\cdot)$ in $\mathbf{L}^1_{loc}(\mathbb{R})$. The convergence is in $C([\delta,T];\mathbf{L}^1_{loc}(\mathbb{R}))$.

We start by checking that all the assumptions in the above theorem are fulfilled for the sequence $\{\tilde{u}^\nu(t,x)\}$. Without loss of generality, we assume that ν satisfies $2^{-\nu}\le\delta\le 2\cdot\lfloor 2^\nu\cdot\delta\rfloor\cdot 2^{-\nu}$.

(3.4): It suffices to show that for any fixed $\delta,R,T>0$, there exists a constant L_δ such that

$$\|\tilde{u}^\nu(t,\cdot)-\tilde{u}^\nu(s,\cdot)\|_{L^1([-R,R])} \le L_\delta|t-s|, \qquad \delta\le s\le t\le T\,. \tag{3.5}$$

Let $s,t\in[\delta,T]$ such that $s=(1-\theta_s)t_i+\theta_s t_{i+1}$ and $t=(1-\theta_t)t_i+\theta_t t_{i+1}$, then $|t-s| = 2^{-\nu}|\theta_t-\theta_s|$ and

$$\begin{aligned}
\|\tilde{u}^\nu(t,\cdot)-\tilde{u}^\nu(s,\cdot)\|_{L^1([-R,R])} &\le |\theta_t-\theta_s|\,\|u^\nu(t_{i+1},\cdot)-u^\nu(t_i,\cdot)\|_{L^1([-R,R])}\\
&\le |\theta_t-\theta_s|(C_{1,\delta}|t_{i+1}-t_i| + C_{2,\delta}2^{-\nu})\\
&= |\theta_t-\theta_s|2^{-\nu}(C_{1,\delta}+C_{2,\delta})\\
&= |t-s|(C_{1,\delta}+C_{2,\delta})
\end{aligned}$$

where we used (2.32). In the general case, $s,t\in[\delta,T]$ with $s=(1-\theta_s)t_i+\theta_s t_{i+1} < t = (1-\theta_t)t_j+\theta_t t_{j+1}$ and $i\ne j$ one obtains

$$\|\tilde{u}^\nu(t,\cdot)-\tilde{u}^\nu(s,\cdot)\|_{\mathbf{L}^1([-R,R])} \;\le\; \|\tilde{u}^\nu(t,\cdot)-\tilde{u}^\nu(t_j,\cdot)\|_{\mathbf{L}^1([-R,R])}$$

$$+ \sum_{i+1\le k\le j-1} \|\tilde u^\nu(t_{k+1},\cdot)-\tilde u^\nu(t_k,\cdot)\|_{\mathbf{L}^1([-R,R])}$$
$$+ \|\tilde u^\nu(t_{i+1},\cdot)-\tilde u^\nu(s,\cdot)\|_{\mathbf{L}^1([-R,R])}$$
$$\le |t-s|(C_{1,\delta}+C_{2,\delta}).$$

Thus choosing $L_\delta = C_{1,\delta}+C_{2,\delta}$ yields (3.5) and hence (3.4).

Observe that due to (2.6), we have for $t=(1-\theta_t)t_i+\theta_t t_{i+1}$ and all $x<y$,

(3.6)

$$\begin{aligned}\tilde u^\nu(t,y)-\tilde u^\nu(t,x) &\le (1-\theta_t)(u^\nu(t_i,y)-u^\nu(t_i,x))+\theta_t(u^\nu(t_{i+1},y)-u^\nu(t_{i+1},x))\\ &\le \Big(\frac{1}{t_i}+2+2t_{i+1}+4t_{i+1}e^{t_{i+1}}\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\Big)(y-x)\\ &\le \Big(\frac{2}{\delta}+2+2(T+1)+4(T+1)e^{T+1}\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\Big)(y-x)\\ &= D_T(y-x)\end{aligned}$$

and hence, as in the proof of Lemma 2.2, we obtain

(3.7)

$$\|\tilde u^\nu(t,x)\|_{\mathbf{L}^\infty(\mathbb{R})} \le \sqrt{2\Big(\frac{2}{\delta}+2+2(T+1)+4(T+1)e^{T+1}\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\Big)e^{T+1}\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}}.$$

Thus choosing $C_T = \sqrt{2(\frac{2}{\delta}+2+2(T+1)+4(T+1)e^{T+1}\|\bar u\|_{\mathbf{L}^1(\mathbb{R})})e^{T+1}\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}}$ finishes the proof of (3.2).

(3.3): Following the proof of [8, Lemma A.1] and applying (3.6) and (3.7), we get,

$$\begin{aligned}\int_{[-R,R]}|\tilde u^\nu(t,x+\xi)-\tilde u^\nu(t,x)|\,dx &\le \text{Tot.Var.}\{\tilde u^\nu(t,\cdot);[-R-|\xi|,R+|\xi|]\}\cdot|\xi|\\ \le |\xi|(\text{Tot.Var.}\{\tilde u^\nu(t,\cdot)-D_T\cdot;[-R-|\xi|,R+|\xi|]\}&+\text{Tot.Var.}\{D_T\cdot;[-R-|\xi|,R+|\xi|]\})\\ &\le |\xi|(2\|\tilde u^\nu(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})}+4D_T(R+|\xi|))\end{aligned}$$

since $\tilde u^\nu - D_T$ is decreasing, due to (3.6), and hence (3.3) is satisfied.

Thus, Theorem 3.1 implies that there exists a subsequence $\nu_j\to\infty$ and a limit function $\tilde u:[\delta,T]\times[-R,R]\to\mathbb{R}$ such that

$$\lim_{j\to\infty}\|\tilde u^{\nu_j}(t,\cdot)-\tilde u(t,\cdot)\|_{\mathbf{L}^1([-R,R])} = 0, \qquad \text{for all } \delta\le t\le T, \tag{3.8}$$

and

$$\lim_{j\to\infty}\|\tilde u^{\nu_j}-\tilde u\|_{\mathbf{L}^1([\delta,T]\times[-R,R])} = 0. \tag{3.9}$$

By construction, see (3.1), we have for any $\delta\le t\le T$, with $t=(1-\theta_t)t_i+\theta_t t_{i+1}$ that

$$\begin{aligned}&\|u^{\nu_j}(t,\cdot)-\tilde u^{\nu_j}(t,\cdot)\|_{\mathbf{L}^1[-R,R])}\\ &\le (1-\theta_t)\|u^{\nu_j}(t,\cdot)-u^{\nu_j}(t_i,\cdot)\|_{\mathbf{L}^1([-R,R])}+\theta_t\|u^{\nu_j}(t,\cdot)-u^{\nu_j}(t_{i+1},\cdot)\|_{\mathbf{L}^1([-R,R])}.\end{aligned}$$

Recalling (2.32), we obtain

$$\|u^{\nu_j}(t,\cdot)-\tilde u^{\nu_j}(t,\cdot)\|_{\mathbf{L}^1[-R,R])} \le L_\delta 2^{-\nu}. \tag{3.10}$$

with $L_\delta := C_{1,\delta}+C_{2,\delta}$. Thus combining (3.8), (3.9) and (3.10), we get

$$\lim_{j\to\infty}\|u^{\nu_j}(t,\cdot)-\tilde u(t,\cdot)\|_{\mathbf{L}^1([-R,R])} = 0, \qquad \text{for all } \delta\le t\le T, \tag{3.11}$$

and

$$\lim_{j\to\infty} \|u^{\nu_j} - \tilde{u}\|_{\mathbf{L}^1([\delta,T]\times[-R,R])} \ = \ 0\,. \tag{3.12}$$

Since (3.11) and (3.12) hold for any $\delta, T, R > 0$, there exists $I \subset \mathbb{N}$ and $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that $\{u^\nu(t,\cdot)\}_{\nu\in I} \to u(t,\cdot)$ in $\mathbf{L}^1_{loc}(\mathbb{R})$ for any $t \geq 0$ and $\{u^\nu\}_{\nu\in I} \to u$ in $\mathbf{L}^1_{loc}([0,\infty[\times\mathbb{R})$. Moreover, by the *Tightness property*, we have that to any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}\setminus[-R_\varepsilon,R_\varepsilon]} |u^\nu(t,x)|\ dx \ \leq \ \varepsilon \quad \text{for all } t \in [0,T], \nu \in I.$$

Thus we get for all $R \geq R_\varepsilon$

$$\begin{aligned}\int_{[-R,R]} |u(t,x)|\ dx \ &= \ \lim_{\nu\in I\to\infty} \int_{[-R,R]} |u^\nu(t,x)|\ dx \\ &\leq \ \lim_{\nu\in I\to\infty} \left(\|u^\nu(t,.)\|_{\mathbf{L}^1(\mathbb{R})} + \varepsilon\right) \\ &\leq \ e^T \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} + \varepsilon.\end{aligned}$$

Since the above estimate is uniform, we can conclude that $u(t,.) \in \mathbf{L}^1(\mathbb{R})$ for all $t \geq 0$. Let $t \in [0,T]$, then there exists $R^1_\varepsilon > 0$ such that

$$\int_{\mathbb{R}\setminus[-R^1_\varepsilon,R^1_\varepsilon]} |u(t,x)|\ dx \ \leq \ \varepsilon.$$

Set $\overline{R}_\varepsilon \doteq \min\{R_\varepsilon, R^1_\varepsilon\}$, we then have

$$\int_{\mathbb{R}\setminus[-\overline{R}_\varepsilon,\overline{R}_\varepsilon]} |u(t,x) - u^\nu(t,x)|\ dx \ \leq \ 2\varepsilon \qquad \text{for all } \nu \in I\,,$$

which implies

$$\lim_{\nu\in I\to\infty} \ \|u(t,\cdot) - u^\nu(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ 2\varepsilon\,.$$

Therefore, $u^\nu(t,\cdot)$ converges to $u(t,\cdot)$ in $\mathbf{L}^1(\mathbb{R})$ for all $t \geq 0$.

Recalling (2.5) and (2.6), we have for all $t > 0$,

$$\|u(t,\cdot)\|_{L^1(\mathbb{R})} \leq e^t \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\ , \tag{3.13}$$

and

$$u(t,x_2) - u(t,x_1) \ \leq \ \left[\frac{1}{t} + 2 + 2t + 4te^t \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\right](x_2 - x_1), \quad \text{for all } x_1 < x_2\,. \tag{3.14}$$

In particular,

$$\|u(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ \sqrt{2e^t \cdot \left[\frac{1}{t} + 2 + 2t + 4te^t \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\right] \cdot \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}, \quad \text{for all } t > 0\,,$$

and thus u is in $\mathbf{L}^\infty_{loc}(]0,\infty[, \mathbf{L}^\infty(\mathbb{R}))$.

2. The map $t \to u(t,\cdot)$ is continuous from $[0,T[$ to $\mathbf{L}^1(\mathbb{R})$. By the *Tightness Property* for $\{u^\nu(t,x)\}_{\nu\in\mathbb{N}}$, we have that to any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\|u^\nu(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}\setminus[-R_\varepsilon,R_\varepsilon])} \leq \varepsilon \qquad \text{for all } t \in [0,T[, \nu \in \mathbb{N}. \tag{3.15}$$

Thus for R_ε big enough (3.12) and (3.15) imply for $t \in (\delta, T)$ that

$$\begin{aligned}\|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R}\setminus[-R_\varepsilon,R_\varepsilon])} &= \|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} - \|u(t,\cdot)\|_{\mathbf{L}^1([-R_\varepsilon,R_\varepsilon])} \\ &\leq \lim_{\nu\in I\to\infty}\left(\|u^\nu(t,.)\|_{\mathbf{L}^1(\mathbb{R})} + \varepsilon - \|u^\nu(t,.)\|_{\mathbf{L}^1([-R_\varepsilon,R_\varepsilon])}\right) \\ &\leq \lim_{\nu\in I\to\infty}\left(\|u^\nu(t,.)\|_{\mathbf{L}^1(\mathbb{R}\setminus[-R_\varepsilon,R_\varepsilon])} + \varepsilon\right) \\ &\leq 2\varepsilon.\end{aligned}$$

Therefore, for any fixed $\delta > 0$ and for any $s, t \in (\delta, T)$ we obtain

$$\begin{aligned}\|u(t,\cdot) - u(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} &\leq \|u(t,\cdot) - u(s,\cdot)\|_{\mathbf{L}^1([-R_\varepsilon,R_\varepsilon])} + 2\varepsilon \\ &= \lim_{\nu\in I\to\infty}\|u^\nu(t,\cdot) - u^\nu(s,\cdot)\|_{\mathbf{L}^1([-R_\varepsilon,R_\varepsilon])} + 2\varepsilon \\ &\leq C_{1,\delta}\cdot|t-s| + 2\varepsilon\,,\end{aligned}$$

where we applied (2.32) in the last step. This implies that $u(t,\cdot)$ is continuous from $(0,T)$ to $\mathbf{L}^1(\mathbb{R})$.

On the other hand, the continuity also holds at $t = 0$, i.e.,

$$\lim_{t\to 0^+}\|u(t,\cdot) - \bar{u}\|_{\mathbf{L}^1(\mathbb{R})} = 0\,. \tag{3.16}$$

Indeed, for any $\nu \in \mathbb{N}$ and $t > 0$, we have

$$\begin{aligned}&\|S^B_{t-t_i}(u^\nu(t_i))(\cdot) - S^B_{t-t_{i-1}}(u^\nu(t_{i-1}))(\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \\ &\quad= \left\|S^B_{t-t_i}(u^\nu(t_i))(\cdot) - S^B_{t-t_i}(u^\nu(t_i-))(\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})} \\ &\qquad\leq \|u^\nu(t_i,\cdot) - u^\nu(t_i-,\cdot)\|_{L^1(\mathbb{R})} \leq 2^{-\nu}e^t\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\end{aligned}$$

for all $i \in \{1, 2, \ldots, \lfloor 2^\nu\cdot t\rfloor\}$. Thus,

$$\begin{aligned}\left\|u^\nu(t,\cdot) - S^B_t(\bar{u})(\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})} &\leq \sum_{i=1}^{\lfloor 2^\nu\cdot t\rfloor\}}\left\|S^B_{t-t_i}(u^\nu(t_i))(\cdot) - S^B_{t-t_{i-1}}(u^\nu(t_{i-1}))(\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})} \\ &\leq te^t\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\;.\end{aligned}$$

Since $u^\nu(t,\cdot)$ converges to $u(t,\cdot)$ in $\mathbf{L}^1(\mathbb{R})$ for all $t \geq 0$, we obtain that

$$\|u(t,\cdot) - S^B_t(\bar{u})(\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq te^t\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}$$

and in particular,

$$\lim_{t\to 0^+}\left\|u(t,\cdot) - S^B_t(\bar{u})\right\|_{\mathbf{L}^1(\mathbb{R})} = 0\,.$$

Therefore, (3.16) follows from the continuity of Burgers semigroup S^B_t at time $t = 0$.

3. Weak entropy condition. We show that $u(t,x)$ satisfies the entropy condition (1.4). Let $\eta(u) = |u-k|$ and $q(u) = \operatorname{sign}(u-k)\left(\frac{u^2}{2} - \frac{k^2}{2}\right)$. Additional define $\eta_\delta(u) = \sqrt{(u-k)^2 + \delta^2}$ and denote by $q_\delta(u)$ the solution to

$$q'_\delta(u) = \eta'_\delta(u)u \qquad q_\delta(k) = 0. \tag{3.17}$$

Then we get for any $\phi \in C_1^c(]0,\infty[\times\mathbb{R},\mathbb{R})$, since $u(t,x)$ is uniformly bounded on the support of ϕ according to (3.14), that

$$
\begin{aligned}
\iint [|u-k|\phi_t + \operatorname{sign}(u-k)\left(\frac{u^2}{2}-\frac{k^2}{2}\right)\phi_x]dxdt \\
&= \iint [\eta(u)\phi_t + q(u)\phi_x]dxdt \\
&= \lim_{\delta\to 0}\iint [\eta_\delta(u)\phi_t + q_\delta(u)\phi_x]dxdt \\
&= \lim_{\delta\to 0}\lim_{j\to\infty}\iint [\eta_\delta(u^{\nu_j})\phi_t + q_\delta(u^{\nu_j})\phi_x]dxdt \\
&= \lim_{\delta\to 0}\lim_{j\to\infty}\sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}}[\eta_\delta(u^{\nu_j})\phi_t + q_\delta(u^{\nu_j})\phi_x]dxdt \\
&\geq \lim_{\delta\to 0}\lim_{j\to\infty}\sum_{i=0}^{\infty}\int_{\mathbb{R}}[\eta_\delta(u^{\nu_j}(t_{i+1}-,x))\phi(t_{i+1},x) - \eta_\delta(u^{\nu_j}(t_i+,x)))\phi(t_i,x)]dx \\
&= \lim_{\delta\to 0}\lim_{j\to\infty}\sum_{i=1}^{\infty}\int_{\mathbb{R}}[\eta_\delta(u^{\nu_j}(t_i-,x)) - \eta_\delta(u^{\nu_j}(t_i+,x))]\phi(t_i,x)dx \\
&= \lim_{\delta\to 0}\iint \frac{-(u(t,x)-k)[G_x * u(t)](x)}{\sqrt{(u-k)^2+\delta^2}}\phi(t,x)dxdt \\
&= -\iint \operatorname{sign}(u(t,x)-k)[G_x * u(t)](x)\phi(t,x)dxdt.
\end{aligned}
$$

4. Lipschitz continuity with respect to time. Let u_1, u_2 be weak entropy solutions of (1.1) with $u_1(0,\cdot) = \bar{u}_1(\cdot)$ and $u_2(0,\cdot) = \bar{u}_2(\cdot)$. We will prove that

$$
\|u_2(t,\cdot) - u_1(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq e^t \cdot \|\bar{u}_2 - \bar{u}_1\|_{\mathbf{L}^1(\mathbb{R})}, \quad \text{for all } t > 0, \tag{3.18}
$$

which implies the uniqueness of the weak entropy solution to (1.1)-(1.3).

Since $t \to u_i(t,\cdot)$ is continuous with values in $L^1(\mathbb{R})$, for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$
\|u_i(t_\varepsilon,\cdot) - \bar{u}_i(\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq \varepsilon \qquad i \in \{1,2\}.
$$

This implies that

$$
\|u_2(t_\varepsilon,\cdot) - u_1(t_\varepsilon,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq 2\varepsilon + \|\bar{u}_2 - \bar{u}_1\|_{\mathbf{L}^1(\mathbb{R})}. \tag{3.19}
$$

On the other hand, since $u_1(t,x)$ and $u_2(t,x)$ are weak entropy solutions, for any $T > 0$ there exists $M_T > 0$ such that for any $t \in [t_\varepsilon, T]$ it holds

$$
\|u_i\|_{\mathbf{L}^\infty([t_\varepsilon,T]\times\mathbb{R})} \leq M_T, \qquad i \in \{1,2\}.
$$

Therefore, one can follow the same argument in the proof of [2, Theorem 6.2] to show that for all $t_\varepsilon \leq s \leq t \leq T$

$$
\begin{aligned}
&\|u_2(t,\cdot) - u_1(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \\
&\quad \leq \|u_2(s,\cdot) - u_1(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} + \int_s^t \|G_x * (u_2(\tau,\cdot) - u_1(\tau,\cdot))\|_{\mathbf{L}^1(\mathbb{R})} d\tau
\end{aligned}
$$

$$\leq \ \|u_2(s,\cdot)-u_1(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}+\int_s^t \|u_2(\tau,\cdot)-u_1(\tau,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}\,d\tau\,.$$

Thus, the function $Z(t):=\|u_2(t,\cdot)-u_1(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}$ satisfies the integral inequality

$$Z(t)\leq Z(t_\varepsilon)+\int_{t_\varepsilon}^t Z(\tau)d\tau,\qquad \text{for all } t_\varepsilon\leq t\leq T\,.$$

Applying the Gronwall's inequality, we get

$$Z(t)\leq e^{(t-t_\varepsilon)}Z(t_\varepsilon),\qquad \text{for all } t_\varepsilon\leq t\leq T\,.$$

Recalling (3.19), we finally obtain that

$$Z(t)\leq e^{(t-t_\varepsilon)}\big[Z(0)+2\varepsilon\big]\leq e^t[Z(0)+2\varepsilon]\qquad \text{for all } \varepsilon>0\,,$$

which yields (3.18). □

4. Local smoothness and wave breaking

In this final section we want to focus on the prediction of wave breaking. In particular, we are interested in identifying for initial data $\bar{u}(\cdot)\in\mathbf{C}^1(\mathbb{R})\cap\mathbf{L}^1(\mathbb{R})$ if wave breaking occurs in the nearby future or not by following solutions along characteristics as long as they exist in the classical sense.

Theorem 4.1. *Let $u(t,x)$ be the weak entropy solution of (1.1) with $u(0,\cdot)=\bar{u}(\cdot)\in\mathbf{C}^1(\mathbb{R})\cap\mathbf{L}^1(\mathbb{R})$. Denote by $x(t)$ the characteristic through $\bar{x}$, i.e.,*

$$x'(t)=u(t,x(t))\qquad x(0)=\bar{x}\,.$$

Then the following statements hold

(i) *$u_x(t,x(t))$ remains bounded for all $t\in[0,T_*[$ where*

(4.1)
$$T_* \;=\; \ln\left(1+\frac{1}{\sqrt{|\bar{u}(\bar{x})|+2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\left(\frac{\pi}{2}+\arctan\left(\frac{\bar{u}'(\bar{x})}{\sqrt{|\bar{u}(\bar{x})|+2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\right)\right)\right)\,.$$

(ii) *If*

$$\bar{u}'(\bar{x}) \;<\; -\frac{1}{2}-\sqrt{|\bar{u}(\bar{x})|+2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}+\frac{1}{4}}\,, \tag{4.2}$$

then $u_x(t,x(t))$ becomes unbounded before time T^* where

$$T^* \;=\; \frac{2}{\left|2\bar{u}'(\bar{x})+1+2\cdot\sqrt{|\bar{u}(\bar{x})|+2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}+\frac{1}{4}}\right|}\,. \tag{4.3}$$

Proof. Let $u(t,x)$ be the weak entropy solution of (1.1) with initial data $u(0,x)=\bar{u}(x)\in\mathbf{C}^1(\mathbb{R})\cap\mathbf{L}^1(\mathbb{R})$. Given $\bar{x}\in\mathbb{R}$ denote by $x(t)$ the characteristic through $\bar{x}$ at time $t=0$, i.e., $x'(t)=u(t,x(t))$ and $x(0)=\bar{x}$. Furthermore let $m=\bar{u}'(\bar{x})$, then we are interested in finding an upper and a lower bound on how long it takes until wave breaking occurs. That is we are going to establish an upper and a lower bound on t^* such that

$$u_x(t,x(t))\to-\infty\qquad t\uparrow t^*.$$

Differentiating (1.1) with respect to x yields

$$(u_x)_t+u(u_x)_x \;=\; -(u_x)^2+[G*u]_{xx} \;=\; -(u_x)^2+[G*u]+u$$

and hence

$$z(t) := u_x(t, x(t))$$

satisfies

$$z'(t) = -z(t)^2 + [G * u(t,\cdot)](x(t)) + u(t, x(t)). \tag{4.4}$$

Since $u(t, x(t))$ satisfies

$$\frac{d}{dt}u(t, x(t)) = [G_x * u(t,\cdot)](x(t)),$$

we have, due to (1.5),

$$\frac{d}{dt}|u(t, x(t))| \leq \|u(t,.)\|_{\mathbf{L}^1(\mathbb{R})} \leq \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} e^t \quad \text{for all } t \in [0, t^*).$$

Thus

$$|u(t, x(t))| \leq |\bar{u}(\bar{x})| + \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})} e^t \leq \Big(|\bar{u}(\bar{x})| + \|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big) \cdot e^t,$$

and

$$\begin{aligned}&|[G * u(t,\cdot)](x(t)) + u(t, x(t))| \\ &\qquad\qquad \leq \|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} + |u(t, x(t))| \leq \Big(|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big) \cdot e^t.\end{aligned}$$

Recalling (4.4), we get for all $t \in [0, t^*)$,

$$-z(t)^2 - \Big(|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big) \cdot e^t \leq z'(t) \leq -z(t)^2 + \Big(|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big) \cdot e^t. \tag{4.5}$$

To derive a lower bound on t^* we look at the subsolution defined through

$$s'(t) = -s(t)^2 - \Big(|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}\Big) \cdot e^t \quad , \quad s(0) = m.$$

This implies that

$$\frac{1}{\sqrt{|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}} \frac{\frac{s'(t)}{\sqrt{|\bar{u}(\bar{x})|+2\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}}{\left(\frac{s(t)}{\sqrt{|\bar{u}(\bar{x})|+2\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\right)^2 + 1} \geq -e^t \qquad \text{for all } t \in [0, t^*)$$

or, equivalently,

$$\frac{1}{\sqrt{|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}} \left(\arctan\left(\frac{s(t)}{\sqrt{|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\right)\right)' \geq -e^t$$

for all $t \in [0, t^*)$. Thus a lower bound on t^* is given by T_l, which is defined implicitly through

$$\frac{1}{\sqrt{|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}} \left(-\frac{\pi}{2} - \arctan\left(\frac{m}{\sqrt{|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\right)\right) \geq 1 - e^{T_l}.$$

Hence

$$T_l \geq \ln\left(1 + \frac{1}{\sqrt{|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}} \left(\frac{\pi}{2} + \arctan\left(\frac{\bar{u}'(\bar{x})}{\sqrt{|\bar{u}(\bar{x})| + 2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\right)\right)\right)$$

which finally implies (4.1) .

To derive an upper bound on t^* we look at the supersolution defined through

$$s'(t) \;=\; -s(t)^2 + \Big(|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\Big)\cdot e^t \quad , \quad s(0) \;=\; m.$$

Let $s_1(t) = s(t)e^{-t}$, then

$$\begin{aligned} s_1'(t) &= s'(t)e^{-t} - s(t)e^{-t} \\ &= -s(t)^2e^{-t} - s(t)e^{-t} + \Big(|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\Big) \\ &= -s_1(t)^2e^{t} - s_1(t) + \Big(|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\Big) \\ &\le -s_1(t)^2 - s_1(t) + \Big(|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\Big). \end{aligned}$$

Observe that $s_1'(t)$ is decreasing if

$$-s_1(t)^2 - s_1(t) + |\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} \;=\; -\Big(s_1(t)+\frac{1}{2}\Big)^2 + \Big(|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}\Big) \;<\; 0\,.$$

Hence if we assume that $s(0) = s_1(0) < -\frac{1}{2} - \sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}$, the function $s(t)$ will be strictly decreasing on $[0, t^*)$ and

$$\frac{1}{\sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}\;\frac{\frac{s_1'(t)}{\sqrt{|\bar u(\bar x)| + 2\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}}{\left(\frac{s_1(t)+\frac{1}{2}}{\sqrt{|\bar u(\bar x)| + 2\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}\right)^2 - 1} \;\le\; -1\,.$$

Equivalently,

$$\frac{1}{2\sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}\left(\ln\left(\frac{s_1(t) + \frac{1}{2} - \sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}{s_1(t) + \frac{1}{2} + \sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}\right)\right)' \le -1.$$

Thus an upper bound on t^* is given by T_u, which is defined through

$$T_u = \frac{1}{2\sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}\ln\left(1 - 2\frac{\sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}{m + \frac{1}{2} + \sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}}\right).$$

Recalling (4.2), we finally obtain

$$T_u \;\le\; \frac{2}{\left|2m + 1 + 2\cdot\sqrt{|\bar u(\bar x)| + 2\,\|\bar u\|_{\mathbf{L}^1(\mathbb{R})} + \frac{1}{4}}\right|} \;=:\; T^*\,.$$

□

As an immediate consequence we obtain the following corollary.

Corollary 4.2. *Let $u(t,x)$ be the weak entropy solution of* (1.1) *with initial data $u(0,\cdot) = \bar u(\cdot) \in \mathbf{C}^1(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$. Additional, let*

$$m \;=\; \inf_{x\in\mathbb{R}} \bar u'(x) \qquad \text{and} \qquad M \;=\; \|\bar u\|_{\mathbf{L}^1(\mathbb{R})}\,.$$

Then the following statements hold.

(i) *$u_x(t,x)$ remains bounded for all $t\in[0,T^-[$ where*

$$T^- \;=\; \ln\left(1+\frac{1}{\sqrt{|m|+3M}}\cdot\left(\frac{\pi}{2}-\arctan\left(\frac{|m|}{\sqrt{2M}}\right)\right)\right).$$

(ii) *If*

$$m \;<\; -1-\sqrt{\frac{5}{2}M+1}\;\le\;0\,, \tag{4.6}$$

then $u_x(t,x)$ becomes unbounded within the time interval $[0,T^+[$ where

$$T^+ \;=\; \frac{2}{\left|2m+1+2\sqrt{|m|+\frac{5}{2}M+\frac{1}{4}}\right|}. \tag{4.7}$$

Proof. Given $\bar{x}\in\mathbb{R}$, denote by $x(t)$ the characteristic through $\bar{x}$, i.e. $x'(t)=u(t,x(t))$ and $x(0)=\bar{x}$. Theorem 4.1 (i) implies that $u_x(t,x(t))$ is bounded for all $t\in[0,t_{\bar{x}}[$ with

(4.8)

$$t_{\bar{x}} \;=\; \ln\left(1+\frac{1}{\sqrt{|\bar{u}(\bar{x})|+2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\left(\frac{\pi}{2}+\arctan\left(\frac{\bar{u}'(\bar{x})}{\sqrt{|\bar{u}(\bar{x})|+2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}}}\right)\right)\right).$$

In particular, we have

$$t_{\bar{x}} \;\ge\; \ln\left(1+\frac{1}{\sqrt{|\bar{u}(\bar{x})|+2M}}\left(\frac{\pi}{2}-\arctan\left(\frac{|m|}{\sqrt{|\bar{u}(\bar{x})|+2M}}\right)\right)\right). \tag{4.9}$$

In addition, it is well-known that

$$\|\bar{u}\|_{\mathbf{L}^\infty(\mathbb{R})} \;\le\; \sqrt{2|m|M} \;\le\; |m|+\frac{M}{2}, \tag{4.10}$$

and hence

$$t_{\bar{x}} \ge \ln\left(1+\frac{1}{\sqrt{|m|+3M}}\left(\frac{\pi}{2}-\arctan\left(\frac{|m|}{\sqrt{2M}}\right)\right)\right). \tag{4.11}$$

Since the right hand side is independent of $\bar{x}$, we have shown the first part.

As far as the second part is concerned, observe first of all that $m<0$, since by assumption $\bar{u}\in\mathbf{C}^1(\mathbb{R})\cap\mathbf{L}^1(\mathbb{R})$. Moreover, (4.6) and (4.10) imply that for all $x\in\mathbb{R}$

$$\frac{1}{2}+\sqrt{|\bar{u}(x)|+\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}+\frac{1}{4}} \;\le\; \frac{1}{2}+\sqrt{|m|+\frac{5}{2}M+\frac{1}{4}} \;<\; |m|.$$

Thus there exists $\bar{x}\in\mathbb{R}$ such that

$$\bar{u}'(\bar{x}) \;<\; -\frac{1}{2}-\sqrt{|\bar{u}(\bar{x})|+\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}+\frac{1}{4}} \quad\text{and}\quad \bar{u}'(x)\;\ge\; m\,, \tag{4.12}$$

and hence according to the proof of Theorem 4.1 wave breaking occurs. Let $\bar{x}\in\mathbb{R}$ such that (4.12) is satisfied, then an upper bound for the maximal time of existence is given by (4.3)

$$T_{\bar{x}} \;=\; \frac{1}{\left|2\bar{u}'(\bar{x})+1+2\cdot\sqrt{|\bar{u}(\bar{x})|+2\,\|\bar{u}\|_{\mathbf{L}^1(\mathbb{R})}+\frac{1}{4}}\right|}. \tag{4.13}$$

Hence, by recalling (4.12), we obtain

$$T_{\bar{x}} \;\leq\; \frac{2}{\left|2m+1+2\sqrt{|m|+\frac{5}{2}M+\frac{1}{4}}\right|}\,, \tag{4.14}$$

where the right hand side is independent of $\bar{x}$ and hence it yields (4.7). □

Finally we want to show that if the $\mathbf{L}^\infty$-norm of the derivative of the initial data $\bar{u}$ is small then the corresponding entropy solution of (1.1) will remain smooth for a long time.

Theorem 4.3. *Let $u(t,x)$ be the weak entropy solution of* (1.1) *with $u(0,\cdot) = \bar{u}(\cdot) \in \mathbf{C}^1(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ and let $m = \|\bar{u}'\|_{\mathbf{L}^\infty(\mathbf{R})}$. Then $\|u_x(t,.)\|_{\mathbf{L}^\infty(\mathbb{R})}$ remains bounded for all $t \in [0, \ln(1+\frac{1}{m})[$.*

Proof. Given any $\bar{x} \in \mathbb{R}$, denote by $x(t)$ the characteristic through $\bar{x}$ at time $t = 0$, i.e., $x'(t) = u(t, x(t))$ and $x(0) = \bar{x}$. Then the function $z(t) = u_x(t, x(t))$ satisfies the differential equation

$$z'(t) \;=\; -z(t)^2 + [G_x * u_x(t,\cdot)](x(t))\,.$$

This implies that

$$\frac{d}{dt}|z(t)| \;\leq\; z^2(t) + \|u_x(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})} \;\leq\; \|u_x(t,\cdot)\|^2_{\mathbf{L}^\infty(\mathbb{R})} + \|u_x(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})}\,.$$

Let $Q(t) \;=\; \|u_x(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})}$, then we have

$$Q'(t) \;\leq\; Q^2(t) + Q(t)$$

or equivalently,

$$\frac{d}{dt}\;\ln\left(\frac{Q(t)}{Q(t)+1}\right) \;\leq\; 1\,.$$

Thus,

$$\ln\left(\frac{Q(t)}{Q(t)+1}\right) + \;\ln\left(\frac{Q(0)+1}{Q(0)}\right) \;\leq\; t\,,$$

which implies that

$$\ln\left(\frac{Q(t)}{Q(t)+1}\right) \;\leq\; t - \ln\left(1+\frac{1}{m}\right).$$

Thus if $Q(t)$ becomes unbounded at time t^*, the left hand side tends to zero as $t \uparrow t^*$ and in particular, $0 \leq t^* - \ln\left(1+\frac{1}{m}\right)$. Or, in other words, $Q(t)$ remains bounded for all $t \in \left[0, \ln\left(1+\frac{1}{m}\right)\right[$. □

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