

Covering numbers for bounded variation functions

Prerona Dutta and Khai T. Nguyen

Department of Mathematics
North Carolina State University.

Emails: pdutta@ncsu.edu, khai@math.ncsu.edu

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Abstract

In this paper, we provide upper and lower estimates of the minimal number of functions needed to represent a bounded variation function with an accuracy of epsilon with respect to \mathbf{L}^1 -distance.

1 Introduction

The ε -entropy has been studied extensively in a variety of literature and disciplines. It plays a central role in various areas of information theory and statistics, including nonparametric function estimation, density information, empirical processes and machine learning (see e.g in [11, 18, 28]). This concept was first introduced by Kolmogorov and Tikhomirov in [25]:

Definition 1.1 *Let (X, d) be a metric space and E a precompact subset of X . For $\varepsilon > 0$, let $\mathcal{N}_\varepsilon(E|X)$ be the minimal number of sets in an ε -covering of E , i.e., a covering of E by subsets of X with diameter no greater than 2ε . Then ε -entropy of E is defined as*

$$\mathcal{H}_\varepsilon(E | X) = \log_2 \mathcal{N}_\varepsilon(E | X).$$

In other words, it is the minimum number of bits needed to represent a point in a given set E in the space X with an accuracy ε with respect to the metric d .

A classical topic in the field of probability is to investigate the metric covering numbers for general classes of real-valued functions \mathcal{F} defined on X under the family of $\mathbf{L}^1(dP)$ where P is a probability distribution on X . Upper bounds in terms of Vapnik-Chervonenkis and pseudo-dimension of the function class were established in [16], and then improved in [28, 18, 19]. Several results on lower bounds were also studied in [24]. Later on, upper and lower estimates of the ε -entropy of \mathcal{F} in $\mathbf{L}^1(dP)$ in terms of a scale-sensitive dimension of the function class were provided in [29, 24], and applied to learning machine.

Thanks to the Helly's theorem, a set of uniformly bounded variation functions is compact

in \mathbf{L}^1 -space. A natural question is to quantify the compactness of such sets by using the ε -entropy. In [24], the authors considered this problem in the scalar case and proved that the ε -entropy of a class of real valued functions of bounded variation in \mathbf{L}^1 is of the order of $\frac{1}{\varepsilon}$. Some related works have been done in the context of density estimation where attention has been given to the problem of finding covering numbers for the classes of densities that are unimodal or nondecreasing in [11, 22]. In the multi-dimensional cases, the covering numbers of convex and uniformly bounded functions were studied in [23]. It was shown that the ε -entropy of a class of convex functions with uniform bound in \mathbf{L}^1 is of the order of $\frac{1}{\varepsilon^{\frac{n}{2}}}$ where n is the dimension of the state variable. The result was previously studied for scalar state variables in [17] and for convex functions that are uniformly bounded and uniformly Lipschitz with a known Lipschitz constant in [13]. These results have direct implications in the study of rates of convergence of empirical minimization procedures (see e.g. in [12, 20] as well as optimal convergence rates in the numerous convexity constrained function estimation problems (see e.g. in [10, 14, 9]).

Recently, the ε -entropy has been used to measure the set of solutions of certain nonlinear partial differential equations. In this setting, it could provide a measure of the order of “resolution” and of the “complexity” of a numerical scheme, as suggested in [26, 27]. Roughly speaking, the order of magnitude of the ε -entropy should indicate the minimum number of operations that one should perform in order to obtain an approximate solution with a precision of order ε with respect to the considered topology. A starting point of this research topic is a result which was obtained in [15] for a scalar conservation law in one dimensional space

$$u_t(t, x) + f(u(t, x))_x = 0, \quad (1.1)$$

with uniformly convex flux f . They showed that the upper bound of the minimum number of functions needed to represent an entropy solution u of (1.1) at any time $t > 0$ with accuracy ε with respect to \mathbf{L}^1 -distance is of the order of $\frac{1}{\varepsilon}$. In [5] a lower bound on such an ε -entropy was established, which is of the same order as of the upper bound in [15]. More generally, the authors in [5] also obtained the same estimate for a system of hyperbolic conservation laws in [6, 7]. In the scalar case, it is well-known that the integral form of an entropy solution of (1.1) is a viscosity solution of the related Hamilton-Jacobi equation. Therefore, it is natural to study the ε -entropy for the set of viscosity solutions to the Hamilton-Jacobi equation

$$u_t(t, x) + H(\nabla_x u(t, x)) = 0, \quad (1.2)$$

with respect to $\mathbf{W}^{1,1}$ -distance in multi-dimensional cases. Most recently, it has been proved in [3] that the minimal number of functions needed to represent a viscosity solution of (1.2) with accuracy ε with respect to the $\mathbf{W}^{1,1}$ -distance is of order of $\frac{1}{\varepsilon^n}$ provided that H is uniformly convex. Here, n is the dimension of the state variable. The same result for when the Hamiltonian depends on the state variable x has also been obtained by the same authors in [4].

Interestingly, the authors in [3] also established an upper bound on the ε -entropy for the class of monotone functions in \mathbf{L}^1 -space. As a consequence of Poincaré-type inequalities, they could obtain the ε -entropy for a class of semi-convex/concave functions in Sobolev $\mathbf{W}^{1,1}$ space. This result somehow extended the one in [23, 17, 13] to a stronger norm, $\mathbf{W}^{1,1}$ -norm instead of \mathbf{L}^1 -norm. Motivated by the results in [24, 23, 17, 13, 3] and a possible application to Hamilton-Jacobi equation with non-strictly convex Hamiltonian, we will provide in the

present paper upper and lower estimates of the ε -entropy for a class of uniformly bounded total variation functions in \mathbf{L}^1 -space in multi-dimensional cases. The precise statement will be stated in Theorem 3.1 in section 3.

2 Notations and preliminaries

Let $n \geq 1$ be an integer and D be a measurable subset of \mathbb{R}^n . Throughout the paper we shall denote by:

- $|\cdot|$ the Euclidean norm in \mathbb{R}^n ;
- $\langle \cdot, \cdot \rangle$ the Euclidean inner product in \mathbb{R}^n ;
- $\text{int}(D)$ the interior of D ;
- ∂D the boundary of D ;
- $\text{Vol}(D)$ the Lebesgue measure of a measurable set $D \subset \mathbb{R}^n$;
- $\mathbf{L}^1(D, \mathbb{R})$ the Lebesgue space of all (equivalence classes of) summable real functions on D , equipped with the usual norm $\|\cdot\|_{\mathbf{L}^1(D)}$;
- $\mathbf{L}^\infty(D, \mathbb{R})$ the space of all essentially bounded real functions on D , and by $\|u\|_{\mathbf{L}^\infty(D)}$ the essential supremum of a function $u \in \mathbf{L}^\infty(D, \mathbb{R})$;
- $\mathcal{C}_c^1(\Omega, \mathbb{R}^n)$, with $\Omega \subset \mathbb{R}^n$ an open set, the set of all continuous differentiable functions from Ω to \mathbb{R}^n with a compact support in Ω ;
- $\chi_D(x) = \begin{cases} 1 & \text{if } x \in D, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus D \end{cases}$ the characteristic function of a subset D of \mathbb{R}^n .
- $\text{Card}(S)$ the number of elements of any finite set S ;
- $[x] \doteq a \doteq \max\{z \in \mathbb{Z} \mid z \leq x\}$ denotes the integer part of x .

We now introduce the concept of functions of bounded variations.

Definition 2.1 *The function $u \in \mathbf{L}^1(\Omega, \mathbb{R})$ is a function of bounded variation on Ω (denoted by $BV(\Omega, \mathbb{R})$) if the distributional derivative of u is representable by a finite Radon measure in Ω , i.e., if*

$$\int_{\Omega} u \cdot \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} \varphi dD_i u \quad \text{for all } \varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}), i \in \{1, 2, \dots, n\}$$

for some Radon measure $Du = (D_1u, D_2u, \dots, D_nu)$. We denote by $|Du|$ the total variation of the vector measure Du , i.e.,

$$|Du|(\Omega) = \sup \left\{ \int_{\Omega} u(x) \text{div}(\phi) \mid \phi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^n), \|\phi\|_{\mathbf{L}^\infty(\Omega)} \leq 1 \right\}.$$

Let's recall a Poincaré-type inequality for bounded total variation functions on convex domain that will be used in the paper. This result is based on [1, theorem 3.2] and on [2, Proposition 3.2.1, Theorem 3.44].

Theorem 2.2 (*Poincaré inequality*) *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, convex set with Lipschitz boundary. For any $u \in BV(\Omega, \mathbb{R})$, it holds*

$$\int_{\Omega} |u(x) - u_{\Omega}| \, dx \leq \frac{\text{diam}(\Omega)}{2} \cdot |Du|(\Omega)$$

where

$$u_{\Omega} = \frac{1}{\text{Vol}(\Omega)} \cdot \int_{\Omega} u(x) \, dx$$

is the mean value of u over Ω .

To complete this section, we will state a result on the ε -entropy for a class of bounded total variation functions in the scalar case using a method similar to the one provided in [8]. Given $L, V, M > 0$, denote by

$$\mathcal{B}_{[L,M,V]} = \left\{ f \in \mathbf{L}^1([0, L], [0, M]) \mid |Df|((0, L)) \leq V \right\}. \quad (2.3)$$

Lemma 2.3 *For all $0 < \varepsilon < \frac{L(M+V)}{6}$, it holds*

$$\mathcal{H}_{\varepsilon}(\mathcal{B}_{[L,M,V]} \mid \mathbf{L}^1([0, L])) \leq 8 \cdot \left\lceil \frac{L(M+V)}{\varepsilon} \right\rceil. \quad (2.4)$$

Proof. For any $f \in \mathcal{B}_{[L,M,V]}$, let $V_f(x)$ be the total variation of f over $[0, x]$. We decompose

$$f(x) = f^+(x) - f^-(x) \quad \text{for all } x \in [0, L].$$

where $f^- = \frac{V_f^-}{2} + \frac{M}{2}$ is a non-decreasing function $[0, L]$ to $[0, \frac{L+M}{2}]$ and $f^+ = \frac{V_f^+}{2} + \frac{M}{2}$ is a nondecreasing function $[0, L]$ to $[\frac{M}{2}, \frac{L+2M}{2}]$. Denote by

$$\mathcal{I} := \left\{ g : [0, L] \rightarrow \left[0, \frac{V+M}{2}\right] \mid g \text{ is nondecreasing} \right\},$$

we then have

$$\mathcal{B}_{[L,M,V]} \subseteq \left(\mathcal{I} + \frac{M}{2} \right) - \mathcal{I} \doteq \left\{ g - h \mid g \in \mathcal{I} + \frac{M}{2} \text{ and } h \in \mathcal{I} \right\}. \quad (2.5)$$

For any $\varepsilon > 0$, it holds

$$\mathcal{N}_{\varepsilon}(\mathcal{B}_{[L,M,V]} \mid \mathbf{L}^1([0, L])) \leq \left[\mathcal{N}_{\frac{\varepsilon}{2}}(\mathcal{I} \mid \mathbf{L}^1([0, L])) \right]^2.$$

Indeed, from the definition 1.1, there exists a set $\mathcal{G}_{\frac{\varepsilon}{2}}$ of $\mathcal{N}_{\frac{\varepsilon}{2}}(\mathcal{I} \mid \mathbf{L}^1([0, L]))$ subsets of $\mathbf{L}^1([0, L])$ such that

$$\mathcal{I} \subseteq \bigcup_{\mathcal{E} \in \mathcal{G}_{\frac{\varepsilon}{2}}} \mathcal{E} \quad \text{and} \quad \text{diam}(\mathcal{E}) = \sup_{h_1, h_2 \in \mathcal{E}} \|h_1 - h_2\|_{\mathbf{L}^1([0, L])} \leq \varepsilon.$$

Thus, (2.5) implies

$$\mathcal{B}_{[L,M,V]} \subseteq \bigcup_{(\mathcal{E}_1, \mathcal{E}_2) \in \mathcal{G}_{\frac{\varepsilon}{2}} \times \mathcal{G}_{\frac{\varepsilon}{2}}} \left[\left(\mathcal{E}_1 + \frac{M}{2} \right) - \mathcal{E}_2 \right].$$

For any two functions

$$f_i = g_i - h_i \in \left(\mathcal{E}_1 + \frac{M}{2} \right) - \mathcal{E}_2 \quad \text{for } i = 1, 2,$$

we have

$$\begin{aligned} \|f_1 - f_2\|_{\mathbf{L}^1([0,L])} &\leq \|g_1 - g_2\|_{\mathbf{L}^1([0,L])} + \|h_1 - h_2\|_{\mathbf{L}^1([0,L])} \\ &\leq \text{diam} \left(\mathcal{E}_1 + \frac{M}{2} \right) + \text{diam}(\mathcal{E}_2) \leq \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

and this implies that

$$\text{diam} \left[\left(\mathcal{E}_1 + \frac{M}{2} \right) - \mathcal{E}_2 \right] \leq 2\varepsilon.$$

By the definition 1.1, we have

$$\mathcal{N}_\varepsilon \left(\mathcal{B}_{[L,M,V]} \mid \mathbf{L}^1([0,L]) \right) \leq \mathcal{N}_{\frac{\varepsilon}{2}}^2(\mathcal{I} \mid \mathbf{L}^1([0,L])).$$

and thus

$$\mathcal{H}_\varepsilon \left(\mathcal{B}_{[L,M,V]} \mid \mathbf{L}^1([0,L]) \right) \leq 2 \cdot \mathcal{H}_{\frac{\varepsilon}{2}} \left(\mathcal{I} \mid \mathbf{L}^1([0,L]) \right). \quad (2.6)$$

Finally, applying [15, Lemma 3.1] for \mathcal{I} , we obtain that for $0 < \varepsilon < \frac{L(M+V)}{6}$, it holds

$$\mathcal{H}_{\frac{\varepsilon}{2}} \left(\mathcal{I} \mid \mathbf{L}^1([0,L]) \right) \leq 4 \cdot \left\lfloor \frac{L(M+V)}{\varepsilon} \right\rfloor,$$

and (2.6) yields (2.4). \square

3 Estimates of the ε -entropy for a class of BV functions

In this section, we establish upper and lower estimates of the ε -entropy for a class of uniformly bounded total variation functions,

$$\mathcal{F}_{[L,M,V]} = \left\{ u \in \mathbf{L}^1([0,L]^n, \mathbb{R}) \mid \|u\|_{\mathbf{L}^\infty([0,L]^n)} \leq M, |Du|((0,L)^n) \leq V \right\}, \quad (3.7)$$

in the $\mathbf{L}^1([0,L]^n, \mathbb{R})$ -space. In particular, it is shown that the minimal number of functions needed to represent a function in $\mathcal{F}_{[L,M,V]}$ with an error ε with respect to \mathbf{L}^1 -distance is of the order of $\frac{1}{\varepsilon^n}$. More precisely, our main result is stated as the following.

Theorem 3.1 *Given $L, M, V > 0$, for every $0 < \varepsilon < \frac{ML^n}{8}$, it holds*

$$\frac{\log_2(e)}{8} \cdot \left\lfloor \frac{VL}{2^{n+2}\varepsilon} \right\rfloor^n \leq \mathcal{H}_\varepsilon \left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0,L]^n) \right) \leq \Gamma_{[n,L,M,V]} \cdot \frac{1}{\varepsilon^n} \quad (3.8)$$

where the constant $\Gamma_{[n,L,M,V]}$ is computed as

$$\Gamma_{[n,L,M,V]} = \frac{8}{\sqrt{n}} (4\sqrt{n}LV)^n + \left(\frac{2^{n+7}V}{M} + 8 \right) \cdot \left(\frac{ML^n}{8} \right)^n.$$

Proof. (*Upper estimate*) Let's first prove the upper-estimate of $\mathcal{H}_\varepsilon \left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0, L]^n) \right)$. The proof is divided into several steps:

1. For any $N \in \mathbb{N}$, we divide the square $[0, L]^n$ into N^n small squares \square_ι for $\iota = (\iota_1, \iota_2, \dots, \iota_n) \in \{0, 1, \dots, N-1\}^n$ such that

$$\square_\iota = \frac{\iota L}{N} + \left(\left[0, \frac{L}{N} \right] \times \left[0, \frac{L}{N} \right] \times \dots \times \left[0, \frac{L}{N} \right] \right) \quad \text{and} \quad \bigcup_{\iota \in \{0,1,2,\dots,N-1\}^n} \square_\iota = [0, L]^n.$$

For any $u \in \mathcal{F}_{[L,M,V]}$, denote by

$$-M \leq u_\iota = \frac{1}{\text{Vol}(\square_\iota)} \int_{\square_\iota} u(x) dx \leq M$$

the average value of u in \square_ι for every $\iota \in \{0, 1, 2, \dots, N-1\}^n$. Let \tilde{u} be a piecewise constant function on $[0, L]^n$ such that

$$\tilde{u}(x) = \begin{cases} u_\iota & \text{for all } x \in \text{int}(\square_\iota), \\ 0 & \text{for all } x \in \bigcup_{\iota \in \{1,2,\dots,N-1\}^n} \partial \square_\iota. \end{cases}$$

Thanks to the Poincaré inequality, we have

$$\int_{\square_\iota} |u(x) - u_\iota| dx \leq \frac{\text{diam}(\square_\iota)}{2} \cdot |Du|(\text{int}(\square_\iota))$$

for all $\iota \in \{0, 1, 2, \dots, N-1\}^n$. Hence, the \mathbf{L}^1 -distance between u and \tilde{u} can be estimated as follows

$$\begin{aligned} \|u - \tilde{u}\|_{\mathbf{L}^1([0,L]^n)} &= \int_{[0,L]^n} |u(x) - \tilde{u}(x)| dx = \sum_{\iota \in \{0,1,2,\dots,N-1\}^n} \int_{\square_\iota} |u(x) - u_\iota| dx \\ &\leq \sum_{\iota \in \{0,1,2,\dots,N-1\}^n} \left(\frac{\text{diam}(\text{int}(\square_\iota))}{2} \cdot |Du|(\text{int}(\square_\iota)) \right) \leq \frac{L\sqrt{n}}{N} \sum_{\iota \in \{0,1,2,\dots,N-1\}^n} |Du|(\square_\iota) \\ &= \frac{L\sqrt{n}}{N} |Du|((0, L)^n) \leq \frac{L\sqrt{n}}{N} \cdot V. \quad (3.9) \end{aligned}$$

2. Let e_1, e_2, \dots, e_n be the standard basis of \mathbb{R}^n where e_i denotes the vector with a 1 in the i -th coordinate and 0's elsewhere. For any $\iota \in \{0, 1, 2, \dots, N-1\}^n$ and $j \in \{1, 2, \dots, n\}$, we estimate $|u_{\iota+e_j} - u_\iota|$ in the following way:

$$\begin{aligned} |u_{\iota+e_j} - u_\iota| &= \left| \frac{1}{\text{Vol}(\square_{\iota+e_j})} \int_{\square_{\iota+e_j}} u(x) dx - \frac{1}{\text{Vol}(\square_\iota)} \int_{\square_\iota} u(x) dx \right| \\ &= \frac{1}{\text{Vol}(\square_\iota)} \cdot \left| \int_{\square_\iota} u\left(x + \frac{L}{N} \cdot e_j\right) - u(x) dx \right| = \frac{1}{\text{Vol}(\square_\iota)} \cdot \left| \int_{\square_\iota} \int_0^{\frac{L}{N}} Du(x + se_j)(e_j) ds dx \right| \\ &\leq \frac{1}{\text{Vol}(\square_\iota)} \cdot \int_0^{\frac{L}{N}} \left| \int_{\square_\iota} Du(x + se_j)(e_j) dx \right| ds \leq \left(\frac{N}{L} \right)^{n-1} \cdot |Du|(\text{int}(\square_\iota \cup \square_{\iota+e_j})). \end{aligned} \quad (3.10)$$

Let us rearrange the index set

$$\{0, 1, 2, \dots, N-1\}^n = \{\kappa^1, \kappa^2, \dots, \kappa^{N^n}\}$$

in the way such that for all $j \in \{1, \dots, N^n - 1\}$, it holds

$$\kappa^{j+1} = \kappa^j + e_k \quad \text{for some } k \in \{1, 2, \dots, n\}.$$

From (3.10) and (3.7), we have

$$\begin{aligned} \sum_{j=1}^{N^n} |u_{\kappa^{j+1}} - u_{\kappa^j}| &\leq \left(\frac{N}{L}\right)^{n-1} \cdot \sum_{j=1}^{N^n} |Du|(\text{int}(\square_{\kappa^j} \cup \square_{\kappa^{j+1}})) \\ &\leq 2 \left(\frac{N}{L}\right)^{n-1} \cdot |Du|((0, L)^n) \leq 2V \left(\frac{N}{L}\right)^{n-1}. \end{aligned} \quad (3.11)$$

To conclude this step, we define the function $f_{u,N} : [0, LN^{n-1}] \rightarrow [-M, M]$ associated to u such that

$$f_{u,N}(x) = u_{\kappa^i} \quad \text{for all } x \in \left[\frac{i \cdot L}{N}, \frac{(i+1) \cdot L}{N}\right), i \in \{1, 2, \dots, N^n - 1\}.$$

Recalling (3.11), we have

$$|Df_{u,N}|((0, LN^{n-1})) \leq 2V \left(\frac{N}{L}\right)^{n-1}. \quad (3.12)$$

3. Let's define

$$L_N := L \cdot N^{n-1}, \quad \beta_N := 2V \left(\frac{N}{L}\right)^{n-1}. \quad (3.13)$$

We introduce the set

$$\begin{aligned} \tilde{\mathcal{F}}_N = \left\{ f : [0, L_N] \rightarrow [-M, M] \mid |Df|((0, L_N)) \leq \beta_N \text{ and} \right. \\ \left. f(x) = f\left(\frac{i \cdot L}{N}\right) \text{ for all } x \in \left[\frac{i \cdot L}{N}, \frac{(i+1) \cdot L}{N}\right) \right\}. \end{aligned}$$

From (3.12), one has

$$f_{u,N} \in \tilde{\mathcal{F}}_N \quad \text{for all } u \in \mathcal{F}_{[L,M,V]}.$$

On the other hand, recalling that

$$\mathcal{B}_{[L_N, 2M, \beta_N]} = \left\{ f \in \mathbf{L}^1([0, L_N], [0, 2M]) \mid |Df|((0, L_N)) \leq \beta_N \right\},$$

we have

$$\tilde{\mathcal{F}}_N \subset \mathcal{B}_{[L_N, 2M, \beta_N]} - M.$$

From Lemma 2.3, for every $0 < \varepsilon' < \frac{L_N(\beta_N + 2M)}{6}$, it holds

$$\mathcal{H}_{\varepsilon'} \left(\mathcal{B}_{[L_N, 2M, \beta_N]} \mid \mathbf{L}^1([0, L_N]) \right) \leq 8 \cdot \left\lfloor \frac{L_N(\beta_N + 2M)}{\varepsilon'} \right\rfloor,$$

and it yields

$$\mathcal{H}_{\varepsilon'} \left(\tilde{F}_N \mid \mathbf{L}^1([0, L_N]) \right) \leq 8 \cdot \left\lfloor \frac{L_N(\beta_N + 2M)}{\varepsilon'} \right\rfloor.$$

By the definition 1.1, there exists a set of $\Gamma_{N,\varepsilon'} = 2^{8 \cdot \left\lfloor \frac{L_N(\beta_N + 2M)}{\varepsilon'} \right\rfloor}$ functions in \tilde{F}_N ,

$$\mathcal{G}_{N,\varepsilon'} = \left\{ g_1, g_2, \dots, g_{\Gamma_{N,\varepsilon'}} \right\} \subset \tilde{F}_N,$$

such that

$$\tilde{F}_N \subset \bigcup_{i=1}^{\Gamma_{N,\varepsilon'}} B(g_i, 2\varepsilon').$$

So for every $u \in \mathcal{F}_{[L,M,V]}$, for its corresponding $f_{u,N}$, $\exists g_{i_u} \in \mathcal{G}_{N,\varepsilon'}$ such that

$$\|f_{u,N} - g_{i_u}\|_{\mathbf{L}^1([0,L_N])} \leq 2\varepsilon'.$$

Let $\mathcal{U}_{N,\varepsilon'}$ be a set of $\Gamma_{N,\varepsilon'}$ functions $u_j^\dagger : [0, L]^N \rightarrow [-M, M]$ defined as follows

$$u_j^\dagger = \begin{cases} 0 & \text{if } x \in \bigcup_{\iota \in \{1,2,\dots,N\}^n} \partial \square_\iota, \\ g_j \left(\frac{i \cdot L}{N} \right) & \text{if } x \in \text{int}(\square_{\kappa^i}), i \in \{1, 2, \dots, N^n\}. \end{cases}$$

Then corresponding to every $u \in \mathcal{F}_{[L,M,V]}$, there exists $u_{i_u}^\dagger \in \mathcal{U}_{N,\varepsilon'}$ for some $i_u \in \{1, 2, \dots, \Gamma_{N,\varepsilon'}\}$ such that

$$\begin{aligned} \|\tilde{u} - u_{i_u}^\dagger\|_{\mathbf{L}^1([0,L]^n)} &= \sum_{i=1}^{N^n} \left| u_{\kappa^i} - g_{i_u} \left(\frac{i \cdot L}{N} \right) \right| \cdot \text{Vol}(\square_{\kappa^i}) \\ &= \sum_{i=1}^{N^n} \left| f_{u,N} \left(\frac{i \cdot L}{N} \right) - g_{i_u} \left(\frac{i \cdot L}{N} \right) \right| \cdot \frac{L}{N} \cdot \frac{L^{n-1}}{N^{n-1}} \\ &= \frac{L^{n-1}}{N^{n-1}} \cdot \|f_{u,N} - g_{i_u}\|_{\mathbf{L}^1([0,L_N])} \leq 2\varepsilon' \cdot \frac{L^{n-1}}{N^{n-1}}. \end{aligned}$$

Combining with (3.9), we obtain

$$\|u - u_{i_u}^\dagger\|_{\mathbf{L}^1([0,L]^n)} \leq \|u - \tilde{u}\|_{\mathbf{L}^1([0,L]^n)} + \|\tilde{u} - u_{i_u}^\dagger\|_{\mathbf{L}^1([0,L]^n)} \leq 2\varepsilon' \cdot \frac{L^{n-1}}{N^{n-1}} + \frac{L\sqrt{n}}{N} \cdot V. \quad (3.14)$$

4. For any $\varepsilon > 0$, we choose

$$N = \left\lfloor \frac{2\sqrt{n}LV}{\varepsilon} \right\rfloor + 1 \quad \text{and} \quad \varepsilon' = \frac{N^{n-1} \cdot \varepsilon}{4L^{n-1}} \quad (3.15)$$

such that

$$\|u - u^\dagger\|_{\mathbf{L}^1([0,L]^n)} \leq 2\varepsilon' \cdot \frac{L^{n-1}}{N^{n-1}} + \frac{L\sqrt{n}}{N} \cdot V \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $u \in \mathcal{F}_{[L,M,V]}$ and for some $u^\dagger \in \mathcal{U}_{N,\varepsilon'}$. From the previous step, it holds

$$\mathcal{F}_{[L,M,V]} \subset \bigcup_{u^\dagger \in \mathcal{U}_{N,\varepsilon'}} \bar{B}(u^\dagger, \varepsilon)$$

provided that

$$\varepsilon' = \frac{N^{n-1}\varepsilon}{4L^{n-1}} \leq \frac{L_N \cdot (\beta_N + 2M)}{6} = \frac{N^{n-1}(VN^{n-1} + ML^{n-1})}{3L^{n-2}}. \quad (3.16)$$

This condition is equivalent to

$$\varepsilon \leq \frac{4}{3} \cdot (LVN^{n-1} + ML^n)$$

From (3.15), one has that the condition (3.16) holds if

$$\varepsilon \leq \frac{4}{3} \cdot \left(\frac{2^{n-1}n^{\frac{n-1}{2}}L^nV^n}{\varepsilon^{n-1}} + ML^n \right). \quad (3.17)$$

Assume that $0 < \varepsilon < \frac{2ML^n}{3} + n^{\frac{n-1}{2n}}LV$, we claim that (3.16) holds. Indeed, if $\frac{2ML^n}{3} > n^{\frac{n-1}{2n}}LV$ then

$$\varepsilon < \frac{2ML^n}{3} + n^{\frac{n-1}{2n}}LV \leq \frac{4ML^n}{3}$$

and it yields (3.17). Otherwise, we have that $\varepsilon < \frac{2ML^n}{3} + n^{\frac{n-1}{2n}}LV \leq 2n^{\frac{n-1}{2n}}LV$. Thus

$$\begin{aligned} \frac{4}{3} \cdot \left(\frac{2^{n-1}n^{\frac{n-1}{2}}L^nV^n}{\varepsilon^{n-1}} + ML^n \right) &\geq \frac{4}{3} \cdot \frac{2^{n-1}n^{\frac{n-1}{2}}L^nV^n}{2^{n-1}n^{\frac{(n-1)^2}{2n}}L^{n-1}V^{n-1}} + \frac{4}{3}ML^n \\ &= \frac{4}{3} \cdot n^{\frac{n-1}{2n}}LV + \frac{4}{3}ML^n. \end{aligned}$$

and this implies (3.17).

To complete the proof, recalling (3.13) and (3.15), we estimate

$$\begin{aligned} \text{card}(\mathcal{U}_{N,\varepsilon'}) &= \Gamma_{N,\varepsilon'} = 2^{8 \cdot \lfloor \frac{L_N(\beta_N + 2M)}{\varepsilon'} \rfloor} = 2^{8 \cdot \lfloor \frac{8}{\varepsilon} \cdot (LVN^{n-1} + ML^n) \rfloor} \\ &\leq 2^{\frac{64}{\varepsilon} \cdot (LV(\lfloor \frac{2\sqrt{n}LV}{\varepsilon} \rfloor + 1)^{n-1} + ML^n)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{H}_\varepsilon \left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0, L]^n) \right) &\leq \frac{64}{\varepsilon} \cdot \left(LV \left(\left\lfloor \frac{2\sqrt{n}LV}{\varepsilon} \right\rfloor + 1 \right)^{n-1} + ML^n \right) \\ &\leq \frac{64}{\varepsilon} \cdot \left(LV \left(\frac{2^{2n-3}n^{\frac{n-1}{2}}L^{n-1}V^{n-1}}{\varepsilon^{n-1}} + 2^{n-2} \right) + ML^n \right) \\ &= \frac{2^{2n+3}n^{\frac{n-1}{2}}L^nV^n}{\varepsilon^n} + \frac{2^{n+4}LV + ML^n}{\varepsilon}. \end{aligned}$$

In particular, if $0 < \varepsilon < \frac{ML^n}{8}$ then

$$\mathcal{H}_\varepsilon \left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0, L]^n) \right) \leq \left[2^{2n+3}n^{\frac{n-1}{2}}L^nV^n + (2^{n+4}LV + ML^n) \cdot \left(\frac{ML^n}{8} \right)^{n-1} \right] \cdot \frac{1}{\varepsilon^n}$$

and it yields the right hand side of (3.8).

(Lower estimate) We are now going to prove the lower estimate of $\mathcal{H}_\varepsilon \left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0, L]^n) \right)$.

1. Again given any $N \in \mathbb{N}$, we divide the square $[0, L]^n$ into N^n small squares \square_ι for $\iota = (\iota_1, \iota_2, \dots, \iota_n) \in \{0, 1, \dots, N-1\}^n$ such that

$$\square_\iota = \frac{\iota L}{N} + \left(\left[0, \frac{L}{N}\right] \times \left[0, \frac{L}{N}\right] \times \dots \times \left[0, \frac{L}{N}\right] \right) \quad \text{and} \quad \bigcup_{\iota \in \{0,1,2,\dots,N-1\}^n} \square_\iota = [0, L]^n.$$

Consider the set of N^n -tuples

$$\Delta_N = \left\{ \delta = (\delta_\iota)_{\iota \in \{0,1,\dots,N-1\}^n} \mid \delta_\iota \in \{0, 1\} \right\}.$$

Given any $h > 0$, for any $\delta \in \Delta_N$, define the function $u_\delta : [0, L]^n \rightarrow \{0, h\}$ such that

$$u_\delta(x) = \sum_{\iota \in \{0,1,\dots,N-1\}^n} h \delta_\iota \cdot \chi_{\text{int}(\square_\iota)}(x) \quad \text{for all } x \in [0, L]^n.$$

One has $u_\delta \in BV((0, L)^n)$ and

$$|Du_\delta|((0, L)^n) \leq \sum_{\iota \in \{0,1,\dots,N-1\}^n} |Du_\delta|(\square_\iota) \leq 2^{n-1} \left(\frac{L}{N} \right)^{n-1} N^n h = (2L)^{n-1} N h.$$

Assume that

$$0 < h \leq \min \left\{ M, \frac{V}{2^{n-1} L^{n-1} N} \right\}, \quad (3.18)$$

we have

$$|Du_\delta|((0, L)^n) \leq (2L)^{n-1} N \cdot \frac{V}{2^{n-1} L^{n-1} N} = V \quad \text{for all } \delta \in \Delta_N,$$

and this implies that

$$\mathcal{G}_{h,N} := \{u_\delta \mid \delta \in \Delta_N\} \subset \mathcal{F}_{[L,M,V]} \quad \text{for all } N \in \mathbb{N}.$$

Hence,

$$\mathcal{N}_\varepsilon \left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0, L]^n) \right) \geq \mathcal{N}_\varepsilon \left(\mathcal{G}_{h,N} \mid \mathbf{L}^1([0, L]^n) \right) \quad \text{for all } \varepsilon > 0. \quad (3.19)$$

Towards an estimate of the covering number $\mathcal{N}_\varepsilon \left(\mathcal{G}_{h,N} \mid \mathbf{L}^1([0, L]^n) \right)$, fixed $\tilde{\delta} \in \Delta_N$, we can define

$$\mathcal{I}_{\tilde{\delta},N}(2\varepsilon) = \left\{ \delta \in \Delta_N \mid \|u_\delta - u_{\tilde{\delta}}\|_{\mathbf{L}^1([0,L]^n)} \leq 2\varepsilon \right\} \quad \text{and} \quad C_N(2\varepsilon) = \text{Card}(\mathcal{I}_{\tilde{\delta},N}(2\varepsilon)) \quad (3.20)$$

since the cardinality of the set $\mathcal{I}_{\tilde{\delta},N}(\varepsilon)$ is independent of the choice $\tilde{\delta} \in \Delta_N$. Observe that for any element of a ε -cover in \mathbf{L}^1 of $\mathcal{G}_{h,N}$ contains at most $C_N(2\varepsilon)$. Since $\text{Card}(\mathcal{G}_{h,N}) = \text{Card}(\Delta_N) = 2^{N^n}$, it holds

$$\mathcal{N}_\varepsilon \left(\mathcal{G}_{h,N} \mid \mathbf{L}^1([0, L]^n) \right) \geq \frac{2^{N^n}}{C_N(2\varepsilon)}. \quad (3.21)$$

2. We now provide an upper bound on $C_N(2\varepsilon)$. For any given pair $\delta, \tilde{\delta} \in \Delta_N$, one has

$$\|u_\delta - u_{\tilde{\delta}}\|_{\mathbf{L}^1([0, L]^n)} = \sum_{\iota \in \{0, 1, \dots, N\}^n} \|u_\delta - u_{\tilde{\delta}}\|_{\mathbf{L}^1(\square_\iota)} = d(\delta, \tilde{\delta}) \cdot \frac{hL^n}{N^n}.$$

where

$$d(\delta, \tilde{\delta}) := \text{Card} \left(\{ \iota \in \{0, 1, \dots, N-1\}^n \mid \delta_\iota \neq \tilde{\delta}_\iota \} \right).$$

From (3.20), we obtain

$$\mathcal{I}_{\tilde{\delta}, N}(2\varepsilon) = \left\{ \delta \in \Delta_N \mid d(\delta, \tilde{\delta}) \leq \frac{2\varepsilon N^n}{hL^n} \right\},$$

and it yields

$$C_N(2\varepsilon) = \text{Card} \left(\mathcal{I}_{\tilde{\delta}, N}(2\varepsilon) \right) \leq \sum_{r=0}^{\lfloor \frac{2\varepsilon N^n}{hL^n} \rfloor} \binom{N^n}{r}.$$

To estimate the last term in the above inequality, let's consider N^n independent random variables with uniform Bernoulli distribution X_1, X_2, \dots, X_{N^n}

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 0) = \frac{1}{2} \quad \text{for all } i \in \{1, 2, \dots, N^n\}.$$

Set $S_{N^n} := X_1 + X_2 + \dots + X_{N^n}$. Observe that for any $k \leq N^n$, we have

$$\sum_{r=1}^k \binom{N^n}{r} = 2^{N^n} \cdot \mathbb{P}(S_{N^n} \leq k).$$

Thanks to Hoeffding's inequality [21, Theorem], for all $\mu \leq \frac{N^n}{2}$, one has

$$\mathbb{P}(S_{N^n} \leq \mathbb{E}[S_{N^n}] - \mu) = \mathbb{P}\left(S_{N^n} \leq \frac{N^n}{2} - \mu\right) \leq \exp\left(-\frac{2\mu^2}{N^n}\right)$$

where $\mathbb{E}[S_{N^n}]$ is the expectation of S_{N^n} . Hence, for every $0 < \varepsilon \leq \frac{hL^n}{8}$ such that $\frac{2\varepsilon N^n}{hL^n} \leq \frac{N^n}{2}$ and $\frac{4\varepsilon}{hL^n} \leq \frac{1}{2}$, it holds

$$\begin{aligned} C_N(2\varepsilon) &\leq \sum_{r=0}^{\lfloor \frac{2\varepsilon N^n}{hL^n} \rfloor} \binom{N^n}{r} = 2^{N^n} \cdot \mathbb{P}\left(S_{N^n} \leq \left\lfloor \frac{2\varepsilon N^n}{hL^n} \right\rfloor\right) \\ &\leq 2^{N^n} \cdot \exp\left(-\frac{2\left(\frac{N^n}{2} - \left\lfloor \frac{2\varepsilon N^n}{hL^n} \right\rfloor\right)^2}{N^n}\right) \leq 2^{N^n} \cdot \exp\left(-\frac{\left(N^n - \frac{4\varepsilon N^n}{hL^n}\right)^2}{2N^n}\right) \\ &= 2^{N^n} \cdot \exp\left(-N^n \cdot \frac{\left(1 - \frac{4\varepsilon}{hL^n}\right)^2}{2}\right) \leq 2^{N^n} \cdot e^{-N^n/8}. \end{aligned}$$

From (3.21) and (3.18), the following holds

$$\mathcal{N}_\varepsilon \left(\mathcal{G}_{h, N} \mid \mathbf{L}^1([0, L]^n) \right) \geq \frac{2^{N^n}}{C_N(2\varepsilon)} \geq e^{\frac{N^n}{8}}$$

provided that

$$0 < h \leq \min \left\{ M, \frac{V}{2^{n-1}L^{n-1}N} \right\} \quad \text{and} \quad 0 < \varepsilon \leq \frac{hL^n}{8}. \quad (3.22)$$

Therefore, for every $0 < \varepsilon < \frac{ML^n}{8}$, by choosing

$$h = \min \left\{ M, \frac{V}{2^{n-1}L^{n-1}N} \right\} \quad \text{and} \quad N \doteq \left\lfloor \frac{VL}{2^{n+2\varepsilon}} \right\rfloor$$

such that (3.22) holds, we obtain that

$$\mathcal{N}_\varepsilon \left(\mathcal{G}_{h,N} \mid \mathbf{L}^1([0, L]^n) \right) \geq \exp \left(\frac{1}{8} \cdot \left\lfloor \frac{VL}{2^{n+2\varepsilon}} \right\rfloor^n \right).$$

Recalling (3.19), we have

$$\mathcal{N}_\varepsilon \left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0, L]^n) \right) \geq \exp \left(\frac{1}{8} \cdot \left\lfloor \frac{VL}{2^{n+2\varepsilon}} \right\rfloor^n \right)$$

and this implies the first inequality in (3.8). \square

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