A model of debt with bankruptcy risk and currency devaluation

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Abstract

The paper studies a system of Hamilton-Jacobi equations, arising from a stochastic optimal debt management problem in an infinite time horizon with exponential discount, modeled as a noncooperative interaction between a borrower and a pool of risk-neutral lenders. In this model, the borrower is a sovereign state that can decide how much to devaluate its currency and which fraction of its income should used to repay the debt. Moreover, the borrower has possibility of going bankrupt at a random time and must declare bankruptcy if the debt reaches to a threshold x^*. When bankruptcy occurs, the lenders only recover a fraction of their capital. To offset the possible loss of part of their investment, the lenders buy bonds at a discounted price which is not given a priori. This leads to a nonstandard optimal control problem. We establish an existence result of solutions to this system and in turn recover optimal feedback payment strategy $u^*(x)$ and currency devaluation $v^*(x)$. In addition, the behavior of (u^*, v^*) near 0 and x^* is studied.

1 Introduction

Consider a system of Hamilton-Jacobi equations

$$\begin{cases} (r+\rho(x))V \;=\; \rho(x)\cdot B + H(x,V',p) + \dfrac{\sigma^2 x^2}{2}\cdot V''\,, \\[2ex] (r+\lambda+v(x))\cdot p-(r+\lambda) \;=\; \rho(x)\cdot[\theta(x)-p] + H_\xi(x,V',p)\cdot p' + \dfrac{(\sigma x)^2}{2}\cdot p''\,, \end{cases} \tag{1.1}$$

motivated by a stochastic optimal debt management problem in infinite time horizon with exponential discount, modeled as a noncooperative interaction between a borrower and a pool of risk-neutral lenders. Here, the independent variable x is the debt-to-income ratio and V is the value function for the borrower who is a sovereign state that can decide the devaluation rate of its currency v and which fraction of its income should used to repay the debt. The borrower has a possibility go to bankrupt with an instantaneous bankruptcy risk ρ, and must declare bankruptcy when the debt-to-income ratio reaches to a threshold $x^* > 0$. The salvage function $0 < \theta \leq 1$ determines the fraction of capital that can be recovered by lenders when bankruptcy occurs. To offset the possible loss of part of their investment, the lenders buy bonds at a discounted rate p which is not given a priori. Rather, it is determined by the expected evolution of the debt-to-income ratio at all future times. Hence it depends globally on the entire feedback controls u and v. This leads to a nonstandard optimal control problem, and a "solution" must be understood as a Nash equilibrium, where the strategy implemented by the borrower represents the best reply to the strategy adopted by the lenders, and conversely.

In the economics literature, some related models of debt and bankruptcy, with a focus on mathematical analysis, can be found in [2, 3, 4, 6, 7]. In [4], the borrower has a fixed income and large values of the debt determine a bankruptcy risk, adding uncertainty to the model. In [7], a numerical analysis was performed of a similar model in which the uncertainty comes not from bankruptcy risk but from random evolution of the borrower's income. An analytical study of a slight variant of the model was performed in [3] where no currency devaluation is available to the government. The authors constructed optimal feedback solutions in the stochastic case and provided an explicit formula for the optimal strategy in the deterministic case. Moreover, their analysis also shows how the expected total cost of servicing the total debt together with the bankruptcy cost are affected by different choices of x^*. Interestingly, under a natural assumption, the possibility of a *"Ponzi scheme"*, where the old debt is serviced by initiating more and more new loans, can be ruled out. This study is continued by adding the option for currency devaluation in [6]. Recently, a stochastic model with no currency devaluation but with uncertainty coming from bankruptcy risk was analyzed in [2].

The aim of the present paper is to provide a detailed mathematical analysis of a more general model of debt and bankruptcy. When the currency devaluation is not an option for the borrower ($v \equiv 0$), or the bankruptcy risk

$$\rho(x) \;=\; \begin{cases} 0 & \text{if } x < x^* \\ +\infty & \text{if } x = x^*, \end{cases}$$

our model reduces to the one analyzed in [2] or [6], respectively. We establish the existence result for the system of Hamilton-Jacobi equations (1.1). In turn, this yields the existence of optimal feedback controls u^*, v^* which minimize the expected cost to the borrower. More precisely, let $L : [0,1[\to [0,+\infty[$ be the cost for the borrower to implement the control strategy $u(\cdot)$ and let $c : [0, v_{\max}[\to [0,+\infty[$ be a social cost resulting by devaluation, i.e., the increasing cost of the welfare and of the imported goods.

- Given the debt-to-income ratio threshold x^* and the bankruptcy risk function ρ with $\lim_{x\to x^*-} \rho(x) = +\infty$, if L and c are strictly convex then the system (1.1) admits a $\mathcal{C}^2$ solution (V,p) in $[0,x^*]$. This implies that an optimal feedback solution to the model of debt and bankruptcy is

$$u^*(x) \;=\; \underset{w\in[0,1]}{\operatorname{argmin}} \left\{ L(w) - w\cdot\frac{V(x)}{p(x)} \right\}, \qquad v^*(x) \;=\; \underset{v\in[0,v_{\max}]}{\operatorname{argmin}} \left\{ c(v) - vxV'(x) \right\}.$$

Since ρ goes to $+\infty$ as x tends to x^*, the system (1.1) is not uniformly elliptic at 0 and x^*. To handle this difficulty, the classical idea is to construct solutions of approximate systems as steady states of corresponding auxiliary parabolic systems. In order to obtain a solution to the original system (1.1), we derive explicit a priori estimates on the derivatives of approximate solutions. As a consequence, the devaluation of currency is not optimal when the debt-to-income ratio x is sufficiently small. In addition, we provide a lower and an upper bounds for the value function V as a sub- and super-solution to the first equation of (1.1). Relying on these bounds, we show that when x is sufficiently close to x^*

- if the risk of bankruptcy ρ slowly approaches to infinity, i.e.,

$$\int_0^{x^*} \rho(t)\ dt < +\infty,$$

  then the optimal strategy of borrower involves continuously devaluating its currency and making payment,

- conversely, if the risk of bankruptcy ρ quickly approaches to infinity , i.e.

$$\lim_{x\to x^*-} \rho(x)(x^*-x)^2 \;=\; +\infty,$$

  then any action to reduce the debt is not optimal.

The remainder of the paper is organized as follows. In Section 2, we provide a more detailed description of the model and derive the equations satisfied by the value function V and the discounted bond price p. In Section 3, we begin by showing existence of optimal controls in feedback form. Finally, we close by performing a more detailed analysis of the behavior of the optimal feedback controls near x^*, under certain assumptions on the bankruptcy risk.

2 Model derivation and system of Hamilton-Jacobi equations

2.1 Model derivation

We introduce our optimal debt management problem in infinite time horizon, modeled as a noncooperative interaction between a borrower and a pool of risk-neutral lenders. Let $v(t)$ be the devaluation rate at time t, regarded as an control. The total income Y is governed by a stochastic process

$$dY \;=\; (\mu + v(t))Y(t)dt + \sigma Y(t)dW,$$

where W is a Brownian motion on a filtered probability space, $\sigma > 0$ is the volatility, and μ is the average growth rate of the economy. We denote by $X(t)$ the total debt of a borrower, financed by issuing bonds, and

- r = the interest rate paid on bonds;
- λ = the rate at which the borrower pays back the principal.

When an investor buys a bond of unit nominal value, he will receive a continuous stream of payments with intensity $(r+\lambda)e^{-\lambda t}$. If no bankruptcy occurs, the payoff for an investor will thus be

$$\int_0^{\infty} e^{-rt}(r+\lambda)e^{-\lambda t}\, dt \;=\; 1.$$

Otherwise, a lender recovers only a fraction $\theta \in [0, 1]$ of his outstanding capital which depends on the total amount of debt at the time on bankruptcy. To offset this possible loss, the investor buys a bond with unit nominal value at a discounted price $p \in [0,1]$. Hence, as in [7], the total debt evolves according to

$$\dot{X}(t) \;=\; -\lambda X(t) + \frac{(\lambda+r)X(t) - U(t)}{p(t)},$$

where $U(t)$ is the rate of payments that the borrower chooses to make to the lenders at time t. By Itō's formula, one derives the stochastic evolution of the the debt-to-income ratio $x = X/Y$

$$dx(t) \;=\; \left[\left(\frac{\lambda+r}{p(t)} - \lambda + \sigma^2 - \mu - v(t)\right)x(t) - \frac{u(t)}{p(t)}\right] dt - \sigma\, x(t)\, dW, \tag{2.1}$$

where $u = U/Y \in [0,1]$ is the portion of the total income allocated to reduce the debt.

In this model, we assume there exists a threshold $x^* > 0$ beyond which bankruptcy immediately occurs. Define T_{x^*} as the time when the borrower's debt first reaches x^*

$$T_{x^*} \;:=\; \inf\{t > 0\,;\;\; x(t) = x^*\}\,. \tag{2.2}$$

As in [2], the borrower may go bankrupt at random time $\mathcal{T}_B$ before T_{x^*}. If at time τ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then the probability that bankruptcy will occur shortly after time τ is measured by

$$\text{Prob.}\;\Big\{\mathcal{T}_B \in [\tau, \tau+\varepsilon] \;\Big|\; \mathcal{T}_B > \tau, x(\tau) = y\Big\} \;=\; \rho(y)\cdot\varepsilon + o(\varepsilon)\,.$$

Here, $\rho : [0, x^*[\,\to [0, +\infty[$ is the instantaneous bankruptcy risk, an increasing function of x with $\lim_{x\to x^{*-}} \rho(x) = +\infty$. Hence, the probability that the borrower is not yet bankrupt at time $t > 0$ is computed as

$$\text{Prob.}\;\{\mathcal{T}_B > t\} \;=\; \begin{cases} \exp\left\{-\displaystyle\int_0^t \rho(x(\tau))\; d\tau\right\} & \text{if} \quad t \;<\; T_{x^*}, \\[2ex] 0 & \text{if} \quad t \geq T_{x^*}\,. \end{cases} \tag{2.3}$$

Let $\theta(x)$ be the salvage rate that determines the fraction of the outstanding capital that can be recovered by lenders if bankruptcy occurs when the debt has size x. As in [3, 4, 7], the discounted bond price is uniquely determined by the competition of a pool of risk-neutral lenders

$$\begin{aligned} p \;=\; E\Big[\int_0^{\mathcal{T}_B} (r+\lambda)\exp\left\{-\int_0^\tau \big(\lambda + r + v(x(s))\big)\, ds\right\} d\tau \\ + \exp\left\{-\int_0^{\mathcal{T}_B} (r+\lambda+v(x(\tau))d\tau\right\}\cdot\theta(x^*)\Big]. \end{aligned} \tag{2.4}$$

Given an initial size x_0 of the debt, the borrower wants to find a pair of optimal controls (u, v) which minimizes his total expected cost, exponentially discounted in time:

$$\text{Minimize } J[x_0, u, v] \;=\; E\left[\int_0^{\mathcal{T}_B} e^{-rt}\cdot L(u(t)) + c(v(t))dt + e^{-r\mathcal{T}_B} B\right]_{x(0)=x_0}, \tag{2.5}$$

where $c(v)$ is the social cost resulting from devaluation, $L(u)$ is the cost to the borrower for putting income towards paying the debt, and B is the cost of bankruptcy.

To complete this subsection, we introduce standard assumptions in our model. Concerning the functions θ, ρ, L and c, we shall assume

(A1) *The map* $\theta : [0, x^*] \to]0,1]$ *is non-increasing, continuous, and locally Lipschitz in* $[0, x^*[$.

(A2) *The function* $\rho : [0, x^*[\to [0, +\infty[$ *is continuously differentiable for* $x \in [0, x^*[$, *and satisfies*

$$\rho(0) \;=\; 0, \qquad \rho'(x) \;\geq\; 0 \qquad \text{and} \qquad \lim_{x \to x^*} \rho(x) \;=\; +\infty.$$

(A3) *The function* $(L, c) : [0,1[\times[0, v_{\max}[\to [0, +\infty[\times[0, +\infty[$ *is twice continuously differentiable such that*

$$L'(u),\ c'(v) \;>\; 0, \qquad L''(u),\ c''(v) \;\geq\; \delta_0$$

and

$$L(0) \;=\; c(0) \;=\; 0, \qquad \lim_{u \to 1} L(u) \;=\; +\infty, \qquad \lim_{v \to v_{\max}} c(v) \;=\; +\infty$$

for some constant $\delta_0 > 0$ *and* $v_{\max} \geq 0$.

2.2 System of Hamilton-Jacobi equations

The stochastic control system (2.1)–(2.4) is not standard. Indeed, the discount price p in (2.4) depends on the debt-to-income ratio not only at the present time t but also at all future times. Therefore, it is natural to look at this model in a feedback form. More precisely, assume that the optimal control has feedback form, so that

$$u \;=\; u^*(x), \qquad v \;=\; v^*(x) \qquad\qquad \text{for} \quad x \in [0, x^*].$$

Definition 2.1 (Stochastic optimal feedback solution)**.** *We say that a triple of functions* $(u^*(x), v^*(x), p(x))$ *provides an optimal solution to the problem of optimal debt management (2.1)–(2.5) if:*

(i) Given the function $p(\cdot)$*, for every initial value* $x_0 \in [0, x^*]$ *the feedback control* (u^*, v^*) *with bankruptcy time* $\mathcal{T}_B$ *as in (2.3) provides an optimal solution to the stochastic control problem (2.5), with dynamics (2.1).*

(ii) Given the feedback control $(u^*(\cdot), v^*(\cdot))$*, the discounted price* p *satisfies (2.4), where* $\mathcal{T}_B$ *is the bankruptcy time (2.3) determined by the dynamics (2.1).*

Let us introduce the associated Hamiltonian function to (2.1)-(2.4)

$$H(x, \xi, p) \;=\; \min_{(u,v) \in [0,1[\times[0, v_{\max}[} \left\{ L(u) + c(v) - \left(\frac{u}{p} + xv \right) \cdot \xi \right\} + \left(\frac{\lambda + r}{p} - \lambda + \sigma^2 - \mu \right) x\, \xi \tag{2.6}$$

and two functions

$$\tilde{u}(\xi, p) \;=\; \underset{u \in [0,1]}{\operatorname{argmin}} \left\{ L(u) - u\, \frac{\xi}{p} \right\}, \qquad \tilde{v}(x, \xi) \;=\; \underset{v \in [0, v_{\max}]}{\operatorname{argmin}} \left\{ c(v) - vx\xi \right\}. \tag{2.7}$$

Under the assumption **(A3)**, a direct computation yields

$$\tilde{v}(x, \xi) = \begin{cases} 0 & \text{if} \quad \xi x \leq c'(0) \\ (c')^{-1}(\xi x) & \text{if} \quad \xi x > c'(0) \end{cases} \quad \text{and} \quad \tilde{u}(\xi, p) = \begin{cases} 0, & \text{if} \quad \frac{\xi}{p} \leq L'(0) \\ (L')^{-1}\left(\frac{\xi}{p}\right), & \text{if} \quad \frac{\xi}{p} > L'(0). \end{cases} \tag{2.8}$$

Assume that the discount bond price, $p(\cdot)$ is given. The value function

$$V(x_0) \;=\; \inf_{u,v} J[x_0,u,v],$$

solves the following second-order ODE

$$(r+\rho(x))V(x) \;=\; \rho(x)\cdot B + H(x,V'(x),p(x)) + \frac{\sigma^2 x^2}{2}\cdot V''(x),$$

with boundary values

$$V(0) \;=\; 0 \qquad \text{and} \qquad V(x^*) \;=\; B.$$

The optimality condition and (2.8) imply that the feedback strategies are recovered by

$$u^*(x) \;=\; \tilde{u}(V'(x),p(x)) \;=\; \begin{cases} 0, & \text{if} \quad \dfrac{V'(x)}{p(x)} \;\leq\; L'(0), \\[2ex] (L')^{-1}\left(\dfrac{V'(x)}{p(x)}\right), & \text{if} \quad \dfrac{V'(x)}{p(x)} \;>\; L'(0), \end{cases} \tag{2.9}$$

and

$$v^*(x) \;=\; \tilde{v}(x,V'(x)) \;=\; \begin{cases} 0, & \text{if} \quad V'(x)x \;\leq\; c'(0), \\[1ex] (c')^{-1}\left(V'(x)x\right) & \text{if} \quad V'(x)x \;>\; c'(0)\,. \end{cases} \tag{2.10}$$

On the other hand, suppose that a pair of optimal feedback controls (u^*,v^*) is known. The Feynman-Kac formula [9] implies that the discounted bond price p is the solution to second-order ODE

$$\begin{aligned}(r+\lambda+v^*(x))\cdot p(x) - (r+\lambda) = \rho(x)\cdot[\theta(x)-p(x)] + \\ H_\xi(x,V'(x),p(x))\cdot p'(x) + \frac{(\sigma x)^2}{2}\cdot p''(x),\end{aligned}$$

with $p(0)=1$ and $p(x^*)=\theta(x^*)$. Therefore, finding an optimal solution to the problem of optimal debt management (2.1)-(2.5) leads to the following system of second order implicit ODEs

$$\begin{cases} (r+\rho(x))V(x) \;=\; \rho(x)\cdot B + H(x,V'(x),p(x)) + \dfrac{\sigma^2 x^2}{2}\cdot V''(x)\,, \\[1ex] (r+\lambda+v(x))\cdot p(x) - (r+\lambda) \;=\; \rho(x)\cdot[\theta(x)-p(x)] + \\ \qquad\qquad H_\xi(x,V'(x),p(x))\cdot p'(x) + \dfrac{(\sigma x)^2}{2}\cdot p''(x)\,, \\[1ex] v(x) \;=\; \operatorname*{argmin}_{w\in[0,v_{\max}]} \left\{c(w) - wxV'(x)\right\}\,, \end{cases} \tag{2.11}$$

with boundary conditions

$$V(0) \;=\; 0, \quad V(x^*) \;=\; B \quad \text{and} \quad p(0) \;=\; 1, \quad p(x^*) \;=\; \theta(x^*). \tag{2.12}$$

To complete this subsection, let us collect some useful properties of Hamiltonian H.

Lemma 2.2. *If (**A3**) holds then H is continuous differentiable and its gradient at points $(x,\xi,p) \in [0,+\infty[\times[0,+\infty[\times]0,1]$ can be expressed by*

$$\begin{cases} H_x(x,\xi,p) &=& \Big[(\lambda+r) - p(\lambda+\mu+\tilde{v}(x,\xi)-\sigma^2)\Big]\cdot\dfrac{\xi}{p}, \\ H_\xi(x,\xi,p) &=& \dfrac{1}{p}\cdot\Big[x\big((\lambda+r)-p(\lambda+\mu+\tilde{v}(x,\xi)-\sigma^2)\big)-\tilde{u}(\xi,p)\Big], \\ H_p(x,\xi,p) &=& (\tilde{u}(\xi,p)-x(\lambda+r))\cdot\dfrac{\xi}{p^2}, \end{cases} \tag{2.13}$$

where the functions $\tilde{u},\tilde{v}$ are defined in (2.8). Furthermore,

(i). for all $\xi \in [0,+\infty[$, the function H satisfies

$$\left(\frac{(\lambda+r)x-1}{p}+(\sigma^2-\lambda-\mu-v(x))x\right)\cdot\xi \;\le\; H(x,\xi,p) \;\le\; \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x\xi\,,$$

$$\frac{(\lambda+r)x-1}{p}+(\sigma^2-\lambda-\mu-v(x))x \;\le\; H_\xi(x,\xi,p) \le \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x\,;$$

(ii). for every $(x,p) \in]0,+\infty[\times]0,+\infty[$ the map $\xi \mapsto H(x,\xi,p)$ is concave down and satisfies

$$H(x,0,p) \;=\; 0, \tag{2.14}$$

$$H_\xi(x,0,p) \;=\; \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x\,. \tag{2.15}$$

Proof. See [6, Lemma B2 and Lemma B3]. □

Corollary 2.1. *Suppose the assumption (**A3**) holds. Then for all $(x,\xi,p,v) \in [0,x^*]\times]0,+\infty[\times[\theta_{\min},+\infty[\times[0,v_{\max}]$, it holds that*

$$|H(x,\xi,p)| \;\le\; K_1\cdot\xi, \qquad \textit{and} \qquad |H_\xi(x,\xi,p)| \;\le\; K_1,$$

where the constant $\theta_{\min} > 0$ is defined in (3.1) and

$$K_1 \;:=\; \max\left\{\left(\frac{\lambda+r}{\theta_{\min}}+\sigma^2\right)x^*,\; {\theta_{\min}}^{-1}+(\lambda+\mu+v_{\max})\,x^*\right\}. \tag{2.16}$$

Proof. By Lemma 2.2 we compute an upper bound

$$H(x,\xi,p) \;\le\; \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x\xi \;\le\; \left(\frac{\lambda+r}{\theta_{\min}}+\sigma^2\right)x^*\xi \;\le\; K_1\cdot\xi$$

and lower bound

$$\begin{aligned} H(x,\xi,p) \;&\ge\; \left(\frac{(\lambda+r)x-1}{p}+(\sigma^2-\lambda-\mu-v(x))x\right)\cdot\xi \\ &\ge\; -\left(\frac{1}{\theta_{\min}}+(\lambda+\mu+v_{\max})x^*\right)\cdot\xi \;\ge\; -K_1\cdot\xi, \end{aligned}$$

providing an uniform bound on H with respect to x and p. A similar analysis and application of Lemma 2.2 provides the uniform bound on H_ξ. □

3 Optimal feedback solutions

In this section, we will construct a solution of the system of Hamilton-Jacobi equations (2.11) with boundary conditions (2.12) for a given bankruptcy threshold x^*. In turn, this result yields the existence of optimal feedback controls for the problem of debt management (2.1)–(2.5). Finally, we show how the bankruptcy risk affects the behavior of the optimal feedback control as the debt tends to x^*.

3.1 Existence results for system of Hamilton-Jacobi equations

Before stating our main result, let us introduce the constant which will be a lower bound of the discount bond price p,

$$\theta_{\min} \ := \ \min\left\{\theta(x^*), \frac{r+\lambda}{r+\lambda+v_{\max}}\right\}. \tag{3.1}$$

Theorem 3.1. *Under the standard assumptions* **(A1) - (A3)**, *the system of second order ODEs* (2.11) *with boundary conditions* (2.12) *admits a solution* $(V,p)\,:\,[0,x^*]\to[0,B]\times\left[\theta_{\min},1\right]$ *of class* C^2 *such that* V *is monotone increasing and*

$$v(x) \ = \ \arg\min_{w\geq 0}\{c(w)-wxV'\} \ = \ 0 \qquad \textit{for all} \quad x\in\left[0,\frac{c'(0)}{M^*}\right]. \tag{3.2}$$

for a constant M^* *which can be explicitly computed .*

It is well-known (see [5, Theorem 4.1] or [8, Theorem 11.2.2]) that having constructed a solution (V,p) to the boundary value problem (2.11)-(2.12), then (u^*,v^*) in (2.9)-(2.10) is an optimal solution to the problem of optimal debt management (2.1)-(2.5). As a consequence of Theorem 3.1, we obtain the following result.

Corollary 3.1. *Under the same assumptions of Theorem 3.1, the debt management problem (2.1)-(2.5) admits an optimal control strategy in feedback form. Moreover, there exists a threshold such that the optimal control strategy does not use currency devaluation for values below that threshold.*

Toward the proof of Theorem 3.1, we introduce a system of second order implicit ODEs which approximates (2.11). More precisely, for any given $\varepsilon>0$, let ρ_ε be a monotone increasing Lipschitz function on $[0,x^*]$ defined by

$$\rho_\varepsilon(x) \ = \ \begin{cases}\rho(x) & \text{if} \quad x\in[0,x_\varepsilon]\\[2mm] \dfrac{1}{\varepsilon} & \text{if} \quad x\in[x_\varepsilon,x^*]\end{cases} \qquad \text{with} \qquad x_\varepsilon \ := \ \rho^{-1}\left(\frac{1}{\varepsilon}\right). \tag{3.3}$$

Consider the following system of implicit ODEs

$$\begin{cases}(r+\rho_\varepsilon(x))\cdot V \ = \ \rho_\varepsilon(x)\cdot B + H(x,V',p+\varepsilon)+\left(\dfrac{\sigma^2x^2}{2}+\varepsilon\right)\cdot V''\,,\\[2mm] (r+\lambda+\tilde{v}(x,V'))\cdot p-(r+\lambda) \ = \ \rho_\varepsilon(x)\cdot[\theta(x)-p]\\[2mm] \qquad\qquad\qquad\qquad\qquad\qquad + H_\xi(x,V',p+\varepsilon)\cdot p'+\left(\dfrac{(\sigma x)^2}{2}+\varepsilon\right)\cdot p''\,,\end{cases} \tag{3.4}$$

where the function

$$\tilde{v}(x,\xi) \;=\; \begin{cases} 0, & \text{if} \quad \xi x \le c'(0), \\ (c')^{-1}(\xi x), & \text{if} \quad \xi x > c'(0). \end{cases} \tag{3.5}$$

From the assumption **(A3)**, one can easily get that

$$\tilde{v}(x,\xi) \;\le\; v_{\max}, \qquad |\tilde{v}_x(x,\xi)| \;\le\; \frac{1}{\delta_0}\cdot|\xi| \qquad \text{and} \qquad |\tilde{v}_\xi(x,\xi)| \;\le\; \frac{1}{\delta_0}\cdot|x|. \tag{3.6}$$

We establish an existence result for (3.4) with boundary condition (2.12) by considering the auxiliary parabolic system whose steady states will provide a solution to (3.4). Following [1], we shall construct a compact, convex and positively invariant set of functions $(V,p):[0,x^*]^2 \mapsto [0,B]\times[\theta_{\min},1]$. A topological technique will then yield the existence of solutions $(V_\varepsilon,p_\varepsilon)$ of the system (3.4).

Lemma 3.2. *Assume that* **(A1) - (A3)** *hold. Then the system of ODEs (3.4) with boundary conditions (2.12) admits a C^2 solution $(V_\varepsilon,p_\varepsilon):[0,x^*]^2\to[0,B]\times[\theta_{\min},1]$ such that V_ε is increasing.*

Proof. Given a threshold of bankruptcy $x^*>0$, let's consider the parabolic system with the unknown $\mathbf{V}(t,x)$ and $\mathbf{p}(t,x)$

$$\begin{cases} \mathbf{V}_t \;=\; -\,(r+\rho_\varepsilon(x))\mathbf{V} + \;\rho_\varepsilon(x)\cdot B + H(x,\mathbf{V}_x,\mathbf{p}+\varepsilon) + \Big(\dfrac{\sigma^2x^2}{2}+\varepsilon\Big)\mathbf{V}_{xx}\,, \\ \mathbf{p}_t \;=\; -\,(r+\lambda+\tilde{v}(x,\mathbf{V}_x))\cdot\mathbf{p} + (r+\lambda) + \rho_\varepsilon(x)\cdot[\theta(x)-\mathbf{p}] + \\ \qquad\qquad H_\xi(x,\mathbf{V}_x,\mathbf{p}+\varepsilon)\cdot\mathbf{p}_x + \Big(\dfrac{(\sigma x)^2}{2}+\varepsilon\Big)\cdot\mathbf{p}_{xx}\,, \end{cases} \tag{3.7}$$

with the boundary conditions

$$\begin{cases} \mathbf{V}(t,0) \;=\; 0 \\ \mathbf{V}(t,x^*) \;=\; B \end{cases} \qquad \text{and} \qquad \begin{cases} \mathbf{p}(t,0) \;=\; 1 \\ \mathbf{p}(t,x^*) \;=\; \theta(x^*). \end{cases} \tag{3.8}$$

It is well-known (see [1, Theorem 3]) that the parabolic system (3.7) with initial data

$$\mathbf{V}(0,x) \;=\; V_0(x) \qquad \text{and} \qquad \mathbf{p}(0,x) \;=\; p_0(x), \tag{3.9}$$

admits a unique solution $(\mathbf{V}(t,x),\mathbf{p}(t,x))\in C^2([0,T]\times[0,x^*])\times C^2([0,T]\times[0,x^*])$ for any $T>0$. Adopting a semigroup notation, we denote by $S_t(V_0,p_0)=(\mathbf{V}(t,\cdot),\mathbf{p}(t,\cdot))$ the solution to the system (3.7) with initial data (3.9).

We claim that the following closed and convex domain in $C^2([0,x^*])\times C^2([0,x^*])$

$$\mathcal{D} \;=\; \Big\{(V,p):[0,x^*]^2\to[0,B]\times[\theta_{\min},1] : (V,p)\in C^2, V'\ge 0, \text{ and (2.12) holds}\Big\},$$

is positively invariant under the semigroup S_t, namely

$$S_t(\mathcal{D}) \;\subseteq\; \mathcal{D} \qquad \text{for all} \;\; t\ge 0\,.$$

Indeed, as in [3], we consider the constant functions $(\mathbf{V}^{\pm}, \mathbf{p}^{\pm})$ defined on $[0,\infty[\times[0,x^*]$ such that

$$(\mathbf{V}^+, \mathbf{p}^+) \ \equiv \ (B, 1) \qquad \text{and} \qquad (\mathbf{V}^-, \mathbf{p}^-) \ \equiv \ (0, \theta_{\min})\,.$$

Recalling (2.14), one has

$$-(r+\rho_\varepsilon(x))\mathbf{V}^+ + \rho_\varepsilon(x)\cdot B + H(x, \mathbf{V}_x^+, \mathbf{p}+\varepsilon) + \Big(\frac{\sigma^2 x^2}{2} + \varepsilon\Big)\mathbf{V}_{xx}^+$$

$$= \ -(r+\rho_\varepsilon(x))\mathbf{V}^+ + \rho_\varepsilon(x)\cdot B \ = \ -rB \ \leq \ 0$$

and

$$-(r+\rho_\varepsilon(x))\mathbf{V}^- + \rho_\varepsilon(x)\cdot B + H(x, \mathbf{V}_x^-, \mathbf{p}+\varepsilon) + \Big(\frac{\sigma^2 x^2}{2} + \varepsilon\Big)\mathbf{V}_{xx}^- \ = \ \rho_\varepsilon(x)\cdot B \ \geq \ 0.$$

This implies that $\mathbf{V}^+$ is a supersolution and $\mathbf{V}^-$ is a subsolution of the first parabolic equation in (3.7). A standard comparison principle [10] yields

$$0 \ = \ \mathbf{V}^-(t,x) \ \leq \ \mathbf{V}(t,x) \ \leq \ \mathbf{V}^+(t,x) \ = \ B \qquad \text{for all } (t,x)\in[0,+\infty[\times[0,x^*].$$

Similarly, from (**A1**)-(**A3**), (3.1) and (3.5), it holds

$$-(r+\lambda+\widetilde{v}(x,\mathbf{V}_x))\cdot\mathbf{p}^+ + (r+\lambda) + \rho_\varepsilon(x)\cdot[\theta(x)-\mathbf{p}^+] + H_\xi(x,\mathbf{V}_x,\mathbf{p}^++\varepsilon)\cdot\mathbf{p}_x^+$$

$$+\Big(\frac{(\sigma x)^2}{2}+\varepsilon\Big)\cdot\mathbf{p}_{xx}^+ \ = \ -\tilde{v}(x,\mathbf{V}_x) + \rho_\varepsilon(x)\cdot[\theta(x)-1] \ \leq \ 0$$

and

$$-(r+\lambda+\widetilde{v}(x,\mathbf{V}_x))\cdot\mathbf{p}^- + (r+\lambda) + \rho_\varepsilon(x)\cdot[\theta(x)-\mathbf{p}^-] + H_\xi(x,\mathbf{V}_x,\mathbf{p}^-+\varepsilon)\cdot\mathbf{p}_x^-$$

$$+\Big(\frac{(\sigma x)^2}{2}+\varepsilon\Big)\cdot\mathbf{p}_{xx}^- \ = \ -(r+\lambda+\widetilde{v}(x,\mathbf{V}_x))\cdot\theta_{\min} + (r+\lambda)$$

$$+\rho_\varepsilon(x)\cdot[\theta(x)-\theta_{\min}] \ \geq \ (r+\lambda) - (r+\lambda+v_{\max})\cdot\theta_{\min} \ \geq \ 0.$$

Thus, $\mathbf{p}^+$ is a supersolution and $\mathbf{p}^-$ is a subsolution of the second parabolic equation in (3.7), and this yields

$$\theta_{\min} \ = \ \mathbf{p}^-(t,x) \ \leq \ \mathbf{p}(t,x) \ \leq \ \mathbf{p}^+(t,x) \ = \ 1 \qquad \text{for all } (t,x)\in[0,+\infty[\times[0,x^*].$$

Setting $\mathbf{W}(t,x) := \mathbf{V}_x(t,x)$ with initial condition $\mathbf{V}(0,x) = V_0(x)\in\mathcal{D}$, we have

$$\lim_{x\to 0^+}\mathbf{W}(t,x)\geq 0, \quad \text{and} \quad \lim_{x\to x^{*-}}\mathbf{W}(t,x)\geq 0, \quad \text{for all } t\in]0,+\infty[\,,$$

and, by definition of $\mathcal{D}$,

$$\mathbf{W}(0,x) \ \geq \ 0, \qquad \text{for all } x\in[0,x^*].$$

Differentiating the first equation in (3.4), we obtain that $\mathbf{W}$ solves the following ODE

$$\mathbf{W}_t \ = \ -(r+\rho_\varepsilon)\mathbf{W} + \rho_\varepsilon'(B-\mathbf{V}) + H_x + H_\xi\mathbf{W}_x + H_p\mathbf{p}_x + \sigma^2 x\mathbf{W}_x + \left(\frac{\sigma^2x^2}{2}+\varepsilon\right)\mathbf{W}_{xx}. \quad (3.10)$$

Since

$$(H_x, H_p)(x, 0, \mathbf{p}+\varepsilon) \ = \ 0 \qquad \text{and} \qquad \mathbf{V}(t,x) \ \leq \ B \qquad \text{for all } (t,x)\in[0,+\infty[\times[0,x^*],$$

one can easily see that the constant function 0 is a subsolution to (3.10). Thus, by [10],

$$\mathbf{W}(t,x) \;\geq\; 0 \qquad \text{for all } (t,x)\in[0,+\infty[\times[0,x^*]\,,$$

yielding the monotone increasing property of $\mathbf{V}$ with respect to x. From the bounds in Lemma 2.2 and the invariance of $\mathcal{D}$, we can apply [1, Theorem 3] to obtain the existence of a steady state solution $(V_\varepsilon,p_\varepsilon)\in\mathcal{D}$ for the system (3.7) which solves (3.4) and (2.12) such that V_ε is monotone increasing. □

In order to obtain a solution to (2.11) from (3.4) by passing ε to $0+$, we derive some *a priori* estimates on the derivatives of $(V_\varepsilon,p_\varepsilon)$.

Lemma 3.3. *Under the same assumptions in Lemma 3.2, let* $(V_\varepsilon,p_\varepsilon)$ *be a solution to* (3.4) *and* (2.12). *Then for every* $0<\varepsilon<1/2$*, it holds*

$$\|V_\varepsilon'\|_{\mathbf{L}^\infty([0,x^*])} \;\leq\; M^* \qquad \text{and} \qquad v_\varepsilon(x) \;=\; 0 \qquad \textit{for all } x\in\left[0,\frac{c'(0)}{M^*}\right], \tag{3.11}$$

where the constant M^* *is explicitly computed by*

$$M^* \;:=\; \max\left\{M_1, \exp\left(\frac{3K_1x^*}{2\sigma^2x_1^2}\right)\cdot\left(\frac{4B}{x^*}+\frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right)\right), \exp\left(\frac{2K_1x^*}{\sigma^2x_1^2}\right)\cdot\left(\frac{4B}{x^*}+\frac{rB}{K_1}\right)\right\},$$

with K_1 *defined in (2.16), and*

$$x_1 \;:=\; \min\left\{\frac{1}{6(\lambda+r+\sigma^2)},\frac{x^*}{2}\right\}, \qquad M_1 \;:=\; 8\left(L\left(\frac{1}{2}\right)+\rho(x_1)\cdot B\right). \tag{3.12}$$

Moreover, for any $\delta\in]0,x^*/2[$*, there exists a constant* $M_\delta>0$ *such that*

$$\|V_\varepsilon''\|_{\mathbf{L}^\infty(]\delta,x^*-\delta[)}+\|p_\varepsilon'\|_{\mathbf{L}^\infty(]\delta,x^*-\delta[)}+\|p_\varepsilon''\|_{\mathbf{L}^\infty(]\delta,x^*-\delta[)} \;\leq\; M_\delta. \tag{3.13}$$

Proof. **1.** Let us first prove (3.11). Let x_1 and M_1 be as in (3.12). From (2.6) and Lemma 3.2, for every $(x,\xi)\in[0,x_1]\times[M_1,\infty[$, it holds

$$\begin{aligned}
H(x,\xi,p_\varepsilon+\varepsilon) \;&\leq\; \min_{u\in[0,1]}\left\{L(u)-\frac{u}{p_\varepsilon+\varepsilon}\xi\right\}+\left(\frac{\lambda+r}{p_\varepsilon+\varepsilon}+\sigma^2\right)x\xi \\
&\leq\; L\left(\frac{1}{2}\right)+\frac{3(\lambda+r+\sigma^2)x-1}{2(p_\varepsilon+\varepsilon)}\cdot\xi \\
&\leq\; L\left(\frac{1}{2}\right)-\frac{\xi}{8} \;\leq\; L\left(\frac{1}{2}\right)-\frac{M_1}{8}\,.
\end{aligned}$$

From the definition of ρ_ε in (3.3), one has

$$\rho_\varepsilon(x) \;\leq\; \rho(x) \qquad \text{for all } x\in[0,x^*], \tag{3.14}$$

and **(A2)** implies that

$$\rho_\varepsilon(x)B+H(x,\xi,p_\varepsilon+\varepsilon) \;\leq\; \rho(x_1)B+L\left(\frac{1}{2}\right)-\frac{M_1}{8} \;<\; 0,$$

for all $(x,\xi)\in[0,x_1]\times[M_1,\infty[$. Since V_ε is non-negative, the first equation of (3.4) yields

$$V_\varepsilon''(x) \;>\; 0 \qquad \text{for all } x\in]0,x_1],$$

provided that $V_\varepsilon'(x) \geq M_1$ for all $x \in [0, x_1]$. Recalling from Lemma 3.2 that V_ε' is non-negative, we have that $\|V_\varepsilon'\|_{\mathbf{L}^\infty([0,x_1])}$ is bounded by M_1 or the maximal of V_ε' in $[0, x^*]$ is obtained at

$$x_m \;:=\; \arg\max_{x\in[0,x^*]} V_\varepsilon'(x) \;>\; x_1\,.$$

Let us establish an upper bound of V_ε' in $[x_1, x^*[$. Since $0 \leq x_1 \leq \frac{x^*}{2}$, by the mean value theorem, there exists a point $x_2 \in]x_1, \frac{3}{4}x^*[$ such that

$$V_\varepsilon'(x_2) \;=\; \frac{V_\varepsilon(\frac{3}{4}x^*) - V_\varepsilon(x_1)}{\frac{3}{4}x^* - x_1} \;\leq\; \frac{B}{\frac{3}{4}x^* - \frac{1}{2}x^*} \;\leq\; \frac{4B}{x^*}\,. \tag{3.15}$$

From the first equation of (3.4) we have

$$V_\varepsilon''(x) \;=\; \frac{2}{\sigma^2x^2 + 2\varepsilon}\cdot\left[rV_\varepsilon(x) + \rho_\varepsilon(x)\left(V_\varepsilon(x) - B\right) - H(x, V_\varepsilon'(x), p_\varepsilon(x) + \varepsilon)\right]. \tag{3.16}$$

Two cases are considered:

- For any $x \in [x_1, x_2]$, from (3.14), **(A2)**, and the above equality we have

$$\begin{aligned} V_\varepsilon''(x) \;&\geq\; \frac{-2}{\sigma^2x^2 + 2\varepsilon}\cdot\left(\rho_\varepsilon(x)B + \left|H(x, V_\varepsilon'(x), p_\varepsilon(x)+\varepsilon)\right|\right) \\ &\geq\; \frac{-2}{\sigma^2x_1^2}\cdot\left(\rho\left(\frac{3x^*}{4}\right)B + \left|H(x, V_\varepsilon'(x), p_\varepsilon(x)+\varepsilon)\right|\right), \end{aligned}$$

  where the last inequality holds because $x_2 \leq \frac{3x^*}{4}$. Since $p_\varepsilon(x) \geq \theta_{\min}$, by Lemma 2.1 it holds that

$$V_\varepsilon''(x) \;\geq\; \frac{-2}{\sigma^2x_1^2}\left(\rho\left(\frac{3x^*}{4}\right)B + K_1\cdot V_\varepsilon'(x)\right),$$

  where the constant K_1 is defined in (2.16). Thus, applying Grönwall's inequality in the interval $[x, x_2]$ with $x \in [x_1, x_2]$, we get

$$V_\varepsilon'(x) \;\leq\; \left(V_\varepsilon'(x_2) + \frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right)\right)\cdot\exp\left(\frac{2K_1}{\sigma^2x_1^2}(x_2 - x)\right) - \frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right).$$

  Recalling (3.15) we obtain that

$$\|V_\varepsilon'\|_{\mathbf{L}^\infty([x_1,x_2])} \;\leq\; \left(\frac{4B}{x^*} + \frac{B}{K_1}\cdot\rho\left(\frac{3x^*}{4}\right)\right)\cdot\exp\left(\frac{3K_1x^*}{2\sigma^2x_1^2}\right). \tag{3.17}$$

- Similarly, for any $x \in [x_2, x^*]$, it holds

$$V_\varepsilon''(x) \;\leq\; \frac{2}{\sigma^2x_2^2}\left(rB + K_1\cdot V_\varepsilon'(x)\right) \;\leq\; \frac{2}{\sigma^2x_1^2}\left(rB + K_1\cdot V_\varepsilon'(x)\right)$$

  and Grönwall's inequality implies that

$$V_\varepsilon'(x) \;\leq\; \left(V_\varepsilon'(x_2) + \frac{rB}{K_1}\right)\cdot\exp\left(\frac{2K_1}{\sigma^2x_1^2}(x - x_2)\right) - \frac{rB}{K_1}.$$

  Thus, (3.15) yields

$$\|V_\varepsilon'\|_{\mathbf{L}^\infty([x_2,x^*])} \;\leq\; \left(\frac{4B}{x^*} + \frac{rB}{K_1}\right)\cdot\exp\left(\frac{2K_1x^*}{\sigma^2x_1^2}\right). \tag{3.18}$$

Therefore, combining (3.12), (3.17), (3.18) and (3.5), we obtain (3.11).

2. For any fixed $0 < \delta < \frac{x^*}{2}$, we will provide uniform bounds on $\|V''_\varepsilon\|_{\mathbf{L}^\infty([\delta,x^*-\delta])}$, $\|p'_\varepsilon\|_{\mathbf{L}^\infty([\delta,x^*-\delta])}$, and $\|p''_\varepsilon\|_{\mathbf{L}^\infty([\delta,x^*-\delta])}$. From (3.16), (3.14) and Lemma 2.1, it holds

$$\begin{aligned}|V''_\varepsilon(x)| &\le \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B + \left|H(x,V'_\varepsilon(x),p_\varepsilon(x)+\varepsilon)\right|\right]\\ &\le \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B + K_1\cdot V'_\varepsilon(x)\right],\end{aligned}$$

for all $x\in[\delta,x^*-\delta]$. Thus, (3.11) implies that

$$\|V''_\varepsilon\|_{\mathbf{L}^\infty([\delta,x^*-\delta])} \le \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B + K_1 M^*\right]. \tag{3.19}$$

Similarly, from the second equation in (3.4), it holds that

$$|p''_\varepsilon(x)| \le \frac{2}{\sigma^2\delta^2}\cdot\left[K_1\cdot|p'_\varepsilon(x)| + (r+\lambda+v_{\max}+\rho(x^*-\delta))\right] \quad \text{for all } x\in[\delta,x^*-\delta]. \tag{3.20}$$

By the mean value theorem, there exists a point $x_3 \in]\delta, x^*-\delta[$ such that

$$\left|p'_\varepsilon(x_3)\right| = \left|\frac{p_\varepsilon(x^*-\delta)-p_\varepsilon(\delta)}{x^*-2\delta}\right| \le \frac{1-\theta_{\min}}{x^*-2\delta}.$$

Applying Grönwall's inequality to (3.20) in the intervals $[\delta,x_3]$ and $[x_3,x^*-\delta]$, yields

$$|p'_\varepsilon(x)| \le K_\delta \qquad \text{for all } x\in[\delta,x^*-\delta],$$

for some constant K_δ depending only on δ. Thus, from (3.20) we get

$$|p''_\varepsilon(x)| \le \frac{2}{\sigma^2\delta^2}\cdot\left[K_1K_\delta + (r+\lambda+v_{\max}+\rho(x^*-\delta))\right] \quad \text{for all } x\in[\delta,x^*-\delta].$$

Combining the two above estimates and (3.19), we obtain (3.13) with the constant

$$M_\delta := \frac{2}{\sigma^2\delta^2}\cdot\left[(r+\rho(x^*-\delta))\cdot B + K_1M^* + K_1K_\delta + (r+\lambda+v_{\max}+\rho(x^*-\delta))\right] + K_\delta.$$

This completes the proof. □

We are ready to prove our first main theorem.

Proof of Theorem 3.1. The proof is divided into three steps:

Step 1. For any $0<\varepsilon<1/2$ sufficiently small, let $(V_\varepsilon,p_\varepsilon)$ be a solution to (3.4) and (2.12) which is constructed in Lemma 3.2. Recalling Lemma 2.2 and (3.11), we obtain that H and H_ξ are Lipschitz continuous on $[\delta,x^*-\delta]\times[0,M^*]\times[\theta_{\min},1]$ for any $\delta\in]0,x^*/2[$. Using the *a priori* estimates (3.11) and (3.13) in Lemma 3.3, and assumptions **(A1)-(A2)**, the system (3.4) implies that the functions V''_ε and p''_ε are also uniformly Lipschitz on $[\delta,x^*-\delta]$. Thus, one can apply the Ascoli-Arzelà Theorem to extract a subsequence $(V_{\varepsilon_n},p_{\varepsilon_n})_{n\ge0}$ with $\lim_{n\to\infty}\varepsilon_n=0$ such that $(V_{\varepsilon_n},p_{\varepsilon_n})$ converges uniformly to (V,p) in $C^2(]\delta,x^*-\delta[)$ for all $\delta>0$, where V,p are twice continuously differentiable and solve the system of ODEs (3.4) on the open interval $]0,x^*[$. Moreover, since V'_ε is uniformly bounded by M^* on $[0,x^*]$,

$$\lim_{n\to\infty}\|V_{\varepsilon_n}-V\|_{\mathbf{L}^\infty([0,x^*])} = 0,$$

which implies that

$$V(0) \;=\; \lim_{n\to\infty} V_{\varepsilon_n}(0) \;=\; 0, \qquad V(x^*) \;=\; \lim_{n\to\infty} V_{\varepsilon_n}(x^*) \;=\; B.$$

Step 2. It remains to check the boundary condition (2.12) for p. Let us first show that

$$\lim_{x\to 0^+} p(x) \;=\; 1\,. \tag{3.21}$$

Given $\varepsilon \in]0, \frac{1}{2}[$, we construct a lower bound p^- of p_ε independent of ε in a neighborhood of 0. From (**A1**), there exists $M > 0$ such that

$$\theta(x) \;\geq\; 1 - Mx \quad \text{for all } x \in [0, x^*/2]. \tag{3.22}$$

Set $\bar{x}_0 := \min\left\{\frac{c'(0)}{M^*}, \frac{x^*}{2}, \frac{1-\theta_{\min}}{M}\right\}$ where M^* is the constant in (3.11). The sub-solution candidate is

$$p^-(x) \;=\; 1 - kx^\gamma,$$

with

$$\gamma \;:=\; \min\left\{1, (r+\lambda)\left(\frac{\lambda+r}{\theta_{\min}} + \sigma^2\right)^{-1}\right\} \qquad \text{and} \qquad k \;:=\; (1-\theta_{\min})\cdot \bar{x}_0^{-\gamma}.$$

Note that $p^-(0) = 1 \leq p_\varepsilon(0)$. By choice of k and $\bar{x}_0$, we have

$$p^-(\bar{x}_0) \;=\; 1 - k\bar{x}_0^\gamma \;=\; 1 - (1-\theta_{\min})\cdot\bar{x}_0^{-\gamma}\cdot\bar{x}_0^\gamma \;=\; \theta_{\min}$$

and from (3.22), it holds for all $x \in [0, \bar{x}_0]$ that

$$p^-(x) \;=\; 1 - (1-\theta_{\min})\cdot\bar{x}_0^{-\gamma}\cdot x^\gamma \;\leq\; 1 - M\bar{x}_0^{1-\gamma}\cdot x^\gamma \;\leq\; 1 - Mx \;\leq\; \theta(x). \tag{3.23}$$

We claim that p^- is a sub-solution of the second equation in (3.4) in the interval $[0, \bar{x}_0]$. Indeed, recalling (3.11) that $v_\varepsilon(x) \equiv 0$ on the region $x \in [0, \bar{x}_0]$, from Lemma 2.2 (i) and (3.23), one estimates

$$\begin{aligned}
(r+\lambda) - &(r+\lambda+v_\varepsilon(x))p^- + \rho_\varepsilon(x)[\theta(x) - p^-(x)] + H_\xi(x, V, p^-)\cdot(p^-)' + \left(\varepsilon + \frac{\sigma^2x^2}{2}\right)(p^-)'' \\
&\geq\; (r+\lambda)kx^\gamma - \gamma k x^{\gamma-1} H_\xi(x,V,p^-) + \left(\varepsilon + \frac{\sigma^2x^2}{2}\right)\gamma(1-\gamma)kx^{\gamma-2} \\
&\geq\; (r+\lambda)kx^\gamma - \gamma k x^{\gamma-1} H_\xi(x,V,p^-) \;\geq\; \left[(r+\lambda) - \left(\frac{\lambda+r}{p^-(x)} + \sigma^2\right)\gamma\right]kx^\gamma \\
&\geq\; \left[(r+\lambda) - \left(\frac{\lambda+r}{p^-(\bar{x}_0)} + \sigma^2\right)\gamma\right]kx^\gamma \;=\; \left[(r+\lambda) - \left(\frac{\lambda+r}{\theta_{\min}} + \sigma^2\right)\gamma\right]kx^\gamma \;\geq\; 0
\end{aligned}$$

for all $x \in [0, \bar{x}_0]$. In turn, we have

$$1 - kx^\gamma \;\leq\; p_\varepsilon(x) \;\leq\; 1 \quad \text{for all } x \in [0, \bar{x}_0]\,, \tag{3.24}$$

and this yields (3.21).

Step 3. To complete the proof, we prove that

$$\lim_{x\to x^{*-}} p(x) \;=\; \theta(x^*). \tag{3.25}$$

Given $\varepsilon \in]0, 1/2[$ sufficiently small, we will provide an upper bound on the $\mathbf{L}^\infty$ distance of p_ε and θ over $[x^* - \delta, x^*]$ denoted by

$$I_\delta \;:=\; \max_{x\in[x^*-\delta,x^*]} \big|p_\varepsilon(x) - \theta(x)\big| \qquad \text{for all } \delta \in [0, x^*/4].$$

For a fixed $0 < \delta < \dfrac{x^*}{4}$, the continuity of p_ε and θ implies that

$$I_\delta \;=\; \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot [\theta(x_m) - p_\varepsilon(x_m)]\,, \tag{3.26}$$

for some $x_m \in [x^* - \delta, x^*]$. Assume that $p_\varepsilon(x_m) \neq \theta(x_m)$. Since $p_\varepsilon(0) = \theta(0) = 1$ and $p_\varepsilon(x^*) = \theta(x^*)$, we can define two points

$$0 \;\le\; x_1^* \;:=\; \max\Big\{x \in [0, x_m[\;\Big|\; p_\varepsilon(x) = \theta(x)\Big\} \;<\; x_m$$

and

$$\frac{3x^*}{4} \;\le\; x_m \;<\; x_2^* \;:=\; \min\Big\{x \in [x_m, x^*] \;\Big|\; p_\varepsilon(x) = \theta(x)\Big\} \;\le\; x^*.$$

Notice that $p_\varepsilon - \theta$ does not change sign in the interval $]x_1^*, x_2^*[$. For simplicity, let us introduce the following function

$$q_\varepsilon(x) \;:=\; \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot p_\varepsilon(x).$$

It is clear that

$$|q'_\varepsilon(x)| \;=\; |p'_\varepsilon(x)| \qquad \text{for all } x \in]x_1^*, x_2^*[.$$

Thus, by the second equation in (3.4) we estimate

$$\begin{aligned} q''_\varepsilon(x) \;&= \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot p''_\varepsilon(x) \\ &\le \left(\frac{2}{2\varepsilon + \sigma^2 x^2}\right) \cdot \Big[r + \lambda + v_{\max} + \big|H_\xi \cdot p'_\varepsilon(x)\big| - \rho_\varepsilon(x) \cdot \big|p_\varepsilon(x) - \theta(x)\big|\Big] \\ &\le \left(\frac{2}{2\varepsilon + \sigma^2 x^2}\right) \cdot \Big[r + \lambda + v_{\max} + \big|H_\xi \cdot q'_\varepsilon(x)\big|\Big] \quad \text{for all } x \in]x_1^*, x_2^*[\,. \end{aligned}$$

Set $\bar{x}_1^* := \max\left\{x_1^*, \dfrac{x^*}{2}\right\}$, we then have

$$q''_\varepsilon(x) \;\le\; \frac{8}{(\sigma x^*)^2} \cdot \big(r + \lambda + v_{\max} + K_1 \cdot |q'_\varepsilon(x)|\big) \quad \text{for all } x \in]\bar{x}_1^*, x_2^*[, \tag{3.27}$$

where K_1 is defined in (2.16). Two cases may occur:

CASE 1: If $x_m - \bar{x}_1^* > \delta$ then

$$\begin{aligned} q_\varepsilon(x_m) - q_\varepsilon(x_m - \delta) \;&=\; \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot [p_\varepsilon(x_m) - p_\varepsilon(x_m - \delta)] \\ &\le\; \operatorname{sign}\big(\theta(x_m) - p_\varepsilon(x_m)\big) \cdot [\theta(x_m) - \theta(x_m - \delta)] \;\le\; \Delta_\delta\theta \cdot \delta, \end{aligned}$$

where $\Delta_\delta\theta = \sup_{x\in[0,x^*-\delta]} \left|\frac{\theta(x+\delta)-\theta(x)}{\delta}\right|$. Thus, by mean value theorem there exists $\bar{x} \in \,]x_m - \delta, x_m[$ such that

$$q'_\varepsilon(\bar{x}) \;=\; \frac{q_\varepsilon(x_m) - q_\varepsilon(x_m - \delta)}{\delta} \;\le\; \Delta_\delta\theta.$$

Let g be solution to the ODE

$$g'(x) \;=\; \frac{8}{(\sigma x^*)^2}\cdot(r+\lambda+v_{\max}+K_1\cdot g(x)), \qquad g(\bar{x}) \;=\; \Delta_\delta\theta \;\geq\; q_\varepsilon'(\bar{x}).$$

Solving the above equation, one gets

$$g(x) \;=\; \left(\frac{r+\lambda+v_{\max}}{K_1}+\Delta_\delta\theta\right)\cdot\exp\left(\frac{8K_1}{(\sigma x^*)^2}\cdot(x-\bar{x})\right)-\frac{r+\lambda+v_{\max}}{K_1} \;\geq\; 0,$$

for all $x \geq \bar{x}$. In particular, it holds that

$$g'(x) \;=\; \frac{8}{(\sigma x^*)^2}\cdot(r+\lambda+v_{\max}+K_1\cdot|g(x)|) \qquad \text{for all } x\in\,]\bar{x},x_2^*[\,.$$

A standard comparison argument yields

$$q_\varepsilon'(x) \;\leq\; g(x) \;\leq\; \left(\frac{r+\lambda+v_{\max}}{K_1}+\Delta_\delta\theta\right)\cdot\exp\left(\frac{4K_1}{\sigma^2x^*}\right) \qquad \text{for all } x\in]\bar{x},x_2^*[.$$

Thus, from (3.26) it holds

$$\begin{aligned}
I_\delta \;&=\; \operatorname{sign}\big(\theta(x_m)-p_\varepsilon(x_m)\big)\cdot(\theta(x_m)-\theta(x_2^*))+q_\varepsilon(x_2^*)-q_\varepsilon(x_m)\\
&\leq\; \sup_{x,y\in[x^*-\delta,x^*]}|\theta(x)-\theta(y)|+\left(\frac{r+\lambda+v_{\max}}{K_1}+\Delta_\delta\theta\right)\cdot\exp\left(\frac{4K_1}{\sigma^2x^*}\right)\cdot\delta.\\
&\leq\; \frac{r+\lambda+v_{\max}}{K_1}\cdot\exp\left(\frac{4K_1}{\sigma^2x^*}\right)\cdot\delta+\left[\exp\left(\frac{4K_1}{\sigma^2x^*}\right)+1\right]\cdot\sup_{x,y\in[x^*-\delta,x^*]}|\theta(x)-\theta(y)|. \qquad (3.28)
\end{aligned}$$

CASE 2: Let us assume that $0 < x_m-\bar{x}_1^* \leq \delta$. Since $\delta < \dfrac{x^*}{4}$ and $x_m \geq \dfrac{3x^*}{4}$, we have that $\bar{x}_1^* > \dfrac{x^*}{2}$ and this yields $\bar{x}_1^* = x_1^*$. Two subcases are considered:

- If $q_\varepsilon(x_m)-q_\varepsilon(x_1^*) \geq 0$ then (3.26) implies that

$$\begin{aligned}
I_\delta \;&=\operatorname{sign}\big(\theta(x_m)-p_\varepsilon(x_m)\big)\cdot[\theta(x_m)-\theta(x_1^*)]+q_\varepsilon(x_1^*)-q_\varepsilon(x_m)\\
&\leq|\theta(x_m)-\theta(x_1^*)| \;\leq\; \sup_{x,y\in[x^*-\delta,x^*]}|\theta(x)-\theta(y)|. \qquad (3.29)
\end{aligned}$$

- Otherwise, if $q_\varepsilon(x_m)-q_\varepsilon(x_1^*) < 0$ then by mean value theorem there exists $\tilde{x}\in\,]x_1^*,x_m[$ such that

$$q_\varepsilon'(\tilde{x}) \;=\; \frac{q_\varepsilon(x_m)-q_\varepsilon(x_1^*)}{x_m-x_1^*} \;<\; 0.$$

  With the same argument in case 1, one can show that

$$q_\varepsilon'(x) \;\leq\; \tilde{g}(x) \;\leq\; \frac{r+\lambda+v_{\max}}{K_1}\cdot\exp\left(\frac{4K_1}{\sigma^2x^*}\right) \qquad \text{for all } x\in]\tilde{x},x_2^*[\,,$$

  where $\tilde{g}$ is the solution to

$$\tilde{g}'(x) \;=\; \frac{8}{(\sigma x^*)^2}\cdot(r+\lambda+v_{\max}+K_1\cdot\tilde{g}(x)), \qquad \tilde{g}(\tilde{x}) \;=\; 0 \;\geq\; q_\varepsilon'(\tilde{x}).$$

  Thus, as in (3.28), it holds

$$I_\delta \;\leq\; \frac{r+\lambda+v_{\max}}{K_1}\cdot\exp\left(\frac{4K_1}{\sigma^2x^*}\right)\cdot\delta+\sup_{x,y\in[x^*-\delta,x^*]}|\theta(x)-\theta(y)|. \qquad (3.30)$$

From (3.28)-(3.30), we obtain that

$$\|p_\varepsilon - \theta\|_{\mathbf{L}^\infty([x^*-\delta,x^*])} \ \leq\ C_1 \cdot \delta + C_2 \cdot \sup_{x,y\in[x^*-\delta,x^*]} |\theta(x)-\theta(y)|,$$

for all $\varepsilon \in \left]0,\frac{1}{2}\right[$, $\delta \in \left]0,\frac{x^*}{4}\right[$ with the constants

$$C_1 \ =\ \frac{r+\lambda+v_{\max}}{K_1}\cdot \exp\left(\frac{4K_1}{\sigma^2 x^*}\right), \qquad C_2 \ =\ \exp\left(\frac{4K_1}{\sigma^2 x^*}\right)+1.$$

In particular,

$$\|p - \theta\|_{\mathbf{L}^\infty(]x^*-\delta,x^*[)} \ \leq\ C_1 \cdot \delta + C_2 \cdot \sup_{x,y\in[x^*-\delta,x^*]} |\theta(x)-\theta(y)|$$

and the uniform continuity of θ yields (3.25). □

3.2 Optimal currency devaluation and payment strategy near x^*

This subsection is devoted to the behavior of optimal feedback controls near the debt threshold x^*. Roughly speaking, let (u^*, v^*, p) be an optimal solution to the problem of optimal debt management (2.1)–(2.5), and V be the corresponding value function. In addition to (3.2), we will show that when the debt x closes to x^*

- *if the risk of bankruptcy ρ slowly approaches to infinity then the optimal strategy of borrower involves continuously devaluating its currency and making payment, i.e. $u^*(x) > 0$ and $v^*(x) > 0$,*
- *conversely, if the risk of bankruptcy ρ quickly approaches to infinity then any action to reduce the debt is not optimal, i.e. $u^*(x) = v^*(x) = 0$.*

Let us first establish upper and lower bounds for V. Recalling that

$$p \ \in\ [\theta_{\min}, 1] \qquad \text{with} \qquad \theta_{\min} \ =\ \min\left\{\theta(x^*), \frac{r+\lambda}{r+\lambda+v_{\max}}\right\},$$

we introduce a non-decreasing function $\beta : [0, x^*[\to [0,+\infty[$ defined by

$$\beta(t) \ =\ \max_{s\in[0,t]}\left[\rho(s)\ln\left(\frac{t}{s}\right)\right] + \frac{\lambda+r}{\theta_{\min}} + \frac{\sigma^2}{2} \ <\ +\infty \qquad \text{for all } t\in[0,x^*[. \tag{3.31}$$

Proposition 3.1. *Under the same assumptions in Theorem 3.1, it holds*

$$V(x) \ \leq\ V_1(x) \ :=\ B\cdot \inf_{t\in[x,x^*[}\left[\frac{\beta(t)}{r\ln\left(\frac{t}{x}\right)+\beta(t)}\right] \qquad \textit{for all } x\in]0,x^*[. \tag{3.32}$$

In addition if we assume that

$$x^* \ \geq\ \frac{2}{r+\lambda} \qquad \text{and} \qquad \lim_{x\to x^*-}\rho(x)(x^*-x)^2 \ =\ +\infty, \tag{3.33}$$

then there exists $x^\diamond \in [x^*/2, x^*[$ *such that*

$$V(x) \;\geq\; V_2(x) \;:=\; B \cdot \left(1 - \frac{\ln\left(\frac{x^*}{x}\right)\left(1 - \frac{x}{x^*}\right)}{\ln\left(\frac{x^*}{x^\diamond}\right)(x^* - x^\diamond)} \right) \qquad \text{for all } x \in [x^\diamond, x^*]\,. \tag{3.34}$$

Proof. **1.** For any given $x_1 \in]0, x^*[$ and $x_2 \in]0, x_1[$, we seek for an upper bound for V as a supersolution to the first equation of (2.11) of the form

$$V_1(x) \;=\; \begin{cases} B & \text{if} \quad x \in [x_1, x^*], \\ B\left(1 - \alpha \ln\left(\frac{x_1}{x}\right)\right) & \text{if} \quad x \in [x_2, x_1], \\ B\left(1 - \alpha \ln\left(\frac{x_1}{x_2}\right)\right) & \text{if} \quad x \in [0, x_2], \end{cases}$$

with $\alpha > 0$ satisfying the following relation

$$-r + \alpha \cdot \left(r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1) \right) \;=\; 0.$$

It is clear that

$$V_1(0) \;=\; B\left(1 - \alpha \ln\left(\frac{x_1}{x_2}\right)\right) \;\geq\; 0 \;=\; V(0) \qquad \text{and} \qquad V_1(x^*) \;=\; B \;=\; V(x^*).$$

For every $x \in]0, x_2[$, it holds

$$\begin{aligned} &-(r + \rho(x))V_1(x) + \rho(x)B + H(x, V_1'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \\ &= \; -(r + \rho(x))V_1(x) + \rho(x)B \;\leq\; B \cdot \left[-r + \alpha(r + \rho(x_2)) \ln\left(\frac{x_1}{x_2}\right) \right] \\ &\leq \; B \cdot \left(-r + \alpha \cdot \left(\beta(x_1) + r \ln\left(\frac{x_1}{x_2}\right) \right) \right) \;=\; 0. \end{aligned}$$

Similarly, for every $x \in]x_1, x^*[$, one has

$$\begin{aligned} &-(r + \rho(x))V_1(x) + \rho(x)B + H(x, V_1'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \\ &\qquad\qquad = \; -(r + \rho(x))V_1(x) + \rho(x)B \;=\; -rB \;<\; 0. \end{aligned}$$

On the other hand, for every $x \in]x_2, x_1[$, we compute

$$V_1'(x) \;=\; \frac{B\alpha}{x} \;>\; 0 \qquad \text{and} \qquad V_1''(x) \;=\; -\frac{B\alpha}{x^2},$$

and use Lemma 2.2 to obtain

$$\begin{aligned} &-(r + \rho(x))V_1(x) + \rho(x)B + H(x, V_1'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \\ &\quad\leq \; -(r + \rho(x))V_1(x) + \rho(x)B + \left(\frac{\lambda + r}{\theta_{\min}} + \sigma^2 \right) x V_1'(x) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \\ &\quad= \; B \cdot \left[-r + \alpha \cdot \left((r + \rho(x)) \ln\left(\frac{x_1}{x}\right) + \frac{\lambda + r}{\theta_{\min}} + \frac{\sigma^2}{2} \right) \right] \\ &\quad\leq \; B \cdot \left[-r + \alpha \cdot \left(r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1) \right) \right] \;=\; 0. \end{aligned}$$

Hence, V_1 is a supersolution of the first equation of (2.11) and

$$V(x) \;\leq\; V_1(x), \qquad \text{for all } x \in [0, x^*].$$

In particular, we have

$$V(x_2) \;\leq\; V_1(x_2) \;=\; B \cdot \left(1 - \frac{r}{r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1)} \cdot \ln\left(\frac{x_1}{x_2}\right)\right),$$

and this implies that

$$V(x_2) \;\leq\; B \cdot \frac{\beta(x_1)}{r \ln\left(\frac{x_1}{x_2}\right) + \beta(x_1)}.$$

Since the above inequality hold for every $x_2 \in]0, x_1[$, one obtains (3.32).

2. We seek for a lower bound for V as a subsolution to the first equation of (2.11) in the interval $[x_1, x^*]$ of the form

$$V_2(x) \;=\; B\left(1 - \alpha_1 \ln\left(\frac{x^*}{x}\right)(x^* - x)\right), \qquad \text{for all } x \in [x_1, x^*],$$

with

$$x_1 \;\in\; [x^*/2, x^*] \qquad \text{and} \qquad \alpha_1 \;=\; \left[\ln\left(\frac{x^*}{x_1}\right)(x^* - x_1)\right]^{-1} \;\geq\; \frac{2}{\ln 2 \cdot x^*}, \tag{3.35}$$

such that $V_2(x_1) = 0$. For every $x \in]x_1, x^*[$, we compute

$$V_2'(x) \;=\; B\alpha_1 \cdot \left(\ln\left(\frac{x^*}{x}\right) + \frac{x^* - x}{x}\right) \;\geq\; 0 \quad \text{and} \quad V_2''(x) \;=\; -\,B\alpha_1\left(\frac{1}{x} + \frac{x^*}{x^2}\right). \tag{3.36}$$

For simplicity, set

$$C_1 \;:=\; \lambda + \mu + v_{\max}\,.$$

Using Lemma 2.2 and the first condition of (3.33), we have

$$H(x, V_2'(x), p(x)) \;\geq\; -\,C_1 V_2'(x)x, \qquad \text{for all } x \in [x_1, x^*]$$

which implies that

$$\begin{aligned} &-(r + \rho(x))V_2(x) + \rho(x)B + H(x, V_2'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_2''(x) \\ &\quad\geq\; -\,(r + \rho(x))V_2(x) + \rho(x)B - C_1 x V_2'(x) + \frac{\sigma^2 x^2}{2} \cdot V_2''(x). \end{aligned}$$

On the other hand, from (3.36), one has

$$C_1 x V_2'(x) \;\leq\; 2B\alpha_1 C_1(x^* - x) \;\leq\; B\alpha_1 C_1 x^*, \qquad \frac{\sigma^2 x^2}{2} \cdot V_2''(x) \;\geq\; -\,B\alpha_1 \sigma^2 x^*,$$

for all $x \in [x_1, x^*]$. Thus,

$$\begin{aligned} &-(r + \rho(x))V_2(x) + \rho(x)B + H(x, V_2'(x), p(x)) + \frac{\sigma^2 x^2}{2} \cdot V_2''(x) \\ &\qquad\geq\; B \cdot \left[\alpha_1\left(\rho(x) \ln\left(\frac{x^*}{x}\right)(x^* - x) - (C_1 + \sigma^2)x^*\right) - r\right]. \end{aligned} \tag{3.37}$$

From (3.33), it holds

$$\lim_{x\to x^*}\rho(x)\ln\left(\frac{x^*}{x}\right)(x^*-x) \;=\; +\infty,$$

and therefore there exists $x^\diamond\in]x^*/2,x^*[$ sufficiently close to x^* such that

$$\frac{2}{\ln 2\cdot x^*}\cdot\left(\rho(x)\ln\left(\frac{x^*}{x}\right)(x^*-x)-(C_1+\sigma^2)x^*\right)-r\;\geq\;0\qquad\text{for all } x\in[x^\diamond,x^*[.$$

In particular, if we choose $x_1=x^\diamond$ then (3.37) and (3.35) imply that

$$-(r+\rho(x))V_2(x)+\rho(x)B+H(x,V_2'(x),p(x))+\frac{\sigma^2x^2}{2}\cdot V_2''(x)\;\geq\;0,$$

for all $x\in[x^\diamond,x^*[$. Since

$$V_2(x^\diamond)\;=\;0\;\leq\;V(x^\diamond)\quad\text{and}\qquad V_2(x^*)\;=\;B\;\leq\;V(x^*),$$

V_2 is a subsolution to the first equation of (2.11) in $[x^\diamond,x^*]$ and thus

$$V(x)\;\geq\;V_2(x),\qquad\text{for all } x\in[x^\diamond,x^*],$$

which yields (3.34). □

From the formula of β in (3.31), one can actually show that β is locally Lipschitz and

$$0\;\leq\;\dot\beta(t)\;\leq\;\frac{\rho(t)}{t}\qquad a.e.\ t\in[x,x^*]. \tag{3.38}$$

Notice that $\lim_{t\to0+}\frac{\rho(t)}{t}=\rho'(0)<+\infty$, if $\int_0^{x^*}\rho(t)dt<+\infty$ then $\int_0^{x^*}\frac{\rho(t)}{t}dt<+\infty$. In this case, we have

$$\sup_{t\in[0,x^*[}\beta(t)\;\leq\;\beta^*:=\int_0^{x^*}\frac{\rho(t)}{t}dt+\frac{\lambda+r}{\theta_{\min}}+\frac{\sigma^2}{2}. \tag{3.39}$$

As a consequence of Proposition 3.1, the followings hold.

Corollary 3.2. *Under the same assumptions in Theorem 3.1, two cases may occur*

(i) Let β^* be as in (3.39). *If*

$$\int_0^{x^*}\rho(t)\ dt\;<\;+\infty\qquad\text{and}\qquad\beta^*<Br\cdot\min\left\{\frac{1}{c'(0)},\frac{1}{L'(0)x^*}\right\},$$

then there exists some $\bar x\in]0,x^*[$ sufficiently close to x^* such that

$$u^*(x)\;>\;0\qquad\text{and}\qquad v^*(x)\;>\;0\qquad\textit{for all } x\in[\bar x,x^*[.$$

(ii) If (3.33) holds then there exists $\hat x\in]0,x^*[$ sufficiently close to x^* such that

$$u^*(x)\;=\;v^*(x)\;=\;0\qquad\textit{for all } x\in[\hat x,x^*[.$$

Proof. **(i).** For every given $x_2 \in]0, x^*[$, (3.32) and (3.39) imply that

$$V(x^*) - V(x_2) \;\geq\; V(x^*) - V_1(x_2) \;\geq\; B \cdot \frac{r\ln(x_1/x_2)}{r\ln(x_1/x_2) + \beta^*} \quad \text{for all } x_1 \in [x_2, x^*[.$$

In particular, we have

$$V(x^*) - V(x_2) \;\geq\; \sup_{x_1 \in [x_2, x^*[} \left[B \cdot \frac{r\ln(x_1/x_2)}{r\ln(x_1/x_2) + \beta^*} \right] \;=\; B \cdot \frac{r\ln(x^*/x_2)}{r\ln(x^*/x_2) + \beta^*}.$$

By mean value theorem, there exists $x_c \in [x_2, x^*]$ such that

$$V'(x_c) \cdot x_c \;\geq\; B \cdot \frac{r\ln(x^*/x_2)}{r\ln(x^*/x_2) + \beta^*} \cdot \frac{x_2}{x^* - x_2}. \tag{3.40}$$

On the other hand, from the first equation of (2.11) and Lemma 2.2, it holds

$$\frac{\sigma^2}{2}[x^2 V''(x) + xV'(x)] \;\geq\; -cxV'(x) - \rho(x)B \qquad \text{for all } x \in]0, x^*[,$$

with constant $c = \frac{r+\lambda}{\theta_{\min}} + \frac{\sigma^2}{2}$. Set $Z(x) = xV'(x)$, $c_1 = \frac{4c}{\sigma^2 x^*}$ and $c_2 = \frac{4B}{\sigma^2 x^*}$, we have

$$Z'(x) \;\geq\; -c_1 Z(x) - c_2\rho(x) \qquad \text{for all } x \in]x^*/2, x^*[.$$

Solving the differential inequality yields

$$Z(x) \;\geq\; e^{c_1(x_0 - x)} \cdot Z(x_0) - c_2 \int_{x_0}^{x} \rho(s)ds \quad \text{for all} \quad \frac{x^*}{2} < x_0 \leq x < x^*.$$

In particular, recalling (3.40), we have

$$Z(x) \;\geq\; Be^{c_1(x_2 - x^*)} \frac{r\ln(x^*/x_2)}{r\ln(x^*/x_2) + \beta^*} \cdot \frac{x_2}{x^* - x_2} - c_2 \int_{x_2}^{x^*} \rho(s)ds \;=:\; I(x_2),$$

for all $x^*/2 < x_2 < x_c \leq x < x^*$. Since $\beta^* < Br \cdot \min\left\{\frac{1}{c'(0)}, \frac{1}{L'(0)x^*}\right\}$ and $\int_0^{x^*} \rho(t)\,dt < +\infty$, it holds

$$\lim_{x_2 \to x^*} I(x_2) \;=\; \frac{Br}{\beta^*} \;>\; \max\left\{c'(0), L'(0)x^*\right\}.$$

Thus, there exists $x_2 \in [x^*/2, x^*[$ sufficiently close to x^* such that

$$Z(x) \;\geq\; I(x_2) \;>\; c'(0) \qquad \text{and} \qquad V'(x) \;\geq\; \frac{I(x_2)}{x} \;>\; L'(0) \;\geq\; L'(0)p(x),$$

for all $x \in [x_c, x^*]$ and, recalling (2.9)-(2.10), this yields (i) for $\bar{x} = x_c$.

(ii). Assuming that (3.33) holds, we recall $x^\diamond$ in Proposition 3.1. For every $x_1 \in [x^\diamond, x^*[$, it holds

$$V'(x_c) \cdot x_c \;=\; \frac{V(x^*) - V(x_1)}{x^* - x_1} \cdot x_c \;\leq\; \frac{B - V_2(x_1)}{x^* - x_1} \cdot x^* \;=\; \frac{B}{\ln\left(\frac{x^*}{x^\diamond}\right)(x^* - x^\diamond)} \cdot \ln\left(\frac{x^*}{x_1}\right), \tag{3.41}$$

for some $x_c \in]x_1, x^*[$. From the first equation of (2.11) and Lemma 2.2, one estimates

$$\frac{\sigma^2}{2}[x^2V''(x) + xV'(x)] \ \leq \ rB + (\lambda + \mu + v_{\max})\, xV'(x) \qquad \text{for all } x \in]x^*/2, x^*[.$$

Recalling that $Z(x) = xV'(x)$, we have

$$Z'(x) \ \leq \ c_3 Z(x) + c_4 \qquad \text{for all } x \in [x^*/2, x^*],$$

with $c_3 := \frac{4rB}{\sigma^2 x^*}$ and $c_4 := \frac{2(\lambda+\mu+v_{\max})}{\sigma^2 x^*}$. Thus,

$$Z(x) \ \leq \ e^{c_3\cdot(x-x_c)} \cdot Z(x_c) + \frac{c_4}{c_3} \cdot \left(e^{c_3\cdot(x-x_c)} - 1\right)$$

and (3.41) implies that

$$Z(x) \ \leq \ \frac{B}{\ln\left(\frac{x^*}{x^\diamond}\right)(x^* - x^\diamond)} \cdot e^{c_3\cdot(x^*-x_1)} \cdot \ln\left(\frac{x^*}{x_1}\right) + \frac{c_4}{c_3} \cdot \left(e^{c_3\cdot(x^*-x_1)} - 1\right) \ =: \ J(x_1),$$

for all $x \in [x_c, x^*[$. Since $\lim_{x_1 \to x^*-} J(x_1) = 0$, there exists $x_1 \in]x^\diamond, x^*[$ such that

$$Z(x) \ \leq \ J(x_1) \ \leq \ c'(0) \quad \text{and} \quad V'(x) \ \leq \ \frac{J(x_1)}{x} \ \leq \ \theta_{\min} L'(0) \ \leq \ L'(0) \cdot p(x),$$

for all $x \in [x_c, x^*[$ and, recalling (2.9)-(2.10), setting $\hat{x} = x_c$ yields (ii). □

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