Hypographs satisfying an external sphere condition and the regularity of the minimum time function *

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Abstract

We prove that if the hypograph of a continuous function f admits at every boundary point a supporting ball then it has ”essentially” positive reach, i.e. the hypograph of the restriction of f outside a closed set of zero measure has (locally) positive reach. Hence such a function enjoys some properties of a concave function, in particular a.e. twice differentiability. We apply this result to a minimum time problem in the case of a nonlinear smooth dynamics and a target satisfying internal sphere condition.

Keywords and phrases: normal vectors, φ-convex (prox-regular, positive reach) sets, internal/external sphere condition.

1 Introduction

The concept of supporting hyperplane is central in Convex Analysis and entails the strong and global regularity properties which are enjoyed by both convex sets and functions. The idea of substituting supporting hyperplanes with supporting spheres was introduced by Federer, in its seminal paper [9], where *sets with positive reach* are introduced and studied. This class of sets was also analyzed independently by several other authors (including Canino [2], Clarke, Stern and Wolenski [3], Poliquin, Rockafellar and Thibaut [12]) under different names, for example *φ-convex* [2], proximally smooth sets [3], and prox-regular sets [12]. One of the main motivations for studying this class of sets is that both convex sets and sets with a $C^{1,1}$ boundary have positive reach.

In [5], G.Colombo and A.Marigonda proved that functions whose hypograph/epigraph has positive reach still enjoy some regularity properties of semi concave/semi convex functions, including twice a.e differentiability, yet not being locally Lipschitz (see Theorem 2.1 below). Moreover, sets with positive reach play an important role for studying the regularity of the minimum time function under weak controllability conditions (i.e., the minimum time function is just continuous). For instance, the minimum time function in

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the case of a linear dynamics and a convex target has epigraph with positive reach (see [7]).

A positive reach set is characterized by a *strong* external sphere condition: at each point on the boundary, *every* proximal normal vector is realized by a ball with locally uniform radius. Since verifying this property is often demanding, finding easy-to-check sufficient conditions for positive reach appears of some interest. In [10], a class of sets which are characterized by a *weak* external sphere condition (at each point on the boundary, there exists *one* proximal normal vector realized by a locally uniform ball) is considered. The authors proved that if a set satisfies this condition and is *wedged* (this concept was introduced by Rockafellar in [11]) then it has positive reach. Wedgedness of a set C is equivalent to the pointedness of the Clarke normal cone to C, i.e. the normal cone does not contain lines (see [4] and [14]). In the recent paper [8], the pointedness assumption for the normal cone to the hypograph of a minimum time function T appears pivotal for computing generalized gradients of T. More precisely, under suitable regularity conditions on the dynamics and on the target, the *proximal supergradient* and the *proximal horizon supergradient* are computed, and the hypograph of T is shown to have positive reach.

Several counterexamples (see. e.g, [10]), though, show that the external sphere condition is in general weaker than positive reach. In particular, in example 2 in [8] a minimum time function whose hypograph satisfies an external sphere condition but has not positive reach everywhere is constructed. Therefore, the problem of understanding whether some convexity features are presented under the weak external sphere condition appears natural. In this paper an answer to this question is provided. Our main result reads -essentially- as follows

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that the hypograph of f satisfies the weak external sphere condition. Then there exists a closed set Γ with zero Lebesgue measure such that the hypograph of the restricted function $f_{\Omega\setminus\Gamma}$ has positive reach.*

Consequently, a function satisfying the assumption of the above theorem admits a second order Taylor expansion around a.e point of its domain, and enjoys several regularity properties inherited by functions whose hypograph has positive reach.

This work was actually motivated by removing the pointedness assumption on the hypograph of the minimum time function T in [8] and so proving regularity properties of T. Indeed, the Corollary 3.1 below is a generalization of Theorem 3.3 in [8] without pointedness assumption.

The paper is organized as follows: §2 is devoted to definitions and basic facts, while §3 contains statements of main results, together with their application. The same section contains also an outline of the proof of Theorem 3.1, which is a localized version of the main result and where all the basic arguments appear. Detailed arguments begin in §4, which contains several lemmas concerning the set of *bad points* (i.e., the normal cone to the hypograph of the function at those points contains at least one line). Section 5 is devoted to proof of Theorem 3.1. On the base of Theorem 3.1, our main theorem will be proved in §6 together with its corollaries. Finally, section 7 gives a general lemma related to pointed cones and two lemmas about restricted functions.

In what follows, sets with positive reach will be denoted by *φ-convex* sets and the *weak* external sphere condition will be simply denoted the external sphere condition.

2 Preliminaries

2.1 Nonsmooth analysis

Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. The hypograph of f is denoted by

$$\text{hypo}(f) \;=\; \{(x,\beta) \mid x \in \Omega, \beta \leq f(x)\}. \tag{2.1}$$

The vector $(-v,\lambda) \in \mathbb{R}^N \times \mathbb{R}$ is a *proximal normal vector* to hypo(f) (we will denote this fact that $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x)))$ at $(x, f(x))$ iff there exists a constant $\sigma > 0$ such that for all $y \in \Omega$ and for all $\beta \leq f(y)$, it holds

$$\langle(-v,\lambda)\ ,\ (y,\beta)-(x,f(x))\rangle \;\leq\; \sigma\ (\|y-x\|^2 + |\beta - f(x)|^2). \tag{2.2}$$

Equivalently, $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ iff there exists a constant $\gamma > 0$ such that

$$B_{N+1}((x,f(x)) + \gamma(-v,\lambda), \gamma\|(-v,\lambda)\|) \cap \text{hypo}(f) = \varnothing \tag{2.3}$$

where

$$B_k(a,r) \;=\; \{z \in \mathbb{R}^k \mid \|z-a\| < r\}$$

is the open ball with center a and radius r in $\mathbb{R}^k$.
Moreover, the vector $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ is *realized by a ball of radius $\rho > 0$* if $(-v,\lambda) \neq 0$ and (2.2) is satisfied for $\sigma = \frac{\|(-v,\lambda)\|}{2\rho}$.

Remark 2.1 *If $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ then $\lambda \geq 0$.*

Associated with hypo(f), one can define some concepts of generalized differential for f at $x \in \Omega$. Let $x \in \Omega$ and $v \in \mathbb{R}^N$, we say that:

1. v is a *proximal supergradient* of f at x ($v \in \partial^P f(x)$) if $(-v,1) \in N^P_{\text{hypo}(f)}(x, f(x))$.

2. v is a *proximal horizon supergradient* of f at x ($v \in \partial^\infty f(x)$) if $(-v,0) \in N^P_{\text{hypo}(f)}(x, f(x))$.

We introduce now two key concepts of our paper.

Definition 2.1 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $f : \Omega \longrightarrow \mathbb{R}$ be a continuous function. Let $\varphi : \Omega \longrightarrow [0.\infty)$ be a continuous function. We say that the set* hypo(f) *is φ-convex if for every $x \in \Omega$, for every $\xi \in N^P_{\text{hypo}(f)}(x, f(x))$ the inequality*

$$\langle \xi\ ,\ (y,\beta)-(x,f(x))\rangle \;\leq \varphi(x)\|\xi\|(\|y-x\|^2 + |\beta - f(x)|^2) \tag{2.4}$$

holds for all $y \in \Omega$ and for all $\beta \leq f(y)$.

In general, upper semicontinuous functions with φ-convex hypograph enjoy several of the regularity properties, except Lipschitz continuity, that semiconcave functions satisfy. We state a result in [5] which collects the main properties.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^N$ be open, and let $f : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be proper, upper semicontinuous, and such that* hypo(f) *is φ-convex for a suitable continuous φ. Then there exists a sequence of sets $\Omega_h \subseteq \Omega$ such that Ω_h is compact in* dom(f) *and*

(1) the union of Ω_h covers $\mathcal{L}^N$-almost all dom(f)*;*

(2) for all $x \in \bigcup_h \Omega_h$ there exist $\delta = \delta(x) > 0$, $L = L(x) > 0$ such that

$$f \text{ is Lipschitz on } B(x,\delta) \text{ with ratio } L, \text{and hence semiconcave on } B(x,\delta). \quad (2.5)$$

Consequently,

(3) f is a.e. Fréchet differentiable and admits a second order Taylor expansion around a.e. point of its domain.

The second concept is weaker

Definition 2.2 *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. Given a continuous function $\theta : \Omega \longrightarrow (0, \infty)$, we say that* hypo$(f)$ *satisfies the θ-external sphere condition if for every $x \in \Omega$, there exists a vector $\xi \in N^P_{\text{hypo}(f)}(x, f(x))$ realized by a ball of radius $\theta(x)$.*

We are now giving some new notations. These notations are concerned with the set of *bad points* where the proximal normal cone of hypo(f) contains at least one line (i.e., it is not pointed). First we introduce two special types of normal vectors, namely

1. Normal vectors which are limit of unique normals at nearby points

$$N^L(x) \;=\; \{\, \xi \in \mathbb{R}^{N+1} | \text{ there exists a sequence } \{x_n\} \text{ converging to } x \text{ such that } i)\ f \text{ is differentiable at } x_n \text{ and } ii)\ \xi = \lim_{n\to\infty} \frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|} \;\}.$$

2. Among them we select the horizontal ones

$$N^L_0(x) \;=\; N^L(x) \cap (-\partial^\infty f(x), 0).$$

We also denote the subspace which is generated by $N^L_0(x)$ as

$$H_0(x) \;= span\{N^L_0(x)\} \;=\; \{\sum_{i=1}^k \alpha_i \xi_i \mid \xi_i \in N^L_0(x) \text{ and } \alpha_i \in \mathbb{R}\},$$

and the positive cone which is generated by $N^L_0(x)$ as

$$H^+_0(x) \;= span^+\{N^L_0(x)\} \;=\; \{\sum_{i=1}^k \alpha_i \xi_i \mid \xi_i \in N^L_0(x) \text{ and } \alpha_i \geq 0\}.$$

3. The largest vector subspace contained in $N^P_{hypo(f)}(x, f(x))$ will be denoted by

$$NL(x) \;=\; \{\, \xi \in N^P_{hypo(f)}(x, f(x)) \mid -\xi \in N^P_{hypo(f)}(x, f(x))\}.$$

   From Remark 2.1, one can see that $NL(x) \subseteq (-\partial^\infty f(x), 0)$.

4. We denote the set of *bad points* of f by

$$BP_f \;=\; \{x \in \Omega \mid NL(x) \neq 0\} \tag{2.6}$$

   At each point $x \in BP_f$, we write BP_f as the union of the two sets

$$BP_f^+(x) \;=\; \{y \in BP_f \mid f(y) \geq f(x)\}$$

$$BP_f^-(x) \;=\; \{y \in BP_f \mid f(y) \leq f(x)\}.$$

2.2 Control theory

The nonlinear control system of the form

$$\begin{cases} \dot{y}(t) \;=\; f(y(t), u(t)) & a.e. \\ u(t) \;\in\; \mathcal{U} & a.e. \\ y(0) \;=\; x, & \end{cases} \tag{2.7}$$

is considered together with the target set $\mathcal{S} \subset \mathbb{R}^N$ which is nonempty and closed.
For a fixed $x \in \mathcal{S}^c = \mathbb{R}^N \setminus \mathcal{S}$, we define

$$\Gamma(x, u) := \min\, \{t \geq 0 \mid y^{x,u}(t) \in \mathcal{S}\}.$$

Of course, $\Gamma(x, u) \in (0, +\infty]$, and $\Gamma(x, u)$ is the time taken for the trajectory $y^{x,u}(\cdot)$ to reach $\mathcal{S}$, provided $\Gamma(x, u) < +\infty$. The *minimum time* $T(x)$ to reach $\mathcal{S}$ from x is defined by

$$T(x) := \inf\, \{\Gamma(x, u) \mid u(\cdot) \in \mathcal{U}_{\text{ad}}\}. \tag{2.8}$$

Our assumptions:

(H1) $\mathcal{U} \subset \mathbb{R}^N$ is compact.

(H2) $f : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^N$ is continuous and satisfies:

$$\|f(x, u) - f(y, u)\| \leq L\, \|x - y\| \quad \forall\, x, y \in \mathbb{R}^N,\ u \in \mathcal{U},$$

for a positive constant L. Moreover, the differential of f with respect to the x variable, $D_x f$, exists everywhere, is continuous with respect to both x and u and satisfies the following Lipschitz condition:

$$\|D_x f(x, u) - D_x f(y, u)\| \leq L_1\, \|x - y\| \quad \forall\, x, y \in \mathbb{R}^N,\ u \in \mathcal{U},$$

for a positive constant L_1.

(H3) The minimum time function $T : \mathbb{R}^N \longrightarrow [0, +\infty)$ is everywhere finite and continuous, (i.e. controllability and small time controllability hold).

(H4) The target $\mathcal{S}$ is nonempty, closed, and satisfies the internal sphere condition of radius $\rho > 0$.

The following result was proved in [8].

Theorem 2.2 *Under the conditions (H1), (H2), (H3) and (H4), together with the further assumption*

$$N^P_{hypo(T)}(x, T(x)) \text{ is pointed for all } x \in \mathcal{S}^c, \tag{2.9}$$

there exists a continuous function $\varphi : \mathcal{S}^c \longrightarrow [0, +\infty)$ such that hypo(T) is φ-convex.

3 Statement of the main results

Our results are the following theorem, together with several corollaries. We recall that the notation BP_f was defined in (2.6).

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(f) satisfies the $\theta-$external sphere condition, where $\theta : \Omega \longrightarrow (0, \infty)$ is continuous. Then*
i) $\Omega_P := \Omega \backslash BP_f$ *is open.*
ii) $\mathcal{L}^N(\Omega \backslash \Omega_P) = 0.$

Corollary 3.1 *Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(f) satisfies the $\theta-$external sphere condition where $\theta : \Omega \longrightarrow [0, \infty)$ is continuous. Then there exists a continuous function $\varphi : \Omega_P \longrightarrow [0, +\infty)$ such that hypo$(f_{|\Omega_P})$ is φ-convex.*

Corollary 3.2 *Let $f : \Omega \longrightarrow \mathbb{R}$ be as in the statement of Theorem 3.1 then f satisfies all of properties in the list of Theorem 2.1.*

In view of Theorem 2.2, we can apply the previous results to the minimum time function.

Corollary 3.3 *Under the conditions (H1), (H2) , (H3), (H4), there exists an open set $\mathcal{S}^c_P \subset \mathcal{S}^c$ such that $\mathcal{L}^N(\mathcal{S}^c \backslash \mathcal{S}^c_P) = 0$ and the restricted continuous function $T_{|\mathcal{S}^c_P} : \mathcal{S}^c_P \longrightarrow [0, +\infty)$ has φ-convex hypograph.*

Corollary 3.4 *Under the conditions (H1), (H2) , (H3), (H4), the minimum time function is twice differentiable almost everywhere in S^c.*

In order to make our proof more clear, we prefer to state our main Theorem in a particular case (local case). The arguments are used in the proof of the main part of the proof of Theorem 3.1.

Theorem 3.2 *Let $f : B_N(0,1) \longrightarrow \mathbb{R}$ be continuous and let $\rho > 0$. Assume that hypo(f) satisfies the $\rho-$ external sphere condition. Then*
i) $BP_f \cup \partial B_N(0,1)$ *is closed.*
ii) $\mathcal{L}^N(BP_f) = 0$.

3.1 Outline of proof of Theorem 3.2

The part (i) is precisely Lemma 4.4.
To prove the part (ii) we will use induction.

For the case $N = 1$. By using Lemma 5.1 and Corollary 4.5 we obtain that the $\mathcal{L}^1$-density of BP_f at x, $D^1_{BP_f}(x) = \lim_{\sigma\to\infty} \frac{\mathcal{L}^1(BP_f \cap B_1(x,\sigma))}{\mathcal{L}^1(B_1(x,\sigma))} = 0$ for all $x \in BP_f$. Therefore, the proof is completed by Lebesgue Theorem.

In order to get the conclusion for $N = k+1$ from inductive assumption for $N = k \geq 1$. We divide the set BP_f into two parts:

The first part is $BP_f^{\zeta^+} \cup BP_f^{\zeta^+}$ *(see the definition of BP_f^{ζ} near Lemma 4.7)* where $\zeta^+ = (0,1)$ and $\zeta^- = (0,-1)$. Using Lemma 4.7, we get $\mathcal{L}^N(BP_f^{\zeta^+} \cup BP_f^{\zeta^+}) = 0$.

To prove $\mathcal{L}^N[BP_f \backslash (BP_f^{\zeta^+} \cup BP_f^{\zeta^+})] = 0$, we notice that Lemma 4.6 can be used at every point in the open set $B_N(0,1)\backslash(BP_f^{\zeta^+} \cup BP_f^{\zeta^+})$. We need to prove for all $B_N(x,r_x) \subset B_N(0,1)\backslash(BP_f^{\zeta^+} \cup BP_f^{\zeta^+})$, it holds $\mathcal{L}^N(BP_f \cap B_N(x,r_x)) = 0$. Three small steps are considered

Step 1: Let $\bar{f} = f_{|B_N(x,r_x)}$. By lemma 4.6, the hypo$(\bar{f}_{x_2})$ *(See the definition of $\bar{f}_{x_2}$ near Lemma 4.6)* satisfies $\theta-$ *external sphere condition* .

Step 2: From Lemma 7.3 and inductive assumption, we get $\mathcal{L}^{N-1}(BP_{\bar{f}_{x_2}}) = 0$.

Step 3: We use Fubini Theorem to complete the proof.

4 Some preparatory lemmas

This section is devoted to several partial results which are needed to prove our main theorem. To simplify our statements, we agree that the continuous function f in this section is defined on $B_N(0,1)$ and hypo(f) satisfies the $\rho-$ *external sphere condition* for a given constant $\rho > 0$.

The first Lemma shows that the proximal normal unit vector to the hypograph of f at $(x, f(x))$ where f is differentiable is unique and is realized by a ball of radius ρ.

Lemma 4.1 *Let x be in $B_N(0,1)$ such that $f(.)$ is differentiable at x. Then $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is the unique proximal normal unit vector to hypo(f) at $(x, f(x))$. Moreover, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is realized by a ball of radius ρ, i.e, for all $y \in B_N(0,1)$ and for all $\beta \leq f(y)$, it holds:*

$$\langle \frac{(-Df(x),1)}{\|(-Df(x),1)\|} \ , \ (y,\beta) - (x, f(x))\rangle \ \leq \ \frac{1}{2\rho} \ (\|y-x\|^2 + \|\beta - f(x)\|^2).$$

Proof
Since $f(.)$ is differentiable at x, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is unique Fréchet normal unit vector to the hypograph of $f(.)$ at $(x, f(x))$. Therefore, since hypo(f) satisfies the $\rho-$ *external sphere condition*, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is the unique proximal normal unit vector to hypo(f) at $(x, f(x))$. Thus, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|} \in N^P_{\text{hypo}(f)}(x, f(x))$ is realized by a ball of radius ρ.

From this lemma and the continuity of f, three corollaries follow.

Corollary 4.1 *Let $x \in B_N(0,1)$ then*

$$N^L(x) \ \subseteq \ N^P_{\text{hypo}(f)}(x, f(x)).$$

More precisely, for each $0 \neq \xi \in N^L(x)$ we have that ξ is a unit proximal normal vector to hypo(f) at $(x, f(x))$ realized by a ball of radius ρ.

Proof
Let $\xi \in N^L(x)$, and take a sequence $\{x_n\}$ converging to x such that f is differentiable at x_n and $\{\frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|}\}$ converges to ξ. By Lemma 4.1, $\frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|} \in N^P_{\text{hypo(f)}}(x_n, f(x_n))$ is realized by a ball of radius ρ, i.e., for all $y \in B_N(0,1)$ and for all $\beta \le f(y)$, it holds

$$\langle \frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|} \ , \ (y,\beta) - (x_n, f(x_n))\rangle \ \le \ \frac{1}{2\rho}\,(\|y-x\|^2 + \|\beta - f(x_n)\|^2). \tag{4.1}$$

By taking n to ∞ in (4.1), the inequality

$$\langle \xi \ , \ (y,\beta) - (x, f(x))\rangle \ \le \ \frac{1}{2\rho}\,(\|y-x\|^2 + \|\beta - f(x)\|^2)$$

holds for all $y \in B_N(0,1)$ and for all $\beta \le f(y)$.
The proof is completed.

Corollary 4.2 *$N^L_0(x)$ is closed for all $x \in B_N(0,1)$. Moreover, if $\xi \in N^L_0(x)$ then ξ is a proximal normal unit vector to hypo(f) at $(x, f(x))$ realized by a ball of radius ρ.*

Proof
Let $\{\xi_n\} \subseteq N^L_0(x)$ converge to $\bar{\xi}$, we need to prove that $\bar{\xi} \in N^L_0(x)$. Indeed, for each n, there exists a sequence $\{x^k_n\}$ converging to x such that f is differentiable at x^k_n and $\{\frac{(-Df(x^k_n),1)}{\|(-Df(x^k_n),1)\|}\}$ converges to a unit vector $\xi_n \in (-\partial^\infty f(x), 0)$. For each n we can take a point $y_n \in \{x^k_n\}$ such that $\|y_n - x\| \le \frac{1}{n}$ and $\|\frac{(-Df(y_n),1)}{\|(-Df(y_n),1)\|} - \bar{\xi}\| \le \frac{1}{n}$. Therefore $\{y_n\}$ and $\{\frac{(-Df(y_n),1)}{\|(-Df(y_n),1)\|}\}$ converge respectively to x and $\bar{\xi}$. This implies that $\bar{\xi} \in N^L(x)$. On the other hand, since $\{\xi_n\} \subseteq N^L_0(x)$ converges to $\bar{\xi}$ we have $\bar{\xi} \in (-\partial^\infty f(x), 0)$. The proof is completed.

With similar proof, we get the third corollary.

Corollary 4.3 *Let $\{x_n\} \in B_N(0,1)$ converge to $x \in B_N(0,1)$ and let $\xi_n \in N_0^L(x_n)$ converge to $\bar{\xi}$, then $\bar{\xi} \in N_0^L(x)$.*

The next Lemma says that if there exists a vector $0 \neq p_0 \in (-\partial^\infty f(x))$ then we can find a vector in $N_0^L(x)$. This vector is found by considering a sequence which converges to x along the ray $\{x + tp_0 \mid t > 0\}$ such that f is differentiable at each point of such sequence. This idea is inspiredly the proof of Lemma 4.7 in [8].

Lemma 4.2 *Let $x \in B_N(0,1)$ such that $\partial^\infty f(x) \neq 0$. Then $N_0^L(x)$ is non empty.*

Proof
Let $0 \neq -p_0 \in \partial^\infty f(x)$. By the definition of $\partial^\infty f(x)$, $(p_0, 0) \in N^P_{\text{hypo(f)}}(x, f(x))$, i.e. there exists a constant $\sigma_0 > 0$ such that

$$\langle (p_0, 0)\ ,\ (y, \beta) - (x, f(x)) \rangle \ \leq\ \sigma_0\ (\|y - x\|^2 + \|\beta - f(x)\|^2) \tag{4.2}$$

for all $y \in B_N(0,1)$ and for all $\beta \leq f(y)$.
Set $x_n = x + \frac{p_0}{n}$. By the Density theorem (see Theorem 1.3.1 in [4]), for each n there exists z_n such that

$$\partial_P f(z_n) \ \neq\ \emptyset \tag{4.3}$$

$$\|z_n - x_n\| \ \leq\ \frac{1}{n^2} \tag{4.4}$$

(4.3) implies that there exists a vector $(\zeta_n, -1)$ which is a proximal normal vector to the epigraph of $f(.)$ at $(z_n, f(z_n))$. Therefore, since hypo(f) satisfies the $\rho-$*external sphere condition* we obtain that $f(.)$ is differentiable at z_n. Recalling Lemma 4.1, for all $z \in B_N(0,1)$ and for all $\beta \leq f(z)$, it holds

$$\langle (-Df(z_n), 1)\ ,\ (z, \beta) - (z_n, f(z_n)) \rangle \ \leq\ \frac{1}{2\rho}\ \|(-Df(z_n), 1)\|\ (\|z_- z_n\|^2 + \|\beta - f(z_n)\|^2). \tag{4.5}$$

Recalling (4.4), $z_n \in B_N(0,1)$ for n large enough. Thus by taking $y = z_n$ in (4.2), we obtain

$$\langle p_0\ ,\ z_n - x \rangle \ \leq\ \sigma_0\ (\|z_n - x\|^2 + \|\beta - f(x)\|^2) \tag{4.6}$$

for all $\beta \leq f(z_n)$.
We have

$$\begin{aligned} \langle p_0\ ,\ z_n - x \rangle &= \langle p_0\ ,\ \frac{p_0}{n} \rangle + \langle p_0\ ,\ z_n - x_n \rangle \\ &= \frac{\|p_0\|^2}{n} + \langle p_0\ ,\ z_n - x_n \rangle. \end{aligned}$$

Combining the above inequality with (4.4), we get

$$\langle p_0\ ,\ z_n - x \rangle \ \geq\ \frac{\|p_0\|^2}{n} - \frac{\|p_0\|}{n^2}. \tag{4.7}$$

Moreover, from (4.4) we get

$$\|z_n - x\| \ = 0(\frac{1}{n}). \tag{4.8}$$

Recalling (4.6), (4.7) and (4.8), for n large enough, the following estimate

$$\frac{\|p_0\|^2}{n} \ \leq 0(\frac{1}{n^2}) + \|\beta - f(x)\|^2 \tag{4.9}$$

holds for all $\beta \leq f(z_n)$.
Therefore, there exists a constant $C > 0$ such that

$$f(x) - f(z_n) \ \geq \ \frac{C}{\sqrt{n}}. \tag{4.10}$$

for n large enough.

We are now going to prove that : $\limsup_{n\to\infty} \|(-Df(z_n), 1)\| = +\infty$.
Assume by contradiction that there exists a constant $K > 0$ such that

$$\|(-Df(z_n), 1)\| \ \leq \ K \qquad \text{for all } n. \tag{4.11}$$

By taking $z = x$ and $\beta = f(x)$ in (4.5) and by recalling (4.11) we have

$$(f(x) - f(z_n))(1 - \frac{K}{2\rho}(f(x) - f(z_n))) \ \leq K(1 + \frac{\|x - z_n\|}{2\rho})\|x - z_n\|. \tag{4.12}$$

for n large enough. Therefore, by (4.10) and (4.8), we get from the above inequality that there exists a constant $C_1 > 0$ such that

$$\frac{1}{\sqrt{n}} \ \leq \ C_1\frac{1}{n}.$$

for n large enough.
This is a contradiction.

We now assume by without of loss of generality that $\lim_{n\to\infty} \frac{(-Df(z_n),1)}{\|(-Df(z_n),1)\|} = (-\overline{\zeta}_0, 0)$. Since $\{z_n\}$ converges to x, we have $(-\overline{\zeta}_0, 0) \in N_0^L(x)$. The proof is completed.

Corollary 4.4 *If $x \in BP_f$ then $N_0^L(x)$ is non empty.*

The following Lemma is a crucial observation. At every *bad point*, we can extract a line from $H_0^+(x) \subseteq NL(x) \subseteq N^P_{hypo(f)(x,f(x))}$. It is also pivotal to prove Lemma 4.4 and Theorem 5.1. The difference between the proof of this Lemma and the proof of the previous Lemma is the way of choosing a sequence which allows us to get a vector in $N_0^L(x)$.

Lemma 4.3 *If $x \in BP_f$ then $H_0^+(x)$ contains at least one line.*

Proof

We recall that by Corollary 4.4, $N_0^L(x)$ is nonempty.

Assume by contradiction that $H_0^+(x)$ does not contain lines. From Corollary 4.2, $N_0^L(x)$ is compact and does not contain 0. Thus by applying Lemma 7.1 for $C = N_0^L(x)$, there exists a constant $\delta_0 > 0$ such that for all $0 \neq \xi_1, \xi_2 \in H_0^+(x)$, it holds

$$\langle \frac{\xi_1}{\|\xi_1\|} \ , \ \frac{\xi_2}{\|\xi_2\|} \rangle > -1 + \delta_0.$$

Therefore, there exist a vector $(v_0, 0) \in H_0(x)$ and a constant $\delta_1 > 0$ such that $v_0 \in \mathbb{R}^N$, $\|v_0\| = 1$ and

$$\langle -(v_0, 0) \ , \ \frac{\xi}{\|\xi\|} \rangle \ \geq \ \delta_1 \quad \text{for all} \ \ 0 \neq \xi \in H_0^+(x). \tag{4.13}$$

Since $x \in BP_f$ (namely, $NL(x)$ contains at least one line) there exists a unit vector $p_0 \in \mathbb{R}^N$ such that $(p_0, 0) \in NL(x)$ and $\langle p_0 \ , \ v_0 \rangle \geq 0$.
Setting $v_1 = v_0 + \frac{\delta_1}{2} p_0$, one can easily get from (4.13) that:

$$\langle -(v_1, 0) \ , \ \frac{\xi}{\|\xi\|} \rangle \ \geq \ \frac{\delta_1}{2} \quad \text{for all} \ \ 0 \neq \xi \in H_0^+(x). \tag{4.14}$$

Setting $x_n = x + \frac{v_1}{n}$. By the Density theorem (see Theorem 1.3.1 in [4]), for each n there exists z_n such that

$$\partial_P f(z_n) \ \neq \ \emptyset \tag{4.15}$$

$$\|z_n - x_n\| \ \leq \ \frac{1}{n^2} \tag{4.16}$$

(4.15) implies that there exists a vector $(\zeta_n, -1)$ which is a proximal normal vector to the epigraph of $f(.)$ at $(z_n, f(z_n))$. Therefore, since hypo(f) satisfies the *ρ-external sphere condition* we obtain that $f(.)$ is differentiable at z_n *(see Proposition 3.15, p.51, [1])*. Recalling Lemma 4.1, for all $z \in B_N(0, 1)$ and for all $\beta \leq f(z)$, it holds

$$\langle (-Df(z_n), 1) \ , \ (z, \beta) - (z_n, f(z_n)) \rangle \ \leq \ \frac{1}{2\rho} \ \|(-Df(z_n), 1)\| \ (\|z_- z_n\|^2 + \|\beta - f(z_n)\|^2). \tag{4.17}$$

On the other hand, since $(p_0, 0) \in NL(x)$, there exists a constant $\sigma_0 > 0$ such that

$$\langle (p_0, 0) \ , \ (y, \beta) - (x, f(x)) \rangle \ \leq \ \sigma_0 \ (\|y - x\|^2 + \|\beta - f(x)\|^2) \tag{4.18}$$

for all $y \in B_N(0, 1)$ and for all $\beta \leq f(y)$.
Recalling (4.16), $z_n \in B_N(0, 1)$ for n large enough. Thus by taking $y = z_n$ in (4.18), we have

$$\langle p_0 \ , \ z_n - x \rangle \ \leq \ \sigma_0 \ (\|z_n - x\|^2 + \|\beta - f(x)\|^2) \tag{4.19}$$

for all $\beta \leq z_n$.
We have

$$\begin{aligned} \langle p_0 \ , \ z_n - x \rangle \quad &= \quad \langle p_0 \ , \ \frac{v_0}{n} \rangle + \langle p_0 \ , \ \frac{\delta_1}{2n} p_0 \rangle + \langle p_0 \ , \ z_n - x_n \rangle \\ &\geq \quad \frac{\delta_1}{2n} + \langle p_0 \ , \ z_n - x_n \rangle. \end{aligned}$$

Combining the above inequality with (4.16), we get

$$\langle p_0\ ,\ z_n - x\rangle\ \geq\ \frac{\delta_1}{2n} - \frac{1}{n^2}. \tag{4.20}$$

Moreover, from (4.16) we get

$$\|z_n - x\|\ = 0(\frac{1}{n}). \tag{4.21}$$

Recalling (4.19), (4.20) and (4.21), for n large enough, the following estimate holds

$$\frac{\delta_1}{2n}\ \leq 0(\frac{1}{n^2}) + \|\beta - f(x)\|^2 \tag{4.22}$$

for all $\beta \leq f(z_n)$.
Therefore, there exists a constant $C > 0$ such that

$$f(x) - f(z_n)\ \geq\ \frac{C}{\sqrt{n}}. \tag{4.23}$$

for n large enough.

We are now going to prove that : $\limsup_{n\to\infty} \|(-Df(z_n), 1)\| = +\infty$.
Assume by contradiction that there exists a constant $K > 0$ such that

$$\|(-Df(z_n), 1)\|\ \leq\ K \qquad \text{for all } n. \tag{4.24}$$

By taking $z = x$ and $\beta = f(x)$ in (4.17) and by recalling (4.24) we have

$$(f(x) - f(z_n))(1 - \frac{K}{2\rho}(f(x) - f(z_n)))\ \leq K(1 + \frac{\|x - z_n\|}{2\rho})\|x - z_n\|. \tag{4.25}$$

for n large enough. Therefore, by (4.23) and (4.21), we get from the above inequality that there exists a constant $C_1 > 0$ such that

$$\frac{1}{\sqrt{n}}\ \leq\ C_1\frac{1}{n}.$$

for n large enough.
This is a contradiction.

We now assume by without of loss of generality that $\lim_{n\to\infty} \frac{(-Df(z_n),1)}{\|(-Df(z_n),1)\|} = (-\overline{\zeta}_0, 0)$.
Moreover, since $\{z_n\}$ converges to x, we have $(-\overline{\zeta}_0, 0) \in N_0^L(x)$.
On the other hand, by (4.23), we can take $z = x$ and $\beta = f(z_n)$ in (4.17) to get

$$\langle\ \frac{(-Df(z_n), 1)}{\|(-Df(z_n), 1)\|}\ ,\ \frac{(x - z_n, 0)}{\|x - z_n\|}\rangle\ \leq\ \frac{\|x - z_n\|}{2\rho}. \tag{4.26}$$

Let n tend to $+\infty$. Recalling (4.21), (4.16) we obtain

$$\langle(-\overline{\zeta_0}, 0)\ ,\ (-v_1, 0)\rangle\ \leq\ 0. \tag{4.27}$$

Since $(-\overline{\zeta}_0, 0) \in N_0^L(x)$, we get a contradiction from (4.27) and (4.14).

Lemma 4.4 $BP_f \cup \partial B_N(0,1)$ *is closed.*

Proof

Letting $\{x_n\} \subseteq BP_f \cup \partial B_N(0,1)$ converge to x, we need to prove that $x \in BP_f \cup \partial B_N(0,1) \subseteq \overline{B}_N(0,1)$.
If $x \in \partial B_N(0,1)$, there is nothing to prove.
If $x \in B_N(0,1)$, we will prove that $x \in BP_f$, namely, $NL(x)$ contains at least one line. Assume by contradiction that $NL(x) = 0$. In particular, $H_0^+(x)$ does not contain lines. Similarly, the previous proof, there exist a vector $(v_0, 0) \in H_0(x)$ and a constant $\delta_1 > 0$ such that $v_0 \in \mathbb{R}^N$, $\|v_0\| = 1$ and

$$\langle -(v_0, 0)\ ,\ \frac{\xi}{\|\xi\|}\rangle \ \geq\ \delta_1 \quad \text{for all}\ \ 0 \neq \xi \in H_0^+(x). \tag{4.28}$$

On the other hand, since $x \in B_N(0,1)$ we have $x_n \in B_N(0,1)$ for n large enough. Thus $x_n \in BP_f$. From Lemma 4.3, for n large enough, $H_0^+(x_n)$ contains at least one line. Therefore, for each n large enough, there exists a vector $\xi_n \in N_0^L(x_n)$ such that

$$\langle -(v_0, 0)\ ,\ \xi_n\rangle\ \leq 0. \tag{4.29}$$

By Corollary 4.2, $\|\xi_n\| = 1$. We assume without of loss of generality that $\lim_{n\to\infty} \xi_n = \bar{\xi}$. Recalling Corollary 4.3, we have that $\bar{\xi} \in N_0^L(x)$.
Moreover, by taking $n \to \infty$ in (4.29) we get

$$\langle -(v_0, 0)\ ,\ \bar{\xi}\rangle\ \leq 0. \tag{4.30}$$

Recalling (4.28), we get a contradiction.

The next Lemma is the first step to prove that the $\mathcal{L}^N - density$ of BP_f at $x \in BP_f$ has zero value.

Lemma 4.5 *Define, for* $x \in BP_f$, $F^+(x) = \{y \in B(0,1)\ |f(y) \geq f(x)\}$. *Then the* $\mathcal{L}^N$*-density of* $F^+(x)$ *at* x *is zero, i.e.,*

$$D^N_{F^+(x)}(x) := \lim_{\delta\to 0} \frac{\mathcal{L}^N(B_N(x,\delta) \cap F^+(x))}{\mathcal{L}^N(B_N(x,\delta))} = 0.$$

Proof

Since $x \in BP_f$ (i.e., $NL(x)$ contains at least one line), there exists $(\zeta_0, 0) \in N^P_{hypo(f)}(x, f(x))$ such that $(-\zeta_0, 0) \in N^P_{hypo(f)}(x, f(x))$ and $\|\zeta_0\| = 1$. Thus there exists a constant $\sigma_0 > 0$ such that for all $y \in B_N(0,1)$ and for all $\beta \leq f(y)$, it holds

$$\begin{cases} \langle(\zeta_0, 0)\ ,\ (y - x, \beta - f(x))\rangle & \leq & \sigma_0\ (\|y-x\|^2 + |\beta - f(x)|^2), \\ \langle(-\zeta_0, 0)\ ,\ (y - x, \beta - f(x))\rangle & \leq & \sigma_0\ (\|y-x\|^2 + |\beta - f(x)|^2). \end{cases} \tag{4.31}$$

Therefore, for all $y \in F^+(x) \cap B_N(x,\delta)$, by taking $\beta = f(x)$ in (4.31) we obtain

$$\begin{cases} \langle\zeta_0\ ,\ y - x\rangle & \leq & \sigma_0\|y-x\|^2\ \leq\ \sigma_0\delta^2, \\ \langle-\zeta_0\ ,\ y - x\rangle & \leq & \sigma_0\|y-x\|^2\ \leq\ \sigma_0\delta^2. \end{cases} \tag{4.32}$$

From (4.32), the set $F^+(x)\cap B_N(x,\delta) \subseteq x+\{t\zeta_0 + v \mid t\in[-\sigma_0\delta^2, \sigma_0\delta^2],\ v\in B_N(0,\delta)\cap\zeta_0^\perp\}$ where $\zeta_0^\perp = \{w\in\mathbb{R}^N \mid \langle w,\zeta_0\rangle = 0\}$. Therefore,

$$D^N_{F^+(x)}(x) := \lim_{\delta\to 0^+}\frac{\mathcal{L}^N(B_N(x,\delta)\cap F^+(x))}{\mathcal{L}^N(B_N(x,\delta))} \le \lim_{\delta\to 0^+}\frac{\sigma_0\delta^{N+1}}{\omega_N\delta^N} = \lim_{\delta\to 0^+}\frac{\sigma_0\delta}{\omega_N} = 0$$

where $\omega_N = \mathcal{L}^N(B_N(0,1))$. The proof is completed.

Since $BP_f^+(x)\subseteq F^+(x)$, the below corollary follows immediately

Corollary 4.5 *If $x\in BP_f$ then the $\mathcal{L}^N$-density of $BP_f^+(x)$ at x*

$$D^N_{BP_f^+(x)}(x) := \lim_{\delta\to 0^+}\frac{\mathcal{L}^N(B_N(x,\delta)\cap BP_f^+(x))}{\mathcal{L}^N(B_N(x,\delta))} = 0.$$

In order to use induction in the proof of Theorem 3.2, we need two following lemmas. In the first Lemma, we are working on the cases $N\ge 2$. For every vector $x\in\mathbb{R}^N$ we rewrite $x=(x_1,x_2)$ where $x_1\in\mathbb{R}^{N-1}$ and $x_2\in\mathbb{R}$. For every $x_2\in(-1,1)$, the function restricted to the first $n-1$ variables, $f_{x_2}: B_{N-1}(0,\sqrt{1-x_2^2})\longrightarrow\mathbb{R}$, is denoted by $f_{x_2}(x_1) = f(x_1,x_2)$ for all $x_1\in B_{N-1}(0,\sqrt{1-x_2^2})$.

Lemma 4.6 *Let $(x_1,x_2)\in B_N(0,1)$ and let (ξ_1,ξ_2,λ) be a proximal normal vector to hypo(f) at $(x_1,x_2,f(x_1,x_2))$ realized by a ball of radius ρ. If $(\xi_1,\lambda)\ne 0$ then (ξ_1,λ) is also a proximal vector to hypo(f_{x_2}) at $(x_1,f_{x_2}(x_1))$ realized by a ball of radius $\frac{\|(\xi_1,\lambda)\|}{\|(\xi_1,\xi_2,\lambda)\|}\rho$.*

Proof

The vector (ξ_1,ξ_2,λ) being a proximal normal to the hypograph of f at $(x_1,x_2)\in B_N(0,1)$ realized by a ball of radius ρ means that for all $(y_1,y_2)\in\mathbb{R}^N$ and for all $\beta\le f(y_1,y_2)$, it holds

$$\langle\frac{(\xi_1,\xi_2,\lambda)}{\|(\xi_1,\xi_2,\lambda)\|}\ ,\ (y_1,y_2,\beta)-(x_1,x_2,f(x_1,x_2))\rangle \ \le\ \frac{1}{2\rho}(\|y_1-x_1\|^2+|y_2-x_2|^2+|\beta-f(x_1,x_2)|^2). \tag{4.33}$$

By taking $y_2=x_2$ in (4.33), and by replacing $f(x_1,x_2)=f_{x_2}(x_1)$, $f(y_1,y_2)=f(y_1,x_2)=f_{x_2}(y_1)$ in (4.33), we obtain that for all $y_1\in B_{N-1}(0,\sqrt{1-x_2^2})$ and for all $\beta\le f_{x_2}(y_1)$, it holds

$$\langle\frac{(\xi_1,\lambda)}{\|(\xi_1,\xi_2,\lambda)\|}\ ,\ (y_1,\beta)-(x_1,f_{x_2}(x_1))\rangle\ \le\ \frac{1}{2\rho}(\|y_1-x_1\|^2+|\beta-f_{x_2}(x_1)|^2). \tag{4.34}$$

Since $(\xi_1,\lambda)\ne 0$, from (4.34) we get that for all $y_1\in B_{N-1}(0,\sqrt{1-x_2^2})$ and for all $\beta\le f_{x_2}(y_1)$, it holds

$$\langle\frac{(\xi_1,\lambda)}{\|(\xi_1,\lambda)\|}\ ,\ (y_1,\beta)-(x_1,f_{x_2}(x_1))\rangle\ \le\ \frac{1}{2\rho\frac{\|(\xi_1,\lambda)\|}{\|(\xi_1,\xi_2,\lambda)\|}}(\|y_1-x_1\|^2+|\beta-f_{x_2}(x_1)|^2).$$

The proof is completed.

The second Lemma is used to treat the case $(\xi_1, \lambda) = 0$ in Lemma 4.6 in the proof of our main theorem. Some notations are needed in this Lemma:
Let ζ be a unit vector in $\mathbb{R}^N$, we denote by:
i) $N_0^\zeta \;=\; \{x \in B_N(0,1) \mid (\zeta, 0) \in N^P_{hypo(f)}(x, f(x))$ is realized by a ball of radius $\rho\}$
ii) $BP_f^\zeta \;=\; BP_f \cap N_0^\zeta$.

Lemma 4.7
i) $BP_f^\zeta \cup \partial B_N(0,1)$ *is closed.*
ii) BP_f^ζ *has zero N-Lebesgue measure.*

Proof
Proof of (i)

By Lemma 5.17, the set $BP_f \cup \partial B_N(0,1)$ is closed. Thus we only need to prove that $N_0^\zeta \cup \partial B_N(0,1)$ is closed.

Let $\{x_n\} \subseteq N_0^\zeta \cup \partial B_N(0,1)$ converge to x, we need to show that $x \in N_0^\zeta \cup \partial B_N(0,1)$.
If $x \in \partial B_N(0,1)$ there is nothing to prove.
If $x \in B_N(0,1)$ then for n large enough we have $x_n \in B_N(0,1)$. Thus $x_n \in N_0^\zeta$, namely, $(\zeta, 0) \in N^P_{hypo(f)}(x_n, f(x_n))$ is realized by a ball of radius ρ, i.e, for all $z \in B_N(0,1)$ and for all $\beta \leq f(z)$, it holds

$$\langle \frac{(\zeta,0)}{\|(\zeta,0)\|}\ ,\ (z,\beta) - (x_n, f(x_n))\rangle \;\leq\; \frac{1}{2\rho}(\|z - x_n\|^2 + |\beta - f(x_n)|^2). \tag{4.35}$$

Since $\{x_n\}$ converges to x and $f(.)$ is continuous, by taking $n \to \infty$ we have

$$\langle \frac{(\zeta,0)}{\|(\zeta,0)\|}\ ,\ (z,\beta) - (x, f(x))\rangle \;\leq\; \frac{1}{2\rho}(\|z - x\|^2 + |\beta - f(x|^2). \tag{4.36}$$

for all $z \in B_N(0,1)$ and for all $\beta \leq f(z)$.
Thus $x \in N_0^\zeta$. The proof is completed.
Proof of (ii)
First, we prove that for all $x \in BP_f^\zeta$, it holds

$$D^N_{BP_f^\zeta}(x) \;=\; \lim_{\delta \to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta) \cap BP_f^\zeta)}{\mathcal{L}^N(B_N(x,\delta))} \leq \frac{1}{2}. \tag{4.37}$$

Indeed, since $BP_f^\zeta \subseteq BP_f$, recalling Corollary 4.5 we obtain

$$D^N_{BP_f^\zeta \cap BP_f^+(x)}(x) = \lim_{\delta \to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta) \cap BP_f^\zeta \cap BP_f^+(x))}{\mathcal{L}^N(B_N(x,\delta))} \;= 0.$$

Thus the inequality (4.37) will hold if

$$D^N_{BP_f^\zeta \cap BP_f^-(x)}(x) \;=\; \lim_{\delta \to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta) \cap BP_f^\zeta \cap BP_f^-(x))}{\mathcal{L}^N(B_N(x,\delta))} \leq \frac{1}{2}. \tag{4.38}$$

If $y \in BP_f^\zeta$, we have $(\zeta, 0) \in N_0^\zeta(y)$. Thus for all $z \in B_N(0,1)$ and for all $\beta \le f(z)$, it holds

$$\langle (\zeta,0)\ ,\ (z-y, \beta - f(y)) \rangle \ \le\ \frac{1}{2\rho}\,(\|z-y\|^2 + |\beta - f(y)|^2). \tag{4.39}$$

Thus, if $y \in BP_f^\zeta \cap BP_f^-(x)$ we can take $z = x$ and $\beta = f(y)$ in 4.39 to get

$$\langle \zeta\ ,\ x-y \rangle \ \le\ \frac{1}{2\rho}\,\|x-y\|^2. \tag{4.40}$$

Therefore, for all $\delta > 0$ small enough, it holds

$$\langle \zeta\ ,\ x-y \rangle \ \le\ \frac{1}{2\rho}\delta^2 \quad \text{for all } y \in [B_N(x,\delta) \cap BP_f^\zeta \cap BP_f^-(x)]. \tag{4.41}$$

(4.41) says that $[B_N(x,\delta) \cap BP_f^\zeta \cap BP_f^-(x)] \subset x + \{t\zeta + v \mid t \in [-\frac{\delta^2}{2\rho}, \delta],\ v \in B_N(0,\delta) \cap \zeta^\perp\}$ where $\zeta^\perp = \{w \in \mathbb{R}^N \mid \langle w, \zeta \rangle = 0\}$. Thus, (4.38) follows. From *(i)*, BP_f^ζ is a Borel set. Moreover, from (4.38), the $\mathcal{L}^N$-density of BP_f^ζ at every point which is in BP_f^ζ is less than $\frac{1}{2}$. Therefore, by Lebesgue theorem we have $\mathcal{L}^N(BP_f^\zeta) = 0$.

5 Proof of Theorem 3.2

5.1 One dimensional case

In this subsection, we are working on $\mathbb{R}$. The function $f(.)$ is defined on $B_1(0,1) = \{x \in \mathbb{R} \mid |x| < 1\}$. Therefore the proximal normal cone $N^P_{hypo(f)}(x, f(x)) \subset \mathbb{R}^2$ contains at most one line.

Lemma 5.1 *For all $x \in BP_f$, we have $N_0^L(x) = \{(1,0),\ (-1,0)\}$.*

Proof
Since $N_0^L(x) \subseteq (-\partial^\infty f(x), 0)$, we have $N_0^L(x) \subseteq \{(t,0) \mid t \in \mathbb{R}\}$. Therefore, from the fact that $\|\xi\| = 1$ for all $\xi \in N_0^L(x)$, we obtain

$$N_0^L(x) \ \subseteq\ \{(1,0),\ (-1,0)\} \tag{5.1}$$

Recalling Lemma 4.3, the set $H_0^+(x) = span^+\{N_0^L(x)\}$ contains at least one line. Thus, the proof is completed by (5.1).

The following statement is a one dimensional version of Theorem 3.2.

Theorem 5.1 *Let $f : B_1(0,1) \longrightarrow \mathbb{R}$ be continuous. Assume that $hypo(f)$ satisfies the $\rho-$external sphere condition. Then*
i) $BP_f \cup \partial B_1(0,1)$ *is closed.*
ii) $\mathcal{L}^1(BP_f) = 0$.

Proof
(i) is the particular case (N=1) of Lemma 4.4 .
Proof of (ii).
We prove first that, for all $x \in BP_f$, the $\mathcal{L}^1$-density of BP_f at x is zero, namely,

$$D^1_{BP_f}(x) := \lim_{\delta \to 0^+} \frac{\mathcal{L}^1(B_1(x,\delta) \cap BP_f)}{\mathcal{L}^1(B_1(x,\delta))} = 0. \tag{5.2}$$

Recalling Corollary 4.5 for N=1, we have

$$D^1_{BP_f^+(x)}(x) = \lim_{\delta \to 0^+} \frac{\mathcal{L}^1(B_1(x,\delta) \cap BP_f^+(x))}{\mathcal{L}^1(B_1(x,\delta))} = 0.$$

Therefore, 5.2 follows from

$$D^1_{BP_f^-(x)}(x) = \lim_{\delta \to 0^+} \frac{\mathcal{L}^1(B_1(x,\delta) \cap BP_f^-(x))}{\mathcal{L}^1(B_1(x,\delta))} = 0. \tag{5.3}$$

From Lemma 5.1, for every $y \in BP_f$, we have $N_0^L(y) = \{(1,0),\ (-1,0)\}$. Thus, for all $y \in BP_f$ it holds

$$\begin{cases} \langle (1,0)\ ,\ (z-y, \beta - f(y)) \rangle & \leq & \frac{1}{2\rho}\,(|z-y|^2 + |\beta - f(y)|^2), \\ \langle (-1,0)\ ,\ (z-y, \beta - f(y)) \rangle & \leq & \frac{1}{2\rho}\,(|z-y|^2 + |\beta - f(y)|^2). \end{cases} \tag{5.4}$$

for all $z \in \overline{B_1(0,1)}$ and for all $\beta \leq f(z)$.
Since $f(y) \leq f(x)$ for all $y \in BP_f^-(x)$, we can take $z = x$ and $\beta = f(y)$ in (5.4) to get

$$|x-y| \ \leq \ \frac{1}{2\rho}\ |x-y|^2 \quad \text{for all } y \in BP_f^-(x). \tag{5.5}$$

Thus $B_1(x,\delta) \cap BP_f^-(x) = \{x\}$ for all $0 < \delta < 2\rho$ and so 5.3 follows.

We are now going to complete the proof of *(ii)*.
Since $BP_f \cup \partial B_1(0,1)$ is closed, BP_f is a Borel set. From 5.2, the $\mathcal{L}^1$-density of BP_f at x has zero value for all $x \in BP_f$. Therefore, by Lebesgue theorem, we have $\mathcal{L}^1(BP_f) = 0$.

5.2 General case

(i) of Theorem 3.2 is precisely Lemma 4.4.

We are going to prove *(ii)* of Theorem 3.2 by induction.
If $N = 1$, (ii) of Theorem 3.2 follows by Theorem 5.1.
Assume that (ii) of Theorem 3.2 holds for $N = k \geq 1$. We prove that (ii) of Theorem 3.2 will hold for $N = k + 1$.
Let $\zeta^+ = (0,1)$ and $\zeta^- = (0,-1)$ be in $\mathbb{R}^{k+1}$. Recalling Lemma 4.7, we obtain that $(BP_f^{\zeta^+} \cup \partial B_{k+1}(0,1))$ and $(BP_f^{\zeta^-} \cup \partial B_{k+1}(0,1))$ are closed. Moreover,

$$\mathcal{L}^{k+1}(BP_f^{\zeta^+}) = \mathcal{L}^{k+1}(BP_f^{\zeta^-}) = 0. \tag{5.6}$$

Set $E = B_{k+1}(0,1)\backslash[N_0^{\zeta^+} \cup N_0^{\zeta^-} \cup \partial B_{k+1}(0,1)]$. One can easily see that E is an open set in $\mathbb{R}^{k+1}$. From (5.6), the conclusion of (ii) of Theorem 3.2 follows from the equality

$$\mathcal{L}^{k+1}(E \cap BP_f) \;=\; 0. \tag{5.7}$$

Recalling Lemma 4.4, $BP_f \cap \partial B_{k+1}(0,1)$ is closed. Thus $E \cap BP_f$ is a Borel set. Therefore, by Lebesgue theorem, (5.7) will follow if for every $x \in E \cap BP_f$, the $\mathcal{L}^{k+1}$-density $D^{k+1}_{E\cap BP_f}(x)$ at x has zero value.

We divide the proof into several steps:

The first step is pivotal *(see the below inequality (5.8))* to show that the restricted functions *(defined before Lemma 4.6)* which are restricted from the function $f_{|B_{k+1}(x,r_x)}$ where $x \in E$, have the hypograph satisfying the $\rho_x - $*external sphere condition.*
Step1:

Let $x \in E$. Since E is open, there exists $r_x > 0$ such that $\overline{B_{k+1}}(x, r_x) \subset E$. By the external sphere assumption on f, for each $y \in B_{k+1}(x, r_x)$, there exists $0 \neq (\xi_1^y, \xi_2^y, \lambda^y) \in N^P_{hypo(f)}(y, f(y))$ realized by a ball of radius ρ where $\xi_1^y \in \mathbb{R}^k$ and $\xi_2^y, \lambda^y \in \mathbb{R}$. We claim that there exists a constant $\alpha_x > 0$ such that

$$\frac{\|(\xi_1^y, \lambda^y)\|}{\|(\xi_1^y, \xi_2^y, \lambda^y)\|} \;\geq\; \alpha_x > 0 \qquad \text{for all } y \in B_{k+1}(x, r_x). \tag{5.8}$$

Assume by contradiction that there exists a sequence $\{y_n\} \subseteq B_{k+1}(x, r_x)$ such that

$$\lim_{n\to\infty} \frac{\|(\xi_1^{y_n}, \lambda^{y_n})\|}{\|(\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})\|} = 0. \tag{5.9}$$

Assuming without loss of generality that $\lim_{n\to\infty} y_n = \bar{y} \in \overline{B_{k+1}}(x, r_x)$ and $\lim_{n\to\infty} \frac{(\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})}{\|(\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})\|} = (\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda})$. From (5.9), one can see that

$$(\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda}) \in \; \{(0,1,0),\; (0,-1,0)\} = \{(\zeta^+, 0),\; (\zeta^-, 0)\}. \tag{5.10}$$

Moreover, $(\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda})$ is a proximal normal vector to hypo(f) at $(\bar{y}, f(\bar{y}))$ realized by a ball of radius ρ. Indeed, since $0 \neq (\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n}) \in N^P_{hypo(f)}(y_n, f(y_n))$ is realized by a ball of radius ρ, we have

$$\langle \frac{(\xi_1^n, \xi_2^n, \lambda^n)}{\|(\xi_1^n, \xi_2^n, \lambda^n)\|} \;,\; (z, \beta) - (y_n, f(y_n))\rangle \;\leq\; \frac{1}{2\rho}(\|z - y_n\|^2 + |\beta - f(y_n)|^2)$$

for all $z \in B_{k+1}(0,1)$ and for all $\beta \leq f(z)$.
By taking $n \to \infty$, we obtain that

$$\langle(\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda}) \;,\; (z, \beta) - (\bar{y}, f(\bar{y}))\rangle \;\leq\; \frac{1}{2\rho}(\|z - \bar{y}\|^2 + |\beta - f(\bar{y})|^2)$$

for all $z \in B_{k+1}(0,1)$ and for all $\beta \leq f(z)$.
Therefore, by (5.10), we get $\bar{y} \in N_0^{\zeta^+} \cup N_0^{\zeta^-}$. This is a contradiction because $\bar{y} \in$

$\overline{B_{k+1}}(x, r_x) \subset E = B_{k+1}(0,1) \backslash [N_0^{\zeta^+} \cup N_0^{\zeta^-} \cup \partial B_{k+1}(0,1)]$.

The second step allows us to make a connection between the set of *bad points* of f and the set of *bad points* of restricted functions of f.

Step 2

Let $x \in E \cap BP_f$. We claim that there exists a line $\{t\xi_x \mid t \in \mathbb{R}\} \subseteq N^P_{hypo(f)}(x)$ such that $\{t\xi_x \mid t \in \mathbb{R}\} \neq \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$.

Assume by contraction, since $x \in BP_f$, i.e. $NL(x) \neq 0$, we have $NL(x) = \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$. Recalling Lemma 4.3, the set $H_0^+(x) \subseteq NL(x)$ contains at least one line. Therefore $H_0^+(x) = \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$ which implies that $(\zeta^+, 0) \in N_0^L(x)$. Recalling Corollary 4.2, $(\zeta^+, 0) \in N^P_{hypo(f)}(x, f(x))$ is realized by a ball of radius ρ. Thus $x \in N_0^{\zeta^+}$ and this is a contradiction because $x \in E$.

In the next step, we are going to prove that $\mathcal{L}^{k+1}(B_{k+1}(x, r_x) \cap BP_f) = 0$ by our inductive assumption .

Step 3:

Let $\bar{f} = f_{|B_{k+1}(x,r_x)} : B_{k+1}(x, r_x) \longrightarrow \mathbb{R}$ be the restricted function of f on $B_{k+1}(x, r_x)$. From Lemma 7.2, the continuous function $\bar{f}$ has the hypo($\bar{f}$) which satisfies $\rho-$*external sphere condition*, and

$$BP_f \cap B_{k+1}(x, r_x) = BP_{\bar{f}}. \tag{5.11}$$

Moreover, two properties which we claimed in Step 1 and Step 2 still hold for the function $\bar{f}$.

Since (5.11) holds, we only need to prove $\mathcal{L}^{k+1}(BP_{\bar{f}}) = 0$.

In order to make the proof more clear, we restate our above problem by replacing $x = 0$, $r_x = 1$ and $\bar{f} = f$. The statement is that

Let $f : B_{k+1}(0,1) \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(f) satisfies $\rho -$ *external sphere condition*. Moreover,

i) For all $y \in B_{k+1}(0,1)$, there exists a vector $0 \neq (\xi_1^y, \xi_2^y, \lambda^y) \in N^P_{hypo(f)}(y, f(y))$ realized by a ball of radius ρ such that

$$\frac{\|(\xi_1^y, \lambda^y)\|}{\|(\xi_1^y, \xi_2^y, \lambda^y)\|} \ \geq \ \alpha_0 > 0. \tag{5.12}$$

ii) For all $x \in BP_f$, there exists a line $\{t\xi_x \mid t \in \mathbb{R}\} \subseteq NL(x)$ such that $\{t\xi_x \mid t \in \mathbb{R}\} \neq \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$.

Then $\mathcal{L}^{k+1}(BP_f) = 0$.

Proof

Since $k \geq 1$, for every $x \in \mathbb{R}^{k+1}$, we write $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}$. For each $x_2 \in (-1,1)$, the restricted function $f_{x_2} : B_k(0, \sqrt{1 - x_2^2}) \longrightarrow \mathbb{R}$ is denoted by $f_{x_2}(x_1) = f(x_1, x_2)$ for all $x_1 \in B_k(0, \sqrt{1 - x_2^2})$.

First, we claim that hypo(f_{x_2}) satisfies $\rho\alpha_0$-*external sphere condition*. Indeed by assumption (i) of the above statement we have that , for each $x_1 \in B_k(0, \sqrt{1 - x_2^2})$, or $(x_1, x_2) \in B_{k+1}(0,1)$, there exists a vector

$$0 \neq (\xi_1^{(x_1,x_2)}, \xi_2^{(x_1,x_2)}, \lambda^{(x_1,x_2)}) \in N^P_{hypo(f)}((x_1, x_2), f(x_1, x_2))$$

realized by a ball of radius ρ such that

$$\frac{\|(\xi_1^{(x_1,x_2)},\lambda^{(x_1,x_2)})\|}{\|(\xi_1^{(x_1,x_2)},\xi_2^{(x_1,x_2)},\lambda^{(x_1,x_2)})\|} \;\geq\; \alpha_0 > 0. \tag{5.13}$$

Recalling Lemma 4.6 for $N = k+1 \geq 2$ and $(\xi_1, \xi_2, \lambda) = (\xi_1^{(x_1,x_2)}, \xi_2^{(x_1,x_2)}, \lambda^{(x_1,x_2)})$, and by (5.13) we obtain that $(\xi_1^{(x_1,x_2)}, \lambda^{(x_1,x_2)})$ is also a proximal normal vector to hypo(f_{x_2}) at $(x_1, f_{x_2}(x_1))$ realized by a ball of radius $\rho\alpha_0$.

Second, we claim that

$$\mathcal{L}^k(BP_{f_{x_2}}) = 0 \quad \text{ for all } x_2 \in (-1,1). \tag{5.14}$$

Indeed, set $\gamma(x_2) = \frac{1}{\sqrt{1-x_2^2}}$ and let $h_{x_2} = f_{x_2}^{\gamma(x_2)}$ be the $\gamma(x_2)$-*stretched function* of f_{x_2} *(see Lemma 7.3)*. By Lemma 7.3 and by the first step, the continuous function $h_{x_2} : B_k(0,1) \longrightarrow \mathbb{R}$ has the hypograph satisfying $\rho_1 -$ *external sphere condition* where $\rho_1 = \rho\alpha_0 \frac{(1-x_2^2)^{\frac{1}{2}}}{(2-x_2^2)^{\frac{3}{2}}}$. Therefore, by inductive assumption, we have

$$\mathcal{L}^k(BP_{h_{x_2}}) = 0. \tag{5.15}$$

Moreover, recalling Corollary 7.1 for $g = f_{x_2}$ and $\gamma = \gamma(x_2)$ we get

$$BP_{h_{x_2}} = (1-x_2^2)^{-\frac{1}{2}} BP_{f_{x_2}}. \tag{5.16}$$

Combining (5.15) and (5.16), we get (5.14).

Thirdly, we claim that

$$BP_f \subseteq \bigcup_{x_2 \in (-1,1)} BP_{f_{x_2}} \times \{x_2\}. \tag{5.17}$$

Assume $x = (x_1, x_2) \in BP_f$. By (ii) there exists a line $\{t\xi_x \mid t \in \mathbb{R}\} \subseteq NL(x) \subseteq (-\partial^\infty f(x), 0)$ such that $\{t\xi_x \mid t \in \mathbb{R}\} \neq \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$ and $\|\xi_x\| = 1$. Therefore, $\xi_x = (\xi_1, \xi_2, 0)$ and $-\xi_x = (-\xi_1, -\xi_2, 0)$ are proximal normal vectors to hypo(f) at $(x, f(x))$ realized by a ball of radius σ where $\sigma > 0$, $0 \neq \xi_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}$ and $\|(\xi_1, \xi_2)\| = 1$. Recalling Lemma 4.6, we obtain that $(\xi_1, 0)$ and $(-\xi_1, 0)$ are proximal normal vectors to the hypograph of f_{x_2} at $(x_1, f_{x_2}(x_1)$. This implies that $N^P_{hypof_{x_2}}(x_1, f_{x_2}(x_1))$ contains the line $\{t(\xi_1, 0) \mid t \in \mathbb{R}\}$. Thus, $x_1 \in BP_{f_{x_2}}$ or $(x_1, x_2) \in (BP_{f_{x_1}}, x_2)$.

Finally, since BP_f is a borel set contained in $B_{k+1}(0,1)$, the indicator function $\mathbf{1}_{BP_f}$ is in $\mathbb{L}^{k+1}(B_{k+1}(0,1))$. From Fubini Theorem, we have

$$\mathcal{L}^{k+1}(BP_f) \;=\; \int_{B_{k+1}(0,1)} \mathbf{1}_{BP_f} dx = \int_{-1}^{1} \int_{B_k(0,\sqrt{1-x_2^2})} \mathbf{1}_{BP_f} dx_1 dx_2 \tag{5.18}$$

Combining the above equality and (5.17), we get

$$\mathcal{L}^{k+1}(BP_f) \leq \int_{-1}^{1} \int_{B_k(0,\sqrt{1-x_2^2})} \mathbf{1}_{BP_{f_{x_2}}} dx_1 dx_2 = \int_{-1}^{1} \mathcal{L}^k(BP_{f_{x_2}}) dx_1. \tag{5.19}$$

The proof is completed using (5.19) and (5.14).

6 Proof of our main results

Proof of Theorem 3.1
Proof of (i)

It is equivalent to prove that $BP_f \cup \partial\Omega \subset \overline{\Omega}$ is closed. Let $\{x_n\} \subseteq BP_f \cup \partial\Omega$ converge to x. We need to show that $x \in BP_f \cup \partial\Omega$.
If $x \in \partial\Omega$, there is nothing to prove.
If $x \in \Omega$, we will prove $x \in BP_f$. Indeed, there exist $r_x > 0$ and $M > 0$ such that $x_n \in B_N(x, r_x) \subset \overline{B}_N(x, r_x) \subset \Omega$ for all $n > M$. From Lemma 7.2, we have $x_n \in BP_{f_{|B_N(x,r_x)}}$ for all $n > M$. On the other hand, from Corollary 7.1, and *(i)* of Theorem 3.2, one can easily see that the set $BP_{f_{|B_N(x,r_x)}} \cup \partial B_N(x, r_x)$ is closed. Therefore, the sequence $\{x_n\}$ converge to $x \in BP_{f_{|B_N(x,r_x)}} \cup \partial B_N(x, r_x)$. Recalling again Lemma 7.2, we obtain $x \in BP_f$.
Proof of (ii)

Since $BP_f \cup \partial\Omega$ is closed, BP_f is a Borel set. Therefore, it is sufficient to prove that for all $x \in BP_f$, the $\mathcal{L}^N - density$ of BP_f at x has zero value, i.e, for all $x \in BP_f$

$$D^N_{BP_f}(x) = \lim_{\delta \to 0} \frac{\mathcal{L}^N(BP_f \cap B_N(x, \delta))}{\mathcal{L}^N(B_N(x, \delta))} = 0. \tag{6.1}$$

Indeed, for all $x \in BP_f \subseteq \Omega$, there exists $r_x > 0$ such that $\overline{B}_N(x, r_x) \subset \Omega$. From Lemma 7.2 , Lemma 7.3, Corollary 7.1 and Theorem 3.2, one can easily get

$$\mathcal{L}^N(BP_f \cap B(x, r_x)) = \mathcal{L}^N(BP_{f_{|B_N(x,r_x)}}) = 0. \tag{6.2}$$

And (6.1) follows.

Proof of Corollary 3.1
From Theorem 3.1 we have
The set Ω_P is open. The function $f_{|\Omega_P} :\longrightarrow \mathbb{R}$ is a continuous function and
i) The set hypo$(f_{|\Omega_P})$ satisfies the $\theta-$ *external sphere condition*.
ii) For every $x \in \Omega_P$, the set $N^P_{hypo(f_{|\Omega_P})}(x, f_{|\Omega_P}(x))$ is pointed.
From [8] or [10], the proof is completed.

Proof of Corollary 3.3

Using the Proposition (3.1) in [8], the hypo(T) satisfies the $\theta-$*external sphere condition*. Applying Corollary 3.1 for $f = T(.)$, we get the conclusion.

7 Appendix

The first Lemma is a geometric Lemma which is needed in the proof of Lemma 4.3 and Lemma 4.4.

Lemma 7.1 *Let $C \in \mathbb{R}^N$ be a compact set which does not contain* 0. *We denote the positive cone generated by C as*

$$H_C^+ = \text{span}^+(C) = \{\sum_{i=1}^{k} \alpha_i c_i \mid c_i \in C \text{ and } \alpha_\text{i} \geq 0\}.$$

Assume that H_C^+ is pointed. Then:
i) H_C^+ is closed.
ii) There exists a constant $\delta_0 > 0$ such that for all $0 \neq x_1, x_2 \in H_C^+$, it holds

$$\langle \frac{x_1}{\|x_1\|} \ , \ \frac{x_2}{\|x_2\|} \rangle > -1 + \delta_0. \tag{7.1}$$

Proof
Proof of (i)
Let a sequence $\{x_n\} \subset H_C^+$ converge to x. We need to prove that $x \in H_C^+$. By Caratheodory Theorem, we can write

$$x_n = \sum_{i=1}^{N+1} \alpha_i^n c_n^i, \text{ where } \alpha_n^i \geq 0, \ c_n^i \in C. \tag{7.2}$$

Assume without loss of generality that $\lim_{n\to\infty} c_n^i = \overline{c^i} \in C$ for all $i \in \{1, 2, .., N+1\}$.
If $\sum_{i=1}^{N+1} \alpha_n^i$ is unbounded, we extract subsequences $\{\alpha_{n_k}^i\} \subseteq \{\alpha_n^i\}$ such that

$$\frac{\alpha_{n_k}^i}{\sum_{i=1}^{N+1} \alpha_{n_k}^i} = \overline{\alpha^i} \geq 0 \ \text{ and } \ \lim_{n_k\to\infty} \sum_{i=1}^{N+1} \alpha_{n_k}^i = +\infty.$$

Therefore, from (7.2) and $\lim_{n\to\infty} x^n = x$ we get

$$\sum_{i=1}^{N+1} \overline{\alpha^i} \ \overline{c^i} = \lim_{n_k\to\infty} \frac{x_{n_k}}{\sum_{i=1}^{N+1} \alpha_{n_k}^i} = 0. \tag{7.3}$$

Note that $\overline{\alpha^i} \geq 0$, $\sum_{i=1}^{N+1} \overline{\alpha^i} = 1$ and $\overline{c^i} \neq 0$, we recall (7.3) to obtain that the cone H_C^+ contains at least one line. This is a contradiction.
Thus $\sum_{i=1}^{N+1} \alpha_n^i$ is bounded. It implies that the sequences $\{\alpha_n^i\}$ are bounded for all $i \in \{1, 2, ..., N+1\}$ since $\alpha_n^i \geq 0$. We extract subsequences $\{\alpha_{n_k}^i\} \subseteq \{\alpha_n^i\}$ such that

$$\lim_{n_k\to\infty} \alpha_{n_k}^i = \overline{\alpha^i} \geq 0 \text{ for all } i \in \{1, 2, ..., N+1\}.$$

From the above equality and (7.2) , we have

$$x = \lim_{n\to\infty} x_n = \lim_{n_k\to\infty} x_{n_k} = \lim_{n_k\to\infty} \sum_{i=1}^{N+1} \alpha_{n_k}^i c_{n_k}^i = \sum_{i=1}^{N+1} \overline{\alpha^i} \ \overline{c^i}.$$

This implies $x \in H_C^+$.
Proof of (ii)
Assume by contradiction that there exist two sequences $\{x_1^n\}$, $\{x_2^n\}$ contained in H_C^+ such that $\|x_1^n\| = \|x_2^n\| = 1$ and

$$\lim_{n\to\infty} \langle x_1^n \ , \ x_2^n \rangle \ = -1. \tag{7.4}$$

Assume without loss of generality that $\lim_{n\to\infty} x_1^n = \overline{x_1}$ and $\lim_{n\to\infty} x_1^n = \overline{x_2}$. Recalling (7.4), we obtain that $-\overline{x_1} = \overline{x_2}$. Moreover, since H_C^+ is closed, we have $x_1, x_2 \in H_C^+$. Therefore H_C^+ contains at least one line. This is a contradiction.

The second Lemma is necessary to use Theorem 3.2 in the proof of our main Theorem.

Lemma 7.2 *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $g : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(g) satisfies the $\theta-$ external sphere condition where $\theta : \Omega \longrightarrow [0, +\infty)$ is continuous. Let, for all $x \in \Omega$, $r_x > 0$ be such that $\overline{B}_N(x, r_x) \subset \Omega$. Then*
i) The hypograph of the restricted function $g_{|B_N(x,r_x)} : B_N(x, r_x) \longrightarrow \mathbb{R}$ satisfies the θ_x- external sphere condition with $\theta_x = \max\{\theta(y) \mid y \in \bar{B}_N(x, r_x)\}$.
ii) $BP_g \cap B_N(x, r_x) = BP_{g_{|B_N(x,r_x)}}$.

Proof
Proof of (i)
Let $z \in B_N(x, r_x)$, there exists a vector $0 \neq \xi \in N^P_{hypo(g)}(z, g(z))$ realized by a ball of radius $\theta(z)$, i.e, for all $y \in \Omega$ and for $\beta \leq g(y)$, it holds

$$\langle \frac{\xi}{\|\xi\|} \ , \ (y, \beta) - (z, g(z)) \rangle \ \leq \ \theta(z) \ (\|y - z\|^2 + |\beta - g(z)|^2). \tag{7.5}$$

Thus, for all $y \in B_N(x, r_x)$ and for all $\beta \leq g_{|B_N(x,r_x)}(y)$, it holds

$$\langle \frac{\xi}{\|\xi\|} \ , \ (y, \beta) - (z, g_{|B_N(x,r_x)}(z)) \rangle \ \leq \ \theta_x \ (\|y - z\|^2 + |\beta - g_{|B_N(x,r_x)}(z)|^2). \tag{7.6}$$

The proof is completed.
Proof of (ii)

It is similar to the previous proof. Indeed, if $0 \neq \xi \in N^P_{hypo(g)}(z, g(z))$ then $0 \neq \xi \in N^P_{hypo(g_{|B_N(x,r_x)})}(z, g_{|B_N(x,r_x)}(z))$. Therefore, $BP_g \cap B_N(x, r_x) \subseteq BP_{g_{|B_N(x,r_x)}}$.

We are going now to prove $BP_{g_{|B_N(x,r_x)}} \subseteq BP_g$. It is sufficient to prove that if $0 \neq \xi \in N^P_{hypo(g_{|B_N(x,r_x)})}(z, g_{|B_N(x,r_x)}(z))$ then $0 \neq \xi \in N^P_{hypo(g)}(z, g(z))$. Indeed, $0 \neq \xi \in N^P_{hypo(g_{|B_N(x,r_x)})}(z, g_{|B_N(x,r_x)}(z))$, i.e, there exists a constant $\sigma > 0$ such that for all $y \in B_N(x, r_x)$ and for all $\beta \leq g_{|B_N(x,r_x)}(y)$, it holds

$$\langle \frac{\xi}{\|\xi\|} \ , \ (y, \beta) - (z, g_{|B_N(x,r_x)}(z)) \rangle \ \leq \ \sigma \ (\|y - z\|^2 + |\beta - g_{|B_N(x,r_x)}(z)|^2). \tag{7.7}$$

Therefore, for all $y \in B_N(x, r_x)$ and for all $\beta \leq g(y)$, it holds

$$\langle \frac{\xi}{\|\xi\|} \ , \ (y, \beta) - (z, g_(z)) \rangle \ \leq \ \sigma \ (\|y - z\|^2 + |\beta - g(z)|^2). \tag{7.8}$$

Since $z \in B_N(x, r_x)$, one can easily get from (7.8) that there exists a constant $\sigma_1 > 0$ such that the inequality

$$\langle \frac{\xi}{\|\xi\|} \ , \ (y, \beta) - (z, g_(z)) \rangle \ \leq \ \sigma_1 \ (\|y - z\|^2 + |\beta - g(z)|^2)$$

holds for all $y \in \Omega$ and for all $\beta \leq g(y)$.
It means that $\xi \in N^P_{hypo(g)}(z, g(z))$. The proof is completed.

The last one is a technical Lemma which is used to simplify our main proofs.

Lemma 7.3 *Let $g : \Omega \longrightarrow \mathbb{R}$ be continuous and let $\gamma > 0$. We denote by $g^\gamma : \gamma\Omega \longrightarrow \mathbb{R}$, the $\gamma-$stretched function of g, as follows:*

$$g^\gamma(y) = g(\frac{y}{\gamma}) \quad \text{ for all } y \in \gamma\Omega.$$

Assume that (ξ, λ) is a proximal normal vector to $hypo(g)$ at $(x, g(x))$ realized by a ball of radius ρ. Then $(\frac{\xi}{\gamma}, \lambda)$ is a proximal normal vector to $hypo(g^\gamma)$ at $(\gamma x, g^\gamma(\gamma x))$ realized by a ball of radius $\rho\frac{\gamma^2}{(1+\gamma^2)^{3/2}}$.

Proof
For all $z \in \Omega$ and for all $\beta \leq g(z)$, it holds

$$\langle \frac{(\xi, \lambda)}{\|(\xi, \lambda)\|} \ , \ (z, \beta) - (x, g(x)) \rangle \ \leq \ \frac{1}{2\rho}(\|z - x\|^2 + |\beta - g(x)|^2).$$

Equivalently, for all $\gamma z \in \gamma\Omega$ and for all $\beta \leq g^\gamma(\gamma z)$, it holds

$$\langle \frac{(\frac{\xi}{\gamma}, \lambda)}{\|(\xi, \lambda)\|} \ , \ (\gamma z, \beta) - (\gamma x, g^\gamma(\gamma x)) \rangle \ \leq \ \frac{1}{2\rho}(\frac{1}{\gamma^2}\|\gamma z - \gamma x\|^2 + |\beta - g^\gamma(\gamma x)|^2). \tag{7.9}$$

Since $\|(\xi, \lambda)\| \leq \sqrt{\gamma^2 + 1} \ \|(\frac{\xi}{\gamma}, \lambda)\|$, one can easily get from (7.9) that for all $\bar{z} = \gamma z \in \gamma\Omega$ and for all $\beta \leq g^\gamma(\bar{z})$, it holds

$$\langle \frac{(\frac{\xi}{\gamma}, \lambda)}{\|(\frac{\xi}{\gamma}, \lambda)\|} \ , \ (\bar{z}, \beta) - (\gamma x, g^\gamma(\gamma x)) \rangle \ \leq \ \frac{1}{2\rho\frac{\gamma^2}{(1+\gamma^2)^{3/2}}} \ (\|\bar{z} - \gamma x\|^2 + |\beta - g^\gamma(\gamma x)|^2) \quad . \tag{7.10}$$

The proof is completed.
The below statement follows immediately from the previous Lemma

Corollary 7.1 *for every $\gamma > 0$, it holds*

$$BP_{g^\gamma} = \gamma BP_g.$$

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