A DEBT MANAGEMENT PROBLEM WITH CURRENCY DEVALUATION

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Abstract. We consider a model of debt management, where a sovereign state trade some bonds to service the debt with a pool of risk-neutral competitive foreign investors. At each time, the government decides which fraction of the gross domestic product (GDP) must be used to repay the debt, and how much to devaluate its currency. Both these operations have the effect to reduce the actual size of the debt, but have a social cost in terms of welfare sustainability. Moreover, at any time the sovereign state can declare bankruptcy by paying a correspondent bankruptcy cost. We show that this optimization problems admits an equilibrium solution, leading to bankruptcy or to a stationary state, depending on the initial conditions.

1. Introduction

According to US Senate Levin-Coburn Report [10], the financial crisis of 2007-2008, which originated the worldwide Great Recession of 2008–2012 and to the European sovereign debt crisis of 2010-2012, and whose effects are still present in many countries, "was not a natural disaster, but the result of high risk, complex financial products, undisclosed conflicts of interest; and the failure of regulators, the credit rating agencies, and the market itself to rein in the excesses of Wall Street." The first part of the report analyze some topic cases of

(1) High Risk Lending;
(2) Regulatory Failure;
(3) Inflated Credit Ratings;
(4) Investment Bank Abuses.

In the final recommendation of the report, a whole section is devoted to the management of high risk lending, in order to prevent abuses.

In the Eurozone, the crisis - whose consequences lasted until 2016 - took the form of a speculative attack to the sovereign debt of some EU countries (Portugal, Ireland, Greece, Spain), but also strongly affects also two major economic powers like Italy and France. The undertaken actions of the EU governments to face the crisis had very high social costs, leading also to an heavy political impact. These considerations lead to the following natural problems:

- to identify suitable tools to estimate the risk of a borrower's bankruptcy (as in the subprime mortgage crisis, which originated);
- to have quantitative tools, relying on reliable prediction of realistic models, which would allow the regulation authority to prevent abuses;
- to provide optimal strategies in the management of sovreign debts.

In [12], the authors introduced a variational model where a government issues nominal defaultable debt and chooses fiscal and monetary policy under discretion. In particular, to reduce the actual size of the debt, the government can choose to devaluate its currency, producing inflaction and thus increasing the welfare cost and negatively affecting the trust of the investors, or to rely only on fiscal policy to serve the debt. The government can also declare the default, which imply to pay a bankruptcy cost due to the temporary exclusion from capital markets, and a drop in the output endowment. The aim is to find a strategy minimizing a cost functional dealing with the trade-off between inflaction, social costs, and debt sustainability and possibly declaring the default if this option would me preferable to continue servicing the debt.

The analysis of the model in [12] was performed by a numerical methods, and as a final conclusion of their analysis, the authors claim that the tool of currency devaluation, though useful in a short-term perspective, is not recommended unless the government is able to make credible commitments about their future inflaction policy. In this sense, it is worth of notice that many countries with limited inflaction credibility, decide either to issue bonds *directly* in a foreign stable currency (e.g., US dollars), or delegates the monetary policy to an independent authority with a strong anti-inflaction commitment (e.g., Eurozone Central Bank).

An analytical study of a variant of the model in [12] was performed in [5], in the case where no currency devaluation is available to the governement, and provided a semi-explicit formula for the optimal strategy in the deterministic case (i.e., when the GDP evolves deterministically).

This paper extend the analytical study of [5], allowing also the possibility of currency devaluation as in [12]. Indeed, from the point of view of the model, one of the most relevant resul is if the initial debt-to-income is sufficiently high, *every* optimal debt-management strategy needed to employ the currency devaluation.

As a real example of such policy, we mention that, according to the World Economic Outlook (April 2019) of International Monetary Fund, Japan leads the ranking of countries with higher debt-to-income ratio with a 237.5%, and it devaluated its currency of about 40% vs US dollars in the period 2012-2016 (USDJPY index).

The paper is structured as follows: in Section 2 we introduce the stochastic model together with the main assumptions, in Section 3 we prove the existence of an equilibrium solution for the stochastic model as the steady state of an auxiliary parabolic system, and study its asymptotic behaviour as the maximum debt-to-income threshold is pushed to $+\infty$. In Section 4 we study the deterministic model obtaining by setting the volatility $\sigma = 0$. In this case we provide a semi-explicit construction for an equilibrium solution, together with a study of its asymptotic behaviour as the maximum debt-to-income threshold is pushed to $+\infty$.

2. A model with stochastic growth

In this section, we develop the model in [5], allowing the possibility of currency devaluation as in [12]. Here the borrower is a sovereign state, that can decide to devaluate its currency (for example, printing more paper money). The total income Y, i.e., the gross national product GDP measured in terms of the floating currency unit, can quickly increase if the currency is devaluated, producing inflaction. It is governed by a stochastic process

$$dY(t) = (\mu + \tilde{v}(t))Y(t)\,dt + \sigma Y(t)\,dW(t),$$

where $W(t)$ is a Brownian motion on a filtered probability space and

- μ is the average growth rate of the economy;

- σ is the volatility;
- $\tilde{v}(t) \geq 0$ is the devaluation rate at time t, regarded as an additional control.

We refer to [12] for a more detailed derivation v in the above system from economic primitives.

Let $X(t)$ be the outstanding stock of nominal government bonds, expressed in the local currency unit. In particular, $X(t)$ represents also the total nominal value of the outstanding debt. To service the debt, the government trades a nominal non-contingent bond with risk-neutral competitive foreign investors. In case of bankruptcy, the lenders recover only a fraction $\theta \in [0,1]$ of their outstanding capital, depending on the total amount of the debt at the time of bankruptcy. To offset this possible loss, the investors buy a bond with unit nominal value at a discounted price $\tilde{p}(t) \in [0,1]$. We denote by $U(t)$ the rate of payments that the borrower chooses to make to the lenders at time t. If this amount is not enough to cover the running interest and pay back part of the principal, new bonds are issued, at the discounted price $p(t)$. As in [5], the nominal value of the outstanding debt thus evolves according to

$$\dot{X}(t) = -\lambda X(t) + \frac{(\lambda + r)X(t) - U(t)}{\tilde{p}(t)}.$$

Here the constants

- λ represents the rate at which the borrower pays back the principal;
- r represents the discount rate.

The debt-to-GDP ratio (DTI) is defined as $x(\cdot) = X(\cdot)/Y(\cdot)$. By Itō's formula [13, 15], the evolution of $x(\cdot)$ is

$$dx(t) = \left[\left(\frac{\lambda + r}{\tilde{p}(t)} - \lambda + \sigma^2 - \mu - \tilde{v}(t)\right) x(t) - \frac{\tilde{u}(t)}{\tilde{p}(t)}\right] dt - \sigma\, x(t)\, dW(t), \tag{2.1}$$

where $\tilde{u} = U/Y \in [0,1]$ is the fraction of the total income allocated to reduce the debt. Throughout the following we will assume that $r > \mu$.

In this model, the borrower has three controls: at each time t he can decide the portion $\tilde{u}(t) \in [0,1]$ of the total income is allocated to repaying the debt, he can decide the devaluation rate $\tilde{v}(t) \in [0, +\infty[$ and he can also decide the time $T_b(x')$ he is going to declare bankruptcy, in correspondence to a DTI value x', paying a bankruptcy cost. Furthermore, we assume that

- there exists a threshold $x^* > 0$ such that if $x(t)$ reaches x^* then the borrower is forced to declare bankruptcy;
- the borrower decides to declare bankruptcy as soon as $x(t)$ reaches x', where $x' \in [0, x^*]$ is an additional control parameter, chosen by the borrower in order to minimize his expected cost.

We are going to consider *control strategy in feedback form*, i.e., $\tilde{u}(t) = u(x(t))$, $\tilde{v}(t) = v(x(t))$, for certain measurable maps $u : [0, x^*] \to [0,1]$ and $v : [0, x^*] \to [0, +\infty[$. The total expected cost to the borrower, exponentially discounted in time, for having implemented the control strategy $(u(\cdot), v(\cdot), x')$ is then given by

$$J[x_0, x', u(\cdot), v(\cdot)] = E\left[\int_0^{T_b(x')} e^{-rt}\Big[L(u(x(t))) + c(v(x(t)))\Big]\, dt + e^{-rT_b(x')}B\right]_{x(0)=x_0},$$

where

- $T_b(x')$ is the random variable bankruptcy time defined by
$$T_b(x') \doteq \inf\{t > 0 : \ x(t) = x'\},$$
where $x(\cdot)$ solves (2.1) with $\tilde{u}(\cdot) = u(x(\cdot))$, $\tilde{v}(\cdot) = v(x(\cdot))$ and $x(0) = x_0$. If (2.1) has no solution, we set $J[x_0, x', u(\cdot), v(\cdot)] = +\infty$. We define also
$$T_b^* \doteq \inf\{t > 0 : \ x(t) = x^*\}. \tag{2.2}$$
- B is the bankruptcy cost, which summarizes the penalties of temporary exclusion from the capital markets, the bad reputation among the investors, and the social costs of the default;
- $c(v) \geq 0$ is a social cost resulting by devaluation, i.e., the increasing cost of the welfare and of the imported goods;
- $L(u) \geq 0$ is the cost for the borrower to implement the control strategy $u(\cdot)$, i.e., adversion toward austerity policies and welfare's budget cuts.

Throughout the paper we will assume the following structural conditions on the functions L, c. More precisely, there exist $\delta_0 > 0$ and $v_{\max} \in]0, +\infty]$ such that

(A1) *the implementing cost function* $L : [0, 1[\to \mathbb{R}$ *is continuous on* $[0, 1[$*, twice continuously differentiable for* $u \in]0, 1[$*, and satisfies*
$$L(0) = 0, \quad L'(u) > 0, \quad L''(u) > \delta_0 > 0 \text{ for } u \in]0, 1[, \text{ and } \lim_{u \to 1^-} L(u) = +\infty.$$

(A2) *the social cost* $c : [0, v_{\max}[\to \mathbb{R}$*, determined by currency devaluation, is continuous on* $[0, v_{\max}[$*, twice continuously differentiable for* $v \in]0, +\infty[$ *and satisfies*
$$c(0) = 0, \quad c'(v) > 0, \quad c''(v) > \delta_0 > 0 \text{ for } v \in]0, v_{\max}[, \text{ and } \lim_{v \to v_{\max}^-} c(v) = +\infty.$$

We extend the definition of the function L, c to lower semicontinuous function defined on the whole of $\mathbb{R}$, keeping the same names, by setting $L(u) = +\infty$ for $u \notin [0, 1[$ and $c(v) = +\infty$ for $v \notin [0, v_{\max}[$. With a slight abuse of notation, we will write $L'(0)$, and $c'(0)$ to denote $\lim_{u \to 0^+} L'(u)$ and $\lim_{v \to 0^+} c'(v)$, respectively.

By a Dynamic Programming argument, it is never convenient for the borrower to declare bankruptcy unless he is not forced to do so, i.e., unless the threshold x^* is reached.

Lemma 2.1. *If (A1)-(A2) hold then for any admissible control strategy* $(u(\cdot), v(\cdot), x')$*, there exists a control strategy* $(\hat{u}(\cdot), \hat{v}(\cdot), x^*)$ *with smaller cost.*

Proof. By contradiction, assume that the borrower implement any strategy $(u(\cdot), v(\cdot), x')$ with $x' < x^*$. We can construct a better strategy simply avoiding to declare bankruptcy at x', and switching off the controls (u, v) after having reached x' until the threshold x^* is reached. Define $(\hat{u}(t), \hat{v}(t)) = (u(t), v(t))$ for $0 \leq t \leq T_b(x')$ and $(\hat{u}(t), \hat{v}(t)) = (0, 0)$ for $T_b(x') \leq t \leq T_b^*$. In this case, recalling that $L(0) = c(0) = 0$, we have
$$\begin{aligned}
J[x_0, x', u(\cdot), v(\cdot)] =& E\left[\int_0^{T_b(x')} e^{-rt}\Big[L(u(x(t))) + c(v(x(t)))\Big]\, dt + e^{-rT_b(x')} B\right]_{x(0)=x_0} \\
=& E\left[\int_0^{T_b^*} e^{-rt}\Big[L(u(x(t))) + c(v(x(t)))\Big]\, dt + e^{-rT_b(x')} B\right]_{x(0)=x_0} \\
\geq& E\left[\int_0^{T_b^*} e^{-rt}\Big[L(\hat{u}(x(t))) + c(\hat{v}(x(t)))\Big]\, dt + e^{-rT_b^*} B\right]_{x(0)=x_0}
\end{aligned}$$

$$=J[x_0, x^*, \hat{u}(\cdot), \hat{v}(\cdot)].$$

□

Thus from now on we will always assume that $x' = x^*$, and the goal of the borrower is to minimize

$$J[x_0, x^*, u(\cdot), v(\cdot)] = E\left[\int_0^{T_b^*} e^{-rt}\Big[L(u(x(t))) + c(v(x(t)))\Big]\,dt + e^{-rT_b^*}B\right]_{x(0)=x_0}. \tag{2.3}$$

To complete the model, we need an equation determining the discounted bond price $\tilde{p}(\cdot)$ in the evolution equation (2.1) for $x(\cdot)$. For every $x > 0$, denote by $\theta(x)$ the salvage rate, i.e., the fraction of the outstanding capital that can be recovered by lenders, if bankruptcy occurs when the debt has size x^*. As in [5], assuming that the investors are risk-neutral, the discounted bond price coincides with the expected payoff to a lender purchasing a coupon with unit nominal value

$$\begin{aligned} p(x_0) = E\Big[&\int_0^{T_b^*}(r+\lambda)\exp\left\{-\int_0^t\big(\lambda + r + v(x(s))\big)\,ds\right\}dt \\ &+ \exp\left\{-\int_0^{T_b^*}(r+\lambda+v(x(t)))dt\right\}\cdot\theta(x^*)\Big]_{x(0)=x_0}, \end{aligned} \tag{2.4}$$

for every initial debt $x_0 \in [0, x^*]$. We then set $\tilde{p}(t) = p(x(t))$.

After having described the model, we introduce the definition of optimal solution in feedback form.

Definition 2.2 (Stochastic optimal feedback solution)**.** *In connection with the above model, we say that a triple of functions $(\bar{u}(\cdot), \bar{v}(\cdot), \bar{p}(\cdot))$ provides an optimal solution to the problem of optimal debt management* (2.1)*)–*(2.4) *if*

(i) Given the function $\bar{p} : [0, x^*] \to [0, 1]$, for every initial value $x_0 \in [0, x^*]$ the feedback control strategy $(\bar{u}(\cdot), \bar{v}(\cdot))$ with stopping time T_b^* as in (2.2) *provides an optimal solution to the stochastic control problem* (2.3) *subject to dynamics* (2.1) *with $\tilde{p}(t) = \bar{p}(x(t))$.*

(ii) Given the feedback control strategy $(\bar{u}(\cdot), \bar{v}(\cdot))$, for every initial value $x_0 \in [0, x^*]$, the discounted price $\bar{p}(x_0)$ satisfies (2.4)*, where T_b^* is the stopping time* (2.2) *determined by the dynamics* (2.1) *with $\tilde{p}(t) = \bar{p}(x(t))$.*

For a given function p, we denote by the *value function* $V : [0, x^*] \to [0, +\infty[$ of the control system (2.1) with $\tilde{p}(t) = p(x(t))$ and cost given by (2.3) is

$$V(x_0) = \inf J[x_0, x^*, u(\cdot), v(\cdot)], \tag{2.5}$$

Under the assumptions **(A1)-(A2)**, the Hamiltonian function $H : [0, x^*] \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ associated to the dynamics (2.1) and to the cost functions L, c in (2.3) is defined by

$$H(x, \xi, p) \doteq -L^\circ\left(\frac{\xi}{p}\right) - c^\circ(x\xi) + \left(\frac{\lambda + r}{p} - \lambda - \mu + \sigma^2\right)x\,\xi, \tag{2.6}$$

where L°, c° are the convex conjugate of L, c respectively (see Appendix A).

The necessary conditions for optimality imply that the value function $V(\cdot)$ solves the second order implicit ODE

$$rV(x) = H(x, V'(x), p(x)) + \frac{\sigma^2 x^2}{2}V''(x).$$

Recalling Lemma B.1 and Lemma B.2, as soon as the value function V is determined, the optimal feedback strategies are

$$u^*(V'(x),p(x)) := \begin{cases} 0, & \text{if} \quad \dfrac{V'(x)}{p(x)} \le L'(0), \\ (L')^{-1}\left(\dfrac{V'(x)}{p(x)}\right), & \text{if} \quad \dfrac{V'(x)}{p(x)} > L'(0), \end{cases} \tag{2.7}$$

and

$$v^*(x,V'(x)) \ := \begin{cases} 0, & \text{if} \quad V'(x)x \le c'(0)\,, \\ (c')^{-1}\left(V'(x)x\right), & \text{if} \quad V'(x)x > c'(0). \end{cases} \tag{2.8}$$

On the other hand, if the feedback optimal controls $\bar{u}(x)$ and $\bar{v}(x)$ are known, then by using Feynman-Kac formula, we obtain the second order nonlinear ODE for the discounted bond price $p(\cdot)$ in (2.4)

$$\begin{aligned} &(r+\lambda+\bar{v}(x))p(x)-(r+\lambda) \\ &\qquad = \left[\left(\frac{\lambda+r}{p(x)}-\lambda-\mu+\sigma^2-\bar{v}(x)\right)x-\frac{\bar{u}(x))}{p(x)}\right]\cdot p'(x)+\frac{(\sigma x)^2}{2}p''(x). \end{aligned}$$

Concerning boundary conditions, we notice that

- since bankruptcy is istantaneously declared at x^*, the optimal cost $V(x^*)$ starting from the DTI level x^* reduces only to the bankruptcy cost B, while if we start from zero DTI level, $x(t)\equiv 0$ is a solution of (2.1) yielding $V(0)=0$ since, for this trajectory bankruptcy never occurs, and so $T_b^*=+\infty$ and we take $u(0)=v(0)=0$ thus the cost is zero.
- with the same argument, at the bankruptcy threshold the cost of a unitary bond $p(x^*)$ must be equal to the salvage rate $\theta(x^*)$ since this is the only money that will be repayed to the lender, while at the zero DTI level, using the same trajectory described in the previous case, we have $p(0)=1$.

Recalling (2.6)-(2.8), and Lemma B.2, we are thus led to the system of second order implicit ODEs

$$\begin{cases} rV(x) = H(x,V'(x),p(x)) + \dfrac{\sigma^2x^2}{2}\cdot V''(x), \\ (r+\lambda+v(x))p(x)-(r+\lambda) = H_\xi(x,V'(x),p(x))\cdot p'(x) + \dfrac{(\sigma x)^2}{2}\cdot p''(x), \\ v(x) = \underset{v\ge 0}{\operatorname{argmin}}\{c(v)-vxV'(x)\}, \end{cases} \tag{2.9}$$

with the boundary conditions

$$V(0)=0, \qquad V(x^*)=B \qquad \text{and} \qquad p(0)=1, \qquad p(x^*)=\theta(x^*). \tag{2.10}$$

Therefore, an optimal feedback solution to the problem of optimal debt management (2.1)–(2.4) will be obtained by solving the above system of ODEs for the value function $V(\cdot)$ and for the discounted bond price $p(\cdot)$ for a given bankruptcy threshold x^*.

3. Stochastic optimal feedback solutions

In this section, we prove the existence of an optimal feedback solution to the problem of optimal debt management (2.1)–(2.4) for a given bankruptcy threshold x^*, and then we will study the asymptotic behavior of the solution as $x^* \to +\infty$.

3.1. Existence of optimal feedback solutions.

Given a bankruptcy threshold x^*, we introduce the constant

$$\theta_{\min} \doteq \min\left\{\theta(x^*), \frac{r+\lambda}{r+\lambda+v_{\max}}\right\}. \tag{3.1}$$

which will be a lower bound of the discount bond price p.

Our main result of this subsection is the following.

Theorem 3.1. *In addition to* ***(A1)-(A2)****, assume that*

(A3) *The volatility satisfies* $\sigma > 0$*, the recover fraction at the bankruptcy threshold satisfies* $\theta(x^*) > 0$ *and the devaluation rate is bounded, i.e.,* $0 < v_{\max} < +\infty$*.*

Then there exists a constant $M^* > 0$ *such that the system of second-order ODEs* (2.9) *with boundary conditions* (2.10) *admits a solution*

$$(V(\cdot), p(\cdot)) : [0, x^*] \to \times[0, B] \times [\theta_{\min}, 1]$$

of class C^2*. Moreover, the function* $V(\cdot)$ *is monotone increasing and* $v(x) = 0$ *for all* $x \in \left[0, \min\left\{\frac{c'(0)}{M^*}, x^*\right\}\right]$.

It is well-known (see Theorem 4.1, p.149, in [9] or Theorem 11.2.2, p. 141, in [13]) that if $(V(\cdot), p(\cdot))$ is a solution to the boundary value problem (2.9)-(2.10), then a standard result in the theory of stochastic optimization implies that the feedback control strategy $(u^*(V'(\cdot), p(\cdot)), v^*(\cdot, V'(\cdot)))$ given by (2.7)-(2.8) is optimal for the problem (2.3) with dynamics (2.1). As a consequence, from Theorem 3.1 we deduce

Corollary 3.2. *Under the same assumptions of Theorem 3.1, the debt management problem* (2.3) *with dynamics* (2.1) *admits an optimal control strategy in feedback form. Moreover, there is a threshold level for DTI such that the optimal control strategy does not use currency devaluation for values of DTI below that threshold.*

The main idea of the proof of Theorem 3.1 is to define a family of auxiliary uniformly parabolic evolutive problems, indexed by a parameter $\varepsilon > 0$, whose steady states will provide an approximate solution of (2.9). For each of such problems, we prove the existence of the steady state, by constructing a compact, convex and positively invariant set of functions $(V, p) : [0, x^*] \mapsto [0, B] \times [\theta_{\min}, 1]$. A topological technique will then yield the existence of a steady state $V_\varepsilon(\cdot)$ for the ε-problem. After having derived some uniform estimates on the approximate solution $V_\varepsilon(\cdot)$, we are able to pass to the limit as $\varepsilon \to 0^+$, obtaining a solution of (2.9). The main difficulty is the degeneration of the coefficient of the second derivative of $V(\cdot)$ in (2.9), which must be tackled by a suitable approximation argument.

Proof of Theorem 3.1. From assumptions **(A1)-(A3)** and Lemma B.2, for all $(x, \xi) \in [0, +\infty[\times\mathbb{R}$ it holds

$$v^*(x,\xi) \le v_{\max}, \qquad |v^*_x(x,\xi)| \le \frac{1}{\delta_0}\cdot|\xi| \qquad \text{and} \qquad |v^*_\xi(x,\xi)| \le \frac{1}{\delta_0}\cdot|x|. \tag{3.2}$$

For any $\varepsilon > 0$, consider the parabolic system in the unknown $\mathbf{V} = \mathbf{V}(t,x)$, $\mathbf{p} = \mathbf{p}(t,x)$

(3.3)
$$\begin{cases} \mathbf{V}_t &= -r\mathbf{V} + H(x, \mathbf{V}_x, \mathbf{p}+\varepsilon) + \left(\varepsilon + \dfrac{(\sigma x)^2}{2}\right)\mathbf{V}_{xx}, \\ \\ \mathbf{p}_t &= (r+\lambda) - (r+\lambda+v^*(x,\mathbf{V}_x))\mathbf{p} + H_\xi(x,\mathbf{V}_x,\mathbf{p}+\varepsilon)\mathbf{p}_x + \left(\varepsilon + \dfrac{(\sigma x)^2}{2}\right)\mathbf{p}_{xx}, \end{cases}$$

together with the boundary conditions

$$\begin{cases} \mathbf{V}(t,0) &= 0, \\ \\ \mathbf{V}(t,x^*) &= B, \end{cases} \qquad \begin{cases} \mathbf{p}(t,0) &= 1, \\ \\ \mathbf{p}(t,x^*) &= \theta(x^*), \end{cases} \tag{3.4}$$

and notice that the system is uniformly parabolic also in a neighborhood of $x = 0$. The remaining part of the proof is divided into three main steps.

Step 1. *The system* (3.3)-(3.4) *admits a steady state solutions* $(V_\varepsilon(\cdot), p_\varepsilon(\cdot))$ *for every given* $\varepsilon > 0$*. Moreover, for* ε *sufficiently small,* $V_\varepsilon(\cdot)$ *is monotone increasing.*

(i). Recalling Theorem 1 in [2], for every pair $(V_0, p_0) \in C^2([0,x^*]) \times C^2([0,x^*])$, the system (3.3)-(3.4) with the initial data

$$\mathbf{V}(0,x) = V_0(x) \qquad \text{and} \qquad \mathbf{p}(0,x) = p_0(x) \tag{3.5}$$

admits a unique solution $(\mathbf{V}(t,x), \mathbf{p}(t,x))$ in $C^2([0,T]\times[0,x^*]) \times C^2([0,T]\times[0,x^*])$. Adopting a semigroup notation, let $t \mapsto S_t(V_0,p_0) := (\mathbf{V}(t,\cdot), \mathbf{p}(t,\cdot))$ be the solution of (3.3)-(3.4) with initial data (3.5). Recalling the definition of $\theta_{\min}$ in (3.1), we claim that the following closed, convex set of C^2 functions

$$\mathcal{D} = \left\{(V_0,p_0) \in C^2([0,x^*];[0,B]) \times C^2([0,x^*];[\theta_{\min},1]) : V_0, p_0 \text{ satisfy } (2.10)\right\}.$$

is positively invariant under the semigroup S_t, namely

$$S_t(\mathcal{D}) \subseteq \mathcal{D}, \text{ for all } t \geq 0.$$

Indeed, consider the constant functions

$$\begin{cases} \mathbf{V}^+(t,x) &= B, \\ \\ \mathbf{V}^-(t,x) &= 0, \end{cases} \qquad \begin{cases} \mathbf{p}^+(t,x) &= 1, \\ \\ \mathbf{p}^-(t,x) &= \theta_{\min}. \end{cases}$$

Recalling Lemma B.3 (2), we have

$$H(x, \mathbf{V}^+_x, \mathbf{p}+\varepsilon) = H(x, \mathbf{V}^-_x, \mathbf{p}+\varepsilon) = H(x, 0, \mathbf{p}+\varepsilon) = 0.$$

It can be easily checked that $\mathbf{V}^+$ is a supersolution and $\mathbf{V}^-$ is a subsolution of the first scalar parabolic equation in (3.3). Moreover, $\mathbf{p}^+$ is a supersolution and $\mathbf{p}^-$ is a subsolution of the second scalar parabolic equation in (3.3). Therefore, for any pair of functions (V_0, p_0) in (3.5) taking values in the box $[0,B] \times [\theta_{\min}, 1]$, the solution of the system (3.3) will satisfy

$$0 \leq \mathbf{V}(t,x) \leq B \qquad \text{and} \qquad \theta_{\min} \leq \mathbf{p}(t,x) \leq 1$$

for all $(t,x) \in [0. +\infty) \times [0,x^*]$.

(ii). Thanks to the bounds of Lemma B.3 (1) and (3.2) we can apply Theorem 3 in [2] and obtain the existence of a steady state $(V_\varepsilon(\cdot), p_\varepsilon(\cdot)) \in \mathcal{D}$ for the system (3.3)-(3.4), i.e., a solution of

$$\begin{cases} -rV + H(x, V', p+\varepsilon) + \left(\varepsilon + \dfrac{(\sigma x)^2}{2}\right) V'' = 0\,, \\ \\ (r+\lambda) - (r+\lambda+v(x))p + H_\xi(x, V', p+\varepsilon)p' + \left(\varepsilon + \dfrac{(\sigma x)^2}{2}\right) p'' = 0\,, \end{cases} \tag{3.6}$$

with

$$v(x) \doteq v^*(x, V'(x)) = \begin{cases} 0, & \text{if } V'(x)x \leq c'(0) \\ \\ (c')^{-1}\left(V'(x)x\right), & \text{if } V'(x)x > c'(0), \end{cases}$$

To complete this step, we show that V_ε is monotone increasing for $\varepsilon > 0$ sufficiently small. Since $V_\varepsilon(0) = 0$, $V_\varepsilon(x^*) = B$ and $V_\varepsilon(x) \in [0, B]$ for all $x \in [0, B]$, it holds

$$\lim_{x \to 0+} V'_\varepsilon(x) \geq 0 \qquad \text{and} \qquad \lim_{x \to x^*-} V'_\varepsilon(x) \geq 0.$$

Assume by a contradiction that V_ε is not monotone increasing. Then there exists $x_0 \in]0, x^*[$ such that

$$V'_\varepsilon(x_0) < 0 \qquad \text{and} \qquad V''_\varepsilon(x_0) = 0.$$

In particular, the first equation of (3.6) implies that

$$H(x_0, V'_\varepsilon(x_0), p_\varepsilon(x_0) + \varepsilon) = rV_\varepsilon(x_0) \geq 0.$$

On the other hand, for any given $0 < \varepsilon < \dfrac{r-\mu}{\lambda+\mu}$, since $p_\varepsilon(x_0) \leq 1$, it holds

$$\frac{\lambda + r}{p_\varepsilon(x_0) + \varepsilon} - (\lambda+\mu) + \sigma^2 \geq \frac{\lambda+r}{1+\varepsilon} - (\lambda+\mu) + \sigma^2 = \frac{r - \mu - (\lambda+\mu)\varepsilon}{1+\varepsilon} + \sigma^2 > 0.$$

Thus, Lemma B.3 (1) yields

$$H(x_0, V'_\varepsilon(x_0), p_\varepsilon(x_0) + \varepsilon) \leq \left(\frac{\lambda+r}{p_\varepsilon(x_0)+\varepsilon} - \lambda - \mu + \sigma^2\right) x_0\, V'_\varepsilon(x_0) < 0,$$

and it yields a contradiction, since the left hand side is nonnegative.

Step 2. *We now derive a priori estimate on these stationary solutions, which will allow to pass to the limit as $\varepsilon \to 0^+$.*

(i). We start providing an upper bound for $\|V'_\varepsilon\|_{L^\infty(]0,x^*[)}$. For any $(x, \xi, p) \in [0, x^*] \times [0, +\infty[\times]0, 1]$, since $c^\circ(v) \geq 0$, $\lambda, \mu \geq 0$, it holds

$$\begin{aligned} H(x, \xi, p+\varepsilon) \;&\leq\; \min_{u \in [0,1]} \left\{ L(u) - \frac{u}{p+\varepsilon}\,\xi \right\} + \left(\frac{\lambda + r}{p+\varepsilon} + \sigma^2\right) x\xi \\ &\leq \frac{2(\lambda + r + \sigma^2)x - 1}{2p + 2\varepsilon} \cdot \xi + L\left(\frac{1}{2}\right). \end{aligned}$$

In particular, for $0 < \varepsilon < 1$, set $\bar{x}_1 \doteq \min\left\{\frac{1}{4(\lambda+r+\sigma^2)}, \frac{x^*}{2}\right\}$ and $\overline{M}_1 \doteq 8L\left(\frac{1}{2}\right) > 0$, it holds

$$H(x,\xi,p+\varepsilon) \le -\frac{1}{4p+4\varepsilon}\cdot\xi + L\left(\frac{1}{2}\right) < 0 \tag{3.7}$$

for all $(x,\xi,p) \in [0,\bar{x}_1]\times[\overline{M}_1,+\infty[\times[0,1]$. As a consequence if $V_\varepsilon'(\tilde{x}) > \overline{M}_1$ for a certain $\tilde{x}\in[0,\bar{x}_1]$, from the first equation of (3.6), we have that $V_\varepsilon''(\tilde{x}) > 0$, and so $\tilde{x}$ cannot be a local maximum for $V_\varepsilon'(\cdot)$. This implies that either $V_\varepsilon'(x) \le \overline{M}_1$ for all $x \in [0,x^*]$, or $V_\varepsilon'(\cdot)$ attains its maximum in $[\bar{x}_1, x^*]$.

Hence, we only need to show that V_ε' is bounded in $[\bar{x}_1,x^*]$. Since $V_\varepsilon'(x)\ge 0$ and $V_\varepsilon(\cdot)$ is bounded from above by B, from the first equation of (3.6) and Lemma B.3 (1), one gets

$$\begin{aligned}|V_\varepsilon''(x)| =& \frac{2}{2\varepsilon+\sigma^2x^2}\cdot\big(|H(x,V_\varepsilon'(x),p_\varepsilon(x))| + rV_\varepsilon(x)\big)\\ \le& \frac{2}{\sigma^2\bar{x}_1^2}\cdot\left[\frac{1+(r+\lambda)x^*}{\theta_{\min}} + (\sigma^2+\mu+\lambda+v_{\max})\cdot x^*\right]\cdot V_\varepsilon'(x) + \frac{2rB}{\sigma^2\bar{x}_1^2}\end{aligned}$$

for all $x\in[\bar{x}_1,x^*]$. On the other hand, by the mean value theorem, there exists a point $\hat{x}\in[\bar{x}_1,x^*]$ where

$$V_\varepsilon'(\hat{x}) = \frac{V_\varepsilon(x^*)-V_\varepsilon(\bar{x}_1)}{x^*-\bar{x}_1} \le \frac{2B}{x^*}$$

By Grönwall's lemma, from the above differential inequality and the estimate on $V_\varepsilon(\hat{x})$, we obtain an upper bound on $V_\varepsilon'(x)$, uniform in ε, for all $x\in[\bar{x}_1,\hat{x}]\cup[\hat{x},x^*]$. Therefore there exists $M^*>0$ which does not depend on ε such that

$$0 \le V_\varepsilon'(x) \le M^* \qquad \text{for all } x \in]0,x^*[. \tag{3.8}$$

As a consequence, (3.1) implies that

$$0 \le v_\varepsilon(x) \doteq v^*(x,V_\varepsilon'(x)) \le (c')^{-1}(M^*x) \qquad \text{for all } x\in[0,x^*] \tag{3.9}$$

and, in particular, we have the following estimate concerning the use of no devaluation in the control strategy

$$v_\varepsilon(x) = 0 \quad \text{for all } x \in \left[0, \min\left\{\frac{c'(0)}{M^*}, x^*\right\}\right].$$

(ii). We now provide uniform bounds on $\|V_\varepsilon''\|_{L^\infty([\delta,x^*[)}$, $\|p_\varepsilon'\|_{L^\infty([\delta,x^*[)}$ and $\|p_\varepsilon''\|_{L^\infty([\delta,x^*[)}$ for any fixed $0<\delta<x^*$. From Lemma B.3 (1) and (3.8), recalling that $p_\varepsilon(x)\ge\theta_{\min}$, there exist $K_1,K_2>0$, which are independent on ε, such that

$$\big|H(x,V_\varepsilon'(x),p_\varepsilon(x))\big| \le K_1 \qquad \text{and} \qquad \big|H_\xi(x,V_\varepsilon'(x),p_\varepsilon(x))\big| \le K_2$$

for all $x\in[0,x^*]$. Thus, the first equation in (3.6) yields

$$\|V_\varepsilon''\|_{L^\infty([\delta,x^*[)} \le \tilde{C}_{1,\delta} \doteq \frac{2rB+2K_1}{\sigma^2\delta^2}. \tag{3.10}$$

Moreover, multiplying the second equation in (3.6) by $p_\varepsilon'(x)$ and using the estimates on H_ξ, we have

$$p_\varepsilon''(x)p_\varepsilon'(x) \le \frac{2K_2}{\sigma^2\delta^2}\cdot\big|p_\varepsilon'(x)\big|^2 + \frac{2(r+\lambda+v_{\max})}{\sigma^2\delta^2}\cdot|p_\varepsilon'(x)| \quad \text{for all } x\in[\delta,x^*).$$

Set $z_\varepsilon(x) = \dfrac{1}{2}\left|p'_\varepsilon(x)\right|^2$, we have for all $x \in [\delta, x^*[$

$$\begin{aligned}\dot z_\varepsilon(x) &\le \frac{4K_2}{\sigma^2\delta^2}\cdot z_\varepsilon(x) + \frac{2\sqrt{2}(r+\lambda+v_{\max})}{\sigma^2\delta^2}\cdot\sqrt{z_\varepsilon(x)}\\ &\le \frac{4K_2}{\sigma^2\delta^2}\cdot z_\varepsilon(x) + \frac{\sqrt{2}(r+\lambda+v_{\max})}{\sigma^2\delta^2}\cdot(z_\varepsilon(x)-1)+1.\end{aligned}$$

By the mean value theorem, there exists a point $\hat{x} \in [\delta, x^*]$ such that

$$z_\varepsilon(\hat{x}) = |p'_\varepsilon(\hat{x})|^2 = \frac{1}{2}\left|\frac{p_\varepsilon(x^*)-p_\varepsilon(\delta)}{x^*-\delta}\right|^2 \le \frac{1}{2(x^*-\delta)^2}.$$

The Grönwall's lemma applied to $z_\varepsilon(\cdot)$, together with the second equation in (3.6), yield

$$\|p'_\varepsilon\|_{L^\infty([\delta,x^*[)} \le \tilde{C}_{2,\delta} \qquad \text{and} \qquad \|p''_\varepsilon\|_{L^\infty([\delta,x^*[)} \le \tilde{C}_{3,\delta} \tag{3.11}$$

for some constants $\tilde{C}_{2,\delta}, \tilde{C}_{3,\delta}$, and the estimate is valid uniformly as $\varepsilon \to 0^+$.

Step 3. *We finally prove the existence of a solution to* (2.9)-(2.10) *by letting* $\varepsilon \to 0^+$.

(i). Recalling (2.6)-(2.7), Lemma B.2, and (3.8), we obtain that H and H_ξ are uniformly Lipschitz on $[\delta, x^*] \times [0, M^*] \times [\theta_{\min}, 1]$. Hence, the functions

$$(V_\varepsilon)'' = \frac{2}{2\varepsilon+\sigma^2x^2}\cdot\left[rV_\varepsilon - H(x,(V_\varepsilon)',p_\varepsilon+\varepsilon)\right]$$

and

$$(p_\varepsilon)'' = \frac{2}{2\varepsilon+\sigma^2x^2}\cdot\left[(r+\lambda)\cdot(p_\varepsilon-1) - H_\xi(x,(V_\varepsilon)',p_\varepsilon+\varepsilon)p'_\varepsilon\right]$$

are also uniformly bounded and uniformly Lipschitz on $[\delta, x^*]$. Thus, by Ascoli-Arzelà theorem, choosing a suitable sequence $\varepsilon_n \to 0^+$, we have the uniform convergence $(V_{\varepsilon_n}, p_{\varepsilon_n}) \to (V, p)$ in $C^2(]\delta, x^*[)$ for all $\delta > 0$, where V, p are twice continuously differentiable and solve the system of ODEs (2.9) on the open interval $]0, x^*[$. Moreover, recalling (3.8), (3.10), and (3.11), it holds

$$\lim_{x\to x^{*-}} V(x) = B, \qquad \lim_{x\to x^{*-}} p(x) = \theta(x^*) \qquad \text{and} \qquad \lim_{x\to 0^+} V(x) = 0.$$

(ii). To complete the proof, we will show that $\lim\limits_{x\to 0^+} p(x) = 1$ by providing a lower bound for $p_\varepsilon(\cdot)$ in a right neighborhood of $x = 0$, independent of ε. Consider the function

$$p^-(x) = 1 - cx^\gamma$$

where

$$c = 1 + \frac{2(M^*)^2}{(r+\lambda)\cdot\delta} + \frac{1}{x^*} \quad \text{and} \quad \gamma = \min\left\{\frac{1}{2}\,,\ \frac{(r+\lambda)}{\left(\frac{\lambda+r}{\theta_{\min}} - \lambda - \mu + \sigma^2\right)}\right\}.$$

We prove that $p^-(\cdot)$ is a lower solution of the second equation of (3.6) in the interval $[0, \bar{x}_0]$ with

$$0 < x_0 = \left(\frac{1-\theta_{\min}}{c}\right)^{\frac{1}{\gamma}} \le \min\{1, x^*\}.$$

Indeed, it is clear that

$$p^-(\bar{x}_0) = \theta_{\min} \leq p_\varepsilon(\bar{x}_0), \quad 1 = p^-(0) = p_\varepsilon(0) \quad \text{and} \quad p^-(x) \geq 0. \tag{3.12}$$

On the other hand, recalling (3.9) and assumption **(A2)**, we have

$$v_\varepsilon(x) = v^*(x, V'_\varepsilon(x)) \leq (c')^{-1}(M^* x) \ \leq \ \frac{(M^*)^2}{\delta} \cdot x \qquad \text{for all } x \in [0, x^*].$$

Thus, by (3.6) and Lemma B.2, we obtain that

$$\begin{aligned}
&(r+\lambda) - (r+\lambda+v_\varepsilon(x))p^- + H_\xi(x, V'_\varepsilon, p^-)p^{-\prime} + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) p^{-\prime\prime} \\
&\quad = - v_\varepsilon(x) + (r+\lambda)cx^\gamma - H_\xi(x, V'_\varepsilon, p^-)\, c\gamma x^{\gamma-1} + c\left(\varepsilon + \frac{(\sigma x)^2}{2}\right)\gamma(1-\gamma)x^{\gamma-2} \\
&\quad \geq (r+\lambda)cx^\gamma - H_\xi(x, V', p^-)\, c\gamma x^{\gamma-1} - \frac{(M^*)^2}{\delta} \cdot x \\
&\quad \geq \frac{r+\lambda}{2} \cdot cx^\gamma - \frac{(M^*)^2}{\delta} \cdot x \geq \frac{(r+\lambda)x^\gamma}{2} \cdot \left(c - \frac{2(M^*)^2}{(r+\lambda)\cdot\delta} \cdot \bar{x}_0^{1-\gamma}\right) \geq 0
\end{aligned}$$

for all $x \in]0, \bar{x}_0]$. Recalling (3.12), a standard comparison argument yields

$$p_\varepsilon(x) \geq p^-(x) = 1 - cx^\gamma \qquad \text{for all } x \in [0, \bar{x}_0].$$

and this implies

$$p(x) = \lim_{\varepsilon_n \to 0+} \ p_{\varepsilon_n}(x) \geq 1 - cx^\gamma \qquad \text{for all } x \in [0, \bar{x}_0].$$

Since $p(x) \in [0,1]$ for all $x \in [0, x^*]$, we conclude that $\lim_{x\to 0^+} p(x) = 1$.

□

3.2. **Dependence on the bankruptcy threshold x^*.** In this subsection, we will study the behavior of the expected total optimal cost for servicing the debt when the maximum size x^* of the DTI, at which bankruptcy is declared, becomes very large. More precisely, for a given $x^* > 0$, let $(V(\cdot, x^*), p(\cdot, x^*))$ be a solution to the system of second order ODEs (2.9) with boundary conditions (3.4). We investigate whether, as $x^* \to \infty$, the value function $V(\cdot, x^*)$ remains strictly positive or approaches zero uniformly on bounded sets.

From the point of view of the model, the latter situation corresponds to a *Ponzi's scheme*, where no actual effort is made by the borrower to repay the lenders, and the debt and its interests are served just by borrowing more money from the investors.

It turns out that a crucial role in the asymptotic behavior of $V(\cdot, x^*)$ as $x^* \to +\infty$ is played by the *speed of decay* of the salvage rate $\theta(x^*)$ as $x^* \to +\infty$, which represents the fraction of the investment that can be recovered by the investors after the bankruptcy (and the unitary bond discounted price at the bankruptcy threshold).

If the salvage rate decay sufficiently slowly, i.e., the lenders can still recover a sufficiently high fraction of their investment after the bankruptcy, then the best choice for the borrower is to implement the Ponzi's scheme. On the other hand, if the salvage rate $\theta(x^*)$ decays sufficiently fast, then Ponzi's scheme is no longer an optimal solution for the borrower.

Theorem 3.3. *Under the same assumptions as in Theorem 3.1, the followings hold:*

(i) if $\liminf_{s\to+\infty} \theta(s)s = +\infty$ *then*

$$\lim_{x^*\to\infty} V(x,x^*) = 0 \qquad \textit{for all } x \in [0,\infty[\tag{3.13}$$

(ii) if $\limsup_{s\to+\infty} \theta(s)s = C_1 < +\infty$ *then*

$$\liminf_{x^*\to\infty} V(x,x^*) \geq \frac{rB}{2(2r+v_{\max}-\mu)} \tag{3.14}$$

$$\textit{for all } x \geq \frac{2\Big(r+v_{\max}+(1+C_1(r+v_{\max}))(r-\mu)\Big)}{r(r-\mu)}.$$

Proof. For any fixed $x^* > 0$, let $(V(\cdot,x^*), p(\cdot,x^*))$ be a solution to (2.9) with boundary conditions (3.4) and set

$$v(x,x^*) \doteq \operatorname*{argmin}_{\omega\geq 0}\{c(\omega) - \omega x V'(x,x^*)\}.$$

With the same argument of the proof of Theorem 3.1, we obtain that

$$(V(x,x^*), p(x,x^*)) \in [0,B] \times [\theta_{\min}, \theta(x^*)],$$

and $V(\cdot,x^*)$ is increasing. Moreover, there exists $M^* > 0$ depending on x^* such that $V'(x,x^*) \leq M^*$ for all $x \in [0,x^*]$ and

$$v(x,x^*) = 0 \qquad \text{for all } x \in \left[0, \bar{x}_0 \doteq \frac{c'(0)}{M^*}\right]. \tag{3.15}$$

In order to achieve (i) and (ii), we construct for upper and lower bounds for $(V(\cdot,x^*), p(\cdot,x^*))$, in the following form

$$V_2(x) \leq V(x,x^*) \leq V_1(x) \text{ and } p_1(x) \leq p(x,x^*) \leq p_2(x)$$

where

- the function $V_1(\cdot)$ and $V_2(\cdot)$ are a supersolution and a subsolution of the first equation in (2.9), respectively.
- the functions $p_1(\cdot)$ and $p_2(\cdot)$ are a subsolution and a supersolution of the second equation in (2.9), respectively.

To this aim, we introduce two constants

$$\gamma \doteq \frac{r+\lambda}{r+\lambda+v_{\max}} \qquad \text{and} \qquad \beta \doteq r+\lambda+v_{\max}.$$

Step 1. We first prove (i). Suppose that

$$\liminf_{s\to+\infty} \theta(s)s = +\infty\,. \tag{3.16}$$

We shall construct a suitable pair of functions $(V_1(\cdot), p_1(\cdot))$. Two cases are considered:

- If $\theta(x^*) \geq \gamma$ then let $V_1(\cdot)$ be the solution to the backward Cauchy problem

  $$rV_1(x) = \left(\beta+\sigma^2\right) xV_1'(x) \qquad \text{with} \qquad V_1(x^*) = B.$$

  Solving the above ODE, we obtain that

  $$V_1(x) = B \cdot \left(\frac{x}{x^*}\right)^{\frac{r}{\beta+\sigma^2}} \qquad \text{for all } x \in [0,x^*].$$

Since $0 < \dfrac{r}{\beta+\sigma^2} < 1$, it holds that $V_1'(x) > 0$ and $V_1''(x) < 0$ for all $x \in]0, x^*[$. Thus, from Lemma B.3 (1), for all $q \geq \gamma$ it holds

$$\begin{aligned} -rV_1(x) + H(x, V_1'(x), q) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \ &\leq \ -rV_1(x) + \left(\frac{r+\lambda}{\gamma} + \sigma^2\right) xV_1'(x) \\ &= \ -rV_1(x) + \left(\beta + \sigma^2\right) xV_1'(x) = 0\,. \end{aligned}$$

In particular, since from Theorem 3.1 and (3.1), it holds

$$p(x, x^*) \geq \theta_{\min} = \min\left\{\theta(x^*), \frac{r+\lambda}{r+\lambda+v_{\max}}\right\} = \gamma,$$

we have

$$-rV_1(x) + H(x, V_1', p(x, x^*)) + \frac{\sigma^2 x^2}{2} \cdot V_1''(x) \leq 0\,.$$

Thus V_1 is a super-solution of the first equation in (2.9). A standard comparison arguments yields

$$V(x, x^*) \leq V_1(x) = B \cdot \left(\frac{x}{x^*}\right)^{\frac{r}{\beta+\sigma^2}} \qquad \text{for all } x \in [0, x^*], \tag{3.17}$$

which proves (i) in this case.

- If $\theta(x^*) < \gamma$ then let $(\tilde{V}_1(\cdot), \tilde{p}_1(\cdot))$ be the solution to the backward Cauchy problem

$$\begin{cases} r\tilde{V}_1(x) &= \left(\dfrac{\lambda+r}{\tilde{p}_1(x)} + \sigma^2\right) x\tilde{V}_1'(x)\,, \\ \beta \cdot (\tilde{p}_1(x) - \gamma) &= \left(\dfrac{\lambda+r}{\tilde{p}_1(x)} + \sigma^2\right) x\tilde{p}_1'(x)\,, \end{cases} \qquad \begin{cases} \tilde{V}_1(x^*) &= B, \\ \tilde{p}_1(x^*) &= \theta(x^*). \end{cases} \tag{3.18}$$

We have that $\tilde{p}_1(\cdot)$ is strictly decreasing and for $x \neq 0$ it solves the implicit equation

$$\tilde{p}_1(x) = \frac{\theta(x^*)x^*}{x} \cdot \left(\frac{\gamma - \tilde{p}_1(x)}{\gamma - \theta(x^*)}\right)^{1+\frac{\sigma^2}{\beta}} \qquad \text{with} \qquad \lim_{x\to 0^+} \tilde{p}_1(x) = \gamma,$$

while $\tilde{V}_1(\cdot)$ is increasing and it can be expressed in terms of $\tilde{p}_1(\cdot)$ as follows

$$\tilde{V}_1(x) = B \cdot \left(\frac{\tilde{p}_1(x) \cdot x}{\theta(x^*) \cdot x^*}\right)^{\frac{r}{\beta+\sigma^2}} \leq B \cdot \left(\frac{x}{\theta(x^*)x^*}\right)^{\frac{r}{\beta+\sigma^2}} \tag{3.19}$$

for all $x \in [0, x^*]$. Using (3.18) and the implicit expression of $\tilde{p}_1(\cdot)$, we obtain

$$\begin{aligned} -1 =& \tilde{p}_1'(x) \cdot \left[\frac{x}{\tilde{p}_1(x)} + \left(1 + \frac{\sigma^2}{\beta}\right) \cdot \frac{x}{\gamma - \tilde{p}_1(x)}\right] \\ =& \tilde{p}_1'(x) \cdot \left[\frac{x}{\tilde{p}_1(x)} + \left(1 + \frac{\sigma^2}{\beta}\right) \cdot \left(\frac{[\theta(x^*)x^*]^{\frac{\beta}{\beta+\sigma^2}}}{\gamma - \theta(x^*)}\right) \cdot \frac{x^{\frac{\sigma^2}{\beta+\sigma^2}}}{\tilde{p}_1(x)^{\frac{\beta}{\beta+\sigma^2}}}\right]. \end{aligned}$$

Since $\tilde{p}_1$ is strictly decreasing, we have $\tilde{p}_1' < 0$, and from the above expression it follows that $\tilde{p}_1'(x)$ is increasing, and so $\tilde{p}_1''(x) > 0$ for all $x \in]0, x^*[$. Hence, from Lemma B.3 (1), it holds

$$(r+\lambda) - (r+\lambda+v(x, x^*))\tilde{p}_1 + H_\xi(x, V'(x, x^*), \tilde{p}_1)\tilde{p}_1' + \frac{\sigma^2 x^2}{2}\tilde{p}_1'' \tag{3.20}$$

$$\geq (r+\lambda)-(r+\lambda+v_{\max})\tilde{p}_1+\left(\frac{\lambda+r}{\tilde{p}_1}+\sigma^2\right)x\tilde{p}_1'+\frac{\sigma^2x^2}{2}\tilde{p}_1''$$
$$=\beta\cdot(\gamma-\tilde{p}_1)+\left(\frac{\lambda+r}{\tilde{p}_1}+\sigma^2\right)x\tilde{p}_1'+\frac{\sigma^2x^2}{2}\tilde{p}_1''=\frac{\sigma^2x^2}{2}\tilde{p}_1''>0$$

for all $x\in]0,x^*[$. On the other hand, recalling $\bar{x}_0=\dfrac{c'(0)}{M^*}$ in (3.15), let $\bar{p}_1(\cdot)$ be the solution of the backward Cauchy problem

$$(r+\lambda)(\bar{p}_1(x)-1)=\left(\frac{\lambda+r}{\bar{p}_1(x)}+\sigma^2\right)x\bar{p}_1'(x)\qquad\text{with}\qquad\bar{p}_1(\bar{x}_0)=\tilde{p}_1(\bar{x}_0)\,.$$

It is clear that

$$\bar{p}_1(x)\geq\tilde{p}_1(x)\qquad\text{for all } x\in[0,\bar{x}_0].$$

Arguing as in the case of $\tilde{p}_1(\cdot)$, we have that $\bar{p}_1$ is decreasing, $\lim_{x\to0+}\bar{p}_1(x)=1$, and

$$\bar{p}_1(x)=\frac{\tilde{p}_1(\bar{x}_0)\bar{x}_0}{x}\cdot\left(\frac{1-\bar{p}_1(x)}{1-\bar{p}_1(\bar{x}_0)}\right)^{\frac{\sigma^2+\lambda+r}{\lambda+r}}\qquad\text{for all } x\in[0,\bar{x}_0].$$

Moreover, $\bar{p}_1''(x)>0$ and

$$(r+\lambda)(1-\bar{p}_1)+H_\xi(x,V'(x,x^*),\bar{p}_1)\bar{p}_1'+\frac{\sigma^2x^2}{2}\bar{p}_1''>0\qquad\text{for all } x\in]0,\bar{x}_0[. \tag{3.21}$$

Let $p_1:[0,x^*]\to\mathbb{R}$ be such that

$$p_1(x)=\begin{cases}\bar{p}_1(x), & \text{for all } x\in[0,\bar{x}_0]\,,\\ \tilde{p}_1(x), & \text{for all } x\in[\bar{x}_0,x^*]\,,\end{cases}$$

we have

$$p_1(0)=p(0,x^*)=1\qquad\text{and}\qquad p_1(x^*)=p(x^*,x^*)=\theta(x^*).$$

Recalling that $v(x,x^*)=0$ for all $x\in[0,x_0]$, (3.20) and (3.21) imply that

$$(r+\lambda)-(r+\lambda+v(x,x^*))p_1+H_\xi(x,V'(x,x^*),p_1)p_1'+\frac{\sigma^2x^2}{2}p_1''>0$$

for all $x\in]0,x_0[\cup]x_0,x^*[$. Thus, $p_1(\cdot)$ is a sub-solution of the second equation in (2.9) and a standard comparison arguments yields

$$p(x,x^*)\geq p_1(x)\geq\tilde{p}_1(x)\qquad\text{for all } x\in[0,x^*]\,. \tag{3.22}$$

To complete this step, we define

$$V_1(x)\doteq\begin{cases}\tilde{V}_1(\bar{x}_1), & \text{for } x\in[0,\bar{x}_1]\\ \tilde{V}_1(x), & \text{for } x\in[\bar{x}_1,x^*]\end{cases}\qquad\text{with}\quad \bar{x}_1\doteq\min\left\{\frac{1}{r+\lambda},\bar{x}_0,x^*\right\}$$

For any $x\in]0,\bar{x}_1[$, it holds that $V_1(x)=\tilde{V}_1(\bar{x}_1)$ and thus

$$-rV_1(x)+H(x,V_1'(x),p(x,x^*))+\frac{\sigma^2x^2}{2}V_1''(x)=-r\tilde{V}_1(x_1)<0\,. \tag{3.23}$$

On the other hand, from Lemma B.2, the map $p \mapsto H(x,\xi,p)$ is monotone decreasing when $\xi \geq 0$ and $x \geq \frac{1}{\lambda+r} \geq \bar{x}_1$. Thus, recalling (3.22), (3.18), and Lemma B.3 (1), we have that all $x \in [\bar{x}_1, x^*]$, it holds

$$\begin{aligned} -rV_1(x) + H(x, V_1'(x), p(x,x^*)) &\leq -rV_1(x) + H(x, V_1'(x), \tilde{p}_1(x)) \\ &= -r\tilde{V}_1(x) + H(x, \tilde{V}_1'(x), \tilde{p}_1(x)) \\ &\leq -r\tilde{V}_1(x) + \left(\frac{r+\lambda}{\tilde{p}_1(x)} + \sigma^2\right) x\tilde{V}_1'(x) = 0. \end{aligned}$$

Differentiating both sides of the first ODE in (3.18), we obtain

$$\left(r - \sigma^2 - \frac{\lambda+r}{\tilde{p}_1(x)} + \frac{(\lambda+r)\tilde{p}_1'(x)}{\tilde{p}_1^2(x)} x\right) \cdot \tilde{V}_1'(x) = \left(\frac{\lambda+r}{\tilde{p}_1(x)} + \sigma^2\right) x\tilde{V}_1''(x)$$

and it yields

$$\tilde{V}_1''(x) < 0 \qquad \text{for all } x \in]0, x^*[\,.$$

Hence, for all $x \in [\bar{x_1}, x^*[$, it holds

$$-rV_1(x) + H(x, V_1'(x), p(x,x^*)) + \frac{\sigma^2 x^2}{2} V_1''(x) \leq \frac{\sigma^2 x^2}{2} V_1''(x) = \frac{\sigma^2 x^2}{2} \tilde{V}_1''(x) < 0$$

Hence, V_1 is a super-solution of the first equation in (2.9) and a standard comparison arguments yields

$$V(x,x^*) \leq V_1(x) = \tilde{V}_1(x) \qquad \text{for all } x \in [\bar{x}_1, x^*].$$

From (3.16) and (3.19), we obtain that

$$\limsup_{x^*\to\infty} V(x,x^*) \leq \limsup_{x^*\to\infty} B \cdot \left(\frac{x}{\theta(x^*)x^*}\right)^{\frac{r}{\beta+\sigma^2}} = 0 \quad \text{for all } x \in [\bar{x}_1, x^*],$$

and the monotone increasing of $V(\cdot, x^*)$ yields (3.13).

Step 2. We now prove (ii). Suppose that

$$\limsup_{s\to+\infty} \theta(s)s = C_1 < +\infty\,. \tag{3.24}$$

We introduce an intermediate point

$$\bar{x}_2 \doteq \frac{1}{r-\mu} + \theta(x^*)x^* \tag{3.25}$$

which depends on x^*. Define

$$p_2(x) = \begin{cases} 1, & \text{for } x \in [0, \bar{x}_2], \\ \dfrac{\bar{x}_2}{x} = \left(\dfrac{1}{r-\mu} + \theta(x^*)x^*\right) \cdot \dfrac{1}{x}, & \text{for } x \in [\bar{x}_2, x^*], \end{cases}$$

we have

$$p_2(0) = 1 = p(0,x^*), \qquad p_2(x^*) = \frac{1}{(r-\mu)x^*} + \theta(x^*) > p(x,x^*),$$

and

$$p_2'(x) = -\frac{p_2(x)}{x}, \qquad p_2''(x) = \frac{2p_2(x)}{x^2} \qquad \text{for all } x \in [\bar{x}_2, x^*].$$

For $x \in]0, \bar{x}_2[$, it holds

$$(r+\lambda) - (r+\lambda+v(x,x^*))p_2(x) + H_\xi(x, V'(x,x^*), p_2(x)) \cdot p_2'(x) + \frac{\sigma^2 x^2}{2} p_2''(x) = -v(x,x^*)p_2(x) = -v(x,x^*) < 0\,.$$

On the other hand, recalling Lemma B.2, we have

$$(r+\lambda) - (r+\lambda+v(x,x^*))p_2(x) + H_\xi(x, V'(x,x^*), p_2(x)) \cdot p_2'(x) + \frac{\sigma^2 x^2}{2} p_2''(x) = \frac{u^*(V'(x,x^*), p_2(x))}{x} - (r-\mu)\cdot p_2(x) \le \frac{1}{x} - (r-\mu)\cdot p_2(x) < 0$$

for all $x \in]\bar{x}_2, x^*[$. Therefore, $p_2(\cdot)$ is a super-solution of the first equation in (2.9) and

$$p(x,x^*) \le p_2(x) \qquad \text{for all } x \in [0,x^*]\,. \tag{3.26}$$

To construct V_2, we define

$$\bar{x}_3 \;:=\; \frac{1+(r+v_{\max})\cdot \bar{x}_2}{r} > \frac{1}{r+\lambda}. \tag{3.27}$$

Notice that for x^* sufficiently large, it holds $\bar{x}_3 < x^*$. In this case, we set

$$V_2(x) = \begin{cases} 0, & \text{for } x \in [0,\bar{x}_3], \\ \left[\dfrac{\bar{x}_2}{\bar{x}_3} - p_2(x)\right]\cdot B = B\bar{x}_2 \cdot \left(\dfrac{1}{\bar{x}_3} - \dfrac{1}{x}\right), & \text{for } x \in [\bar{x}_3, x^*]. \end{cases}$$

For every $x \in]0, \bar{x}_3[$, it holds

$$-rV_2 + H(x, V_2', p(x,x^*)) + \frac{\sigma^2 x^2}{2} V_2'' = H(x, 0, p(x,x^*)) = 0\,. \tag{3.28}$$

For every $x \in]\bar{x}_3, x^*[$, we have

$$V_2'(x) = B\cdot \frac{p_2(x)}{x} > 0, \qquad V_2''(x) = -2B\cdot\frac{p_2(x)}{x^2} < 0\,.$$

Thus, recalling (3.27) and Lemma B.3 (1), we estimate

$$\begin{aligned} -rV_2 + H(x, V_2', p_2) \;&\ge\; -rV_2 + \left(\frac{(\lambda+r)x-1}{p_2} + (\sigma^2-\lambda-\mu-v_{\max})x\right)V_2' \\ &=\; -rV_2 + B\cdot\left(\lambda + r - \frac{1}{x} + (\sigma^2-\lambda-\mu-v_{\max})\cdot p_2\right) \\ &\ge\; -rV_2 + B\cdot\left(r - \frac{1}{x} + (\sigma^2-\mu-v_{\max})\cdot p_2\right) \\ &=\; B\cdot\left(r - \frac{1}{x} - \frac{r\bar{x}_2}{\bar{x}_3} + (r-\mu-v_{\max})\cdot p_2\right) + B\sigma^2 p_2 \\ &\ge\; B\cdot\left(r - \frac{1}{x} - \frac{r\bar{x}_2}{\bar{x}_3} - v_{\max}\cdot p_2\right) - \frac{\sigma^2 x^2}{2}\cdot V_2'' \\ &\ge\; B\cdot\frac{r\bar{x}_3 - (1+(r+v_{\max})\cdot\bar{x}_2)}{\bar{x}_3} - \frac{\sigma^2}{2}\cdot V_2'' = -\frac{\sigma^2 x^2}{2}\cdot V_2'' \end{aligned}$$

and it yields

$$-rV_2 + H(x, V_2', p_2) + \frac{\sigma^2 x^2}{2}\cdot V_2'' \ge 0 \qquad \text{for all } x \in [\bar{x}_3, x^*].$$

From Lemma B.2 and (3.27), the map $p \to H(x, V_2'(x), p)$ is monotone decreasing on $[0,1]$, for all $x \in [\bar{x}_3, x^*]$. Recalling (3.26), we get

$$-rV_2 + H(x, V_2', p(x,x^*)) + \frac{\sigma^2 x^2}{2} V_2'' \geq 0 \qquad \text{for all } x \in]\bar{x}_3, x^*[.$$

Since $V_2(0) = V(x,x^*)$ and $V_2(x^*) < B = V(x,x^*)$, together with (3.28), the function V_2 is a sub-solution of the first equation in (2.9) and

$$\begin{aligned} V(x,x^*) \geq & V_2(x) = B\bar{x}_2 \cdot \left[\frac{1}{\bar{x}_3} - \frac{1}{x}\right] \geq \frac{B\bar{x}_2}{2\bar{x}_3} \\ = & \frac{rB}{2} \cdot \frac{1}{r + v_{\max} + \frac{1}{\bar{x}_2}} \geq \frac{rB}{2(2r + v_{\max} - \mu)} \end{aligned}$$

for all $x \in [2\bar{x}_3, x^*]$. Finally, recalling (3.24), (3.25), and (3.27), we have

$$\limsup_{x^* \to \infty} \bar{x}_3 = \frac{(r + v_{\max} + (1 + C_1(r + v_{\max}))(r - \mu))}{r(r-\mu)}$$

and it yields (ii). □

4. The deterministic case $\sigma = 0$

In the case $\sigma = 0$, the stochastic control system (2.1) reduces to the deterministic one

$$\dot{x}(t) = \left(\frac{\lambda + r}{p(t)} - \lambda - \mu - v(t)\right) x(t) - \frac{u(t)}{p(t)}. \tag{4.1}$$

Here the control $u(t)$ is assumed to be in $[0,1]$ for all $t \geq 0$. The Debt Management Problem can be formulated as follows.

(DMP) *Given an initial value $x(0) = x_0 \in [0, x^*]$ of the DTI, minimize*

$$\int_0^{T_b} e^{-rt}[L(u(t)) + c(v(t))]\, dt + e^{-rT_b} B, \tag{4.2}$$

subject to the dynamics (4.1)*, where the bankruptcy time T_b is defined as in* (2.2)*, while the discount bond price*

$$\begin{aligned} p(t) = & \int_t^{T_b} (r + \lambda) \exp\left\{-\int_0^s \big(\lambda + r + v(\tau)\big)\, d\tau\right\} ds + \\ & + \exp\left\{-\int_t^{T_b} (r + \lambda + v(s))\, ds\right\} \theta(x^*). \end{aligned} \tag{4.3}$$

As in the stochastic case, we will assume that **(A1)-(A2)** hold. Since in this case the optimal feedback control u^*, v^* and the corresponding functions V^*, p^* may be nonsmooth, a concept of equilibrium solution should be more carefully defined.

Definition 4.1 (**Equilibrium solution in feedback form**)**.** *A couple of piecewise Lipschitz continuous functions $(u^*(\cdot), v^*(\cdot))$ and l.s.c. $p^*(\cdot)$ provide an equilibrium solution to the debt management problem (DMP), with continuous value function $V^*(\cdot)$, if*

(i) For every $x_0 \in [0, x^*]$, V^* is the minimum cost for the optimal control problem

$$\textit{minimize:} \int_0^{T_b} e^{-rt}[L(u(t)) + c(v(t))]\, dt + e^{-rT_b} B, \tag{4.4}$$

subject to

$$\dot{x}(t) = \left(\frac{\lambda + r}{p^*(x(t))} - \lambda - \mu - v(t)\right) x(t) - \frac{u(t)}{p^*(x(t))}, \qquad x(0) = x_0, \tag{4.5}$$

where $u : [0, +\infty[\to [0, 1]$ *and* $v : [0, +\infty[\to [0, +\infty[$ *are measurable functions. Moreover, every Carathéodory solution of* (4.5) *with* $(u(t), v(t)) = (u^*(x(t)), v^*(x(t)))$ *is optimal.*

(ii) For every $x_0 \in [0, x^*]$*, there exists at least one solution* $t \mapsto x(t)$ *of the Cauchy problem*

$$\dot{x}(t) = \left(\frac{\lambda + r}{p^*(x(t))} - \lambda - \mu - v^*(x(t))\right) x(t) - \frac{u^*(x(t))}{p^*(x(t))}, \qquad x(0) = x_0, \tag{4.6}$$

such that

$$p^*(x_0) = \int_0^{T_b} (r + \lambda) \exp\left\{-\int_0^t \left(\lambda + r + v^*(x(s)\right) ds\right\} dt + \\ + \exp\left\{-\int_0^{T_b} (r + \lambda + v^*(x(t)))\, dt\right\} \theta(x^*), \tag{4.7}$$

with T_b *as in* (2.2).

4.1. **System of first order Hamilton-Jacobi equations.** In the deterministic case, the Debt Management Problem (4.1)-(4.3) leads to the following implicit system of first order ODEs

$$\begin{cases} rV(x) = H(x, V'(x), p(x)), \\ (r + \lambda + v(x))p(x) - (r + \lambda) = H_\xi(x, V'(x), p(x)) \cdot p'(x), \\ v(x) = \underset{\omega \geq 0}{\operatorname{argmin}}\{c(\omega) - \omega x V'(x)\}, \end{cases} \tag{4.8}$$

with the boundary conditions

$$\begin{cases} V(0) = 0, \\ V(x^*) = B, \end{cases} \qquad \begin{cases} p(0) = 1, \\ p(x^*) = \theta(x^*). \end{cases} \tag{4.9}$$

The Hamiltonian function (2.6) reduces to

$$H(x, \xi, p) \doteq \min_{u \in [0,1]} \left\{L(u) - u\frac{\xi}{p}\right\} + \min_{v \geq 0} \left\{c(v) - vx\xi\right\} + \left(\frac{\lambda + r}{p} - \lambda - \mu\right) x\,\xi. \tag{4.10}$$

Therefore, we have (see Appendix A for the notation)

$$-H(x, \xi, p) \doteq L^\circ\left(\frac{\xi}{p}\right) + c^\circ(x\xi) - \left(\frac{\lambda + r}{p} - \lambda - \mu\right) x\,\xi, \tag{4.11}$$

and we notice that for fixed $x > 0$, $p \in]0, 1]$ the map $\xi \mapsto -H(x, \xi, p)$ is convex and lower semicontinuous. Here L°, c° are the convex conjugate of L and c.

Given $x > 0$, $p \in]0,1]$, $\xi \geq 0$, we denote by $u^*(\xi,p) \in [0,1]$ and $v^*(x,\xi) \in [0,+\infty[$ the unique elements of $\partial L^\circ\left(\dfrac{\xi}{p}\right)$ and $\partial c^\circ(x\xi)$, respectively, provided by Lemma B.1.

$$u^*(\xi,p) \doteq \operatorname*{argmin}_{u\in[0,1]}\left\{L(u) - u\,\frac{\xi}{p}\right\} = \begin{cases}0, & \text{if } 0 \leq \xi < pL'(0),\\ (L')^{-1}(\xi/p), & \text{if } \xi \geq pL'(0) > 0,\end{cases}$$

$$v^*(x,\xi) \doteq \operatorname*{argmin}_{v\geq 0}\Big\{c(v) - vx\xi\Big\} = \begin{cases}0, & \text{if } 0 \leq x\xi < c'(0),\\ (c')^{-1}(x\xi), & \text{if } x\xi \geq c'(0) > 0.\end{cases}$$

In particular,

- for every $p \in]0,1]$ the map $\xi \mapsto u^*(\xi,p)$ is strictly increasing in $[pL'(0),+\infty[$, and $u^*(\cdot,p) \equiv 0$ in $[0,pL'(0)]$;
- for every $\xi \geq 0$ the map $p \mapsto u^*(\xi,p)$ is strictly decreasing in $[\xi/L'(0),1]$, and $u^*(\xi,\cdot) \equiv 0$ in $[0,\xi/L'(0)]$;
- for every $\xi > 0$ the map $x \mapsto v^*(x,\xi)$ is strictly increasing in $[c'(0)/\xi,+\infty[$, and $v^*(\cdot,\xi) \equiv 0$ in $[0,c'(0)/\xi]$;
- for every $x > 0$ the map $\xi \mapsto v^*(x,\xi)$ is strictly increasing in $[c'(0)/x,+\infty[$, and $v^*(x,\cdot) \equiv 0$ in $[0,c'(0)/x]$.

It is proved in Lemma B.2 that the gradient of the Hamiltonian function $H(\cdot)$ at points $(x,\xi,p) \in [0,+\infty[\times[0,+\infty[\times]0,1]$ can be expressed in terms of $u^*(\xi,p)$ and $v^*(x,\xi)$ by

$$\begin{cases} H_x(x,\xi,p) = & \Big[(\lambda+r) - p(\lambda+\mu+v^*(x,\xi))\Big]\cdot\dfrac{\xi}{p},\\ H_\xi(x,\xi,p) = & \dfrac{1}{p}\cdot\Big[x\big((\lambda+r) - p(\lambda+\mu+v^*(x,\xi))\big) - u^*(\xi,p)\Big],\\ H_p(x,\xi,p) = & (u^*(\xi,p) - x(\lambda+r))\cdot\dfrac{\xi}{p^2}.\end{cases} \tag{4.12}$$

The following Lemma will catch some relevant properties of $H(\cdot)$ needed to study the system (4.8).

Lemma 4.2. *Let $x \geq 0$ and $0 < p \leq 1$ be fixed, and set*

$$H^{\max}(x,p) \doteq \max_{\xi\geq 0} H(x,\xi,p).$$

Then

(1) *there exists $\xi^\sharp(x,p) > 0$ such that, given $\eta > 0$, the equation $r\eta = H(x,\xi,p)$ admits*
   - *no solutions $\xi \in [0,+\infty)$ if $r\eta > H^{\max}(x,p)$,*
   - *$\xi^\sharp(x,p)$ as unique solution if $r\eta = H^{\max}(x,p)$,*
   - *exactly two distinct solutions $\{F^-(x,\eta,p), F^+(x,\eta,p)\}$ with*

   $$0 < F^-(x,\eta,p) < \xi^\sharp(x,p) < F^+(x,\eta,p)$$

   if $0 < r\eta < H^{\max}(x,p)$,

(2) *we extend the definition of $\eta \mapsto F^\pm(x,\eta,p)$ by setting*

   $$F^\pm\left(x, \frac{1}{r}\,H^{\max}(x,p), p\right) = \xi^\sharp(x,p),$$

   thus for fixed $x > 0$, $p \in]0,1]$, the maps $\eta \mapsto F^-(x,\eta,p)$ and $\eta \mapsto F^+(x,\eta,p)$ are respectively strictly increasing and strictly decreasing in $[0, H^{\max}(x,p)/r]$.

(3) for all $0 < \eta < H^{\max}(x,p)/r$ *with* $x > 0$ *and* $p \in]0,1]$, *we have*

$$\frac{\partial}{\partial \eta} F^{\pm}(x,\eta,p) = \frac{r}{H_\xi(x, F^{\pm}(x,\eta,p),p)},$$

(4) The map $p \mapsto H^{\max}(x,p)$ *is strictly decreasing on* $]0,1]$ *for every fixed* $x \in]0, x^*[$.

Proof. Since for all fixed $x > 0$, $0 < p \le 1$ we have that $\xi \mapsto H(x,\xi,p)$ is the minimum of a family of affine functions of ξ, we have that the map $\xi \mapsto H(x,\xi,p)$ is concave down. Recalling (4.12), and the monotonicity properties of $u^*(\cdot,p)$ and $v^*(x,\cdot)$, since

- $H_\xi(x,\xi,p) = H_\xi(x,0,p) > 0,$ for all $\xi \in [0, \min\{pL'(0), c'(0)/x\}]$,
- $\xi \mapsto H_\xi(x,\xi,p),$ is strictly decreasing for all $\xi > \min\{pL'(0), c'(0)/x\}$,
- $\lim\limits_{\xi \to +\infty} H_\xi(x,\xi,p) = -\infty$,

we have that $\xi \mapsto H_\xi(x,\xi,p)$ vanishes in at most one point in $[0,+\infty)$, so $\xi \mapsto H(x,\xi,p)$ reaches its maximum value $H^{\max}(x,p)$ on $[0,+\infty)$ at a unique point $\xi^\sharp(x,p)$, moreover it is strictly increasing for $0 < \xi < \xi^\sharp(x,p)$ and strictly decreasing for $\xi > \xi^\sharp(x,p)$, with $\xi^\sharp(x,p) \ge \min\{pL'(0), c'(0)/x\}$.

We define

- the strictly increasing map $\eta \mapsto F^-(x,\eta,p)$, for $0 < \eta < H^{\max}(x,p)/r$, to be the inverse of $\xi \mapsto \frac{1}{r}H(x,\xi,p)$ for $0 < \xi < \xi^\sharp(x,p)$;
- the strictly decreasing map $\eta \mapsto F^+(x,\eta,p)$, for $0 < \eta < H^{\max}(x,p)/r$, to be the inverse of $\xi \mapsto \frac{1}{r}H(x,\xi,p)$ for $\xi > \xi^\sharp(x,p)$.

Set

$$u^\sharp(x,p) \doteq u^*(\xi^\sharp(x,p),p), \qquad v^\sharp(x,p) \doteq v^*(x,\xi^\sharp(x,p)).$$

Recalling (4.12), we have

$$u^\sharp(x,p) = \big[(\lambda + r) - (\lambda + \mu + v^\sharp(x,p))p\big] \cdot x, \tag{4.13}$$

$$H^{\max}(x,p) = L(u^\sharp(x,p)) + c(v^\sharp(x,p)). \tag{4.14}$$

Moreover,

- if $\xi^\sharp(x,p) \ge pL'(0)$ we have

$$\xi^\sharp(x,p) = pL'(u^\sharp(x,p)) = pL'\Big(\big[(\lambda + r) - (\lambda + \mu + v^\sharp(x,p))p\big] \cdot x\Big), \tag{4.15}$$

- if $\xi^\sharp(x,p) \ge c'(0)/x$ we have

$$\xi^\sharp(x,p) = c'(v^\sharp(x,p))/x, \tag{4.16}$$

Conversely, assume that given $x > 0$, $p \in]0,1]$, we have

$$u^*(\xi,p) = x\big((\lambda + r) - p(\lambda + \mu + v^*(x,\xi))\big),$$

then $\xi = \xi^\sharp(x,p)$, $v^*(x,\xi) = v^\sharp(x,p)$, $u^*(\xi,p) = u^\sharp(x,p)$. This follows from the fact that $H_\xi(x,\xi,p) = 0$ iff $\xi = \xi^\sharp(x,p)$.

For any fixed $x \ge 0$ and $0 < p \le 1$, given $\eta > 0$ we consider the equation $r\eta = H(x,\xi,p)$, and all the statements (1-2-3) follows by applying $F^\pm(x,\cdot,p)$ to it. To prove item (4), we notice that

$$\frac{d}{dp}H^{\max}(x,p) = \frac{d}{dp}H(x,\xi^\sharp(x,p),p) = H_p(x,\xi^\sharp(x,p),p).$$

Recalling (4.12), we have

$$\begin{aligned} H_p(x,\xi^\sharp(x,p),p) &= \Big[u^\sharp(x,p)-(r+\lambda)x\Big]\cdot\frac{\xi^\sharp(x,p)}{p} \\ &= -(\lambda+\mu+v^\sharp(x,p))x\xi^\sharp(x,p) < 0\,, \end{aligned}$$

since for $x,p\neq 0$ we have $\xi^\sharp(x,p)>0$. □

Definition 4.3 (Normal form of the system)**.** Given $x>0$, $0<p\le 1$, $0<r\eta\le H^{\max}(x,p)$ we define the maps

$$G^{\pm}(x,\eta,p)=\frac{(r+\lambda+v^*(x,F^{\pm}(x,\eta,p)))p-(r+\lambda)}{H_\xi(x,F^{\pm}(x,\eta,p),p)}. \tag{4.17}$$

Notice that if $rV(x)>H^{\max}(x,p)$, then the first equation of (4.8) has no solution. Otherwise, if $0<rV(x)<H^{\max}(x,p)$ this equation splits into

$$\begin{cases} V'(x)=F^-(x,V(x),p(x)),\\ p'(x)=G^-(x,V(x),p(x)),\end{cases} \qquad \text{or} \qquad \begin{cases} V'(x)=F^+(x,V(x),p(x)),\\ p'(x)=G^+(x,V(x),p(x)).\end{cases}$$

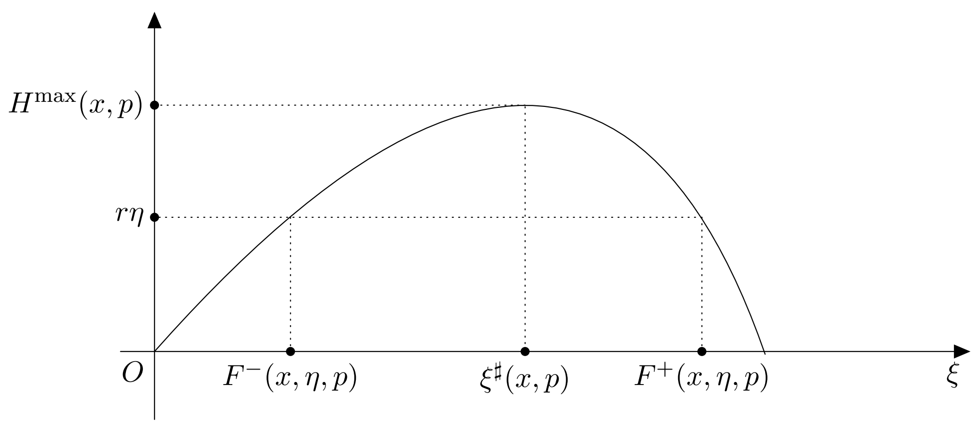

Figure 1. For $x\ge 0$, $p\in]0,1]$, the function $\xi\mapsto H(x,\xi,p)$ has a unique global maximum $H^{\max}(x,p)$ attained at $\xi=\xi^\sharp(x,p)$. For $0<r\eta\le H^{\max}$, the values $F^-(x,\eta,p)\le\xi^\sharp(x,p)\le F^+(x,\eta,p)$ are well defined. Moreover, $F^{\pm}(x,\frac{1}{r}H^{\max}(x,p),p)=\xi^\sharp(x,p)$.

Remark 4.4. Recalling (4.1) and (4.13), we observe that

- The value $V'(x)=F^+(x,V(x),p)\ge\xi^\sharp(x,p)$ corresponds to the choice of an optimal control such that $\dot x(t)<0$. The total debt-to-ratio is descreasing.
- The value $V'(x)=F^-(x,V(x),p)\le\xi^\sharp(x,p)$ corresponds to the choice of an optimal control such that $\dot x(x)>0$. The total debt-to-ratio is increasing.
- When $rV(x)=H^{\max}(x,p)$, then the value

$$V'(x)=F^+(x,V(x),p)=F^-(x,V'(x),p)=\xi^\sharp(x,p)$$

corresponds to the unique control strategy such that $\dot x(t)=0$.

Remark 4.5. We notice that if $0 \leq x\xi < \min\{xpL'(0), c'(0)\}$, since $u^* = v^* = 0$, we have

$$\xi = F^-(x,\eta,p) = \frac{pr\eta}{(\lambda + r - p(\lambda+\mu))x},$$

in particular, if $0 \leq x\xi < \min\{xpL'(0), c'(0)\}$ we have that $\eta \mapsto F^-(x,\eta,p)$ is Lipschitz continuous, uniformly for $(x,p) \in [x_1, x^*] \times [p_1, 1]$, for all $x_1 \in]0, x^*]$, $p_1 \in]0,1]$. If $x\xi > \min\{xpL'(0), c'(0)\}$, we have instead

$$H_{\xi\xi}(x,\xi,p) \leq -\frac{1}{p} \min\left\{\frac{1}{pL''(u^*(\xi,p))}, \frac{x^2 p}{c''(v^*(x,\xi))}\right\}.$$

Lemma 4.6. *Given $x_1 \in]0, x^*]$, $p_1 \in]0,1]$, there exists a constant $C = C(x_1, p_1)$ such that*

$$|F^-(x,\eta_1,p) - F^-(x,\eta_2,p)| \leq C \cdot |\eta_1 - \eta_2|^{1/2},$$

for all $x \in [x_1, x^*]$, $p \in [p_1, 1]$, $0 < \eta_1, \eta_2 \leq \frac{1}{r} H^{\max}(x,p)$.

Proof. We distinguish two cases:

(1) if $0 \leq x\xi < \min\{xpL'(0), c'(0)\}$, since $u^* = v^* = 0$, we have

$$\xi = F^-(x,\eta,p) = \frac{pr\eta}{(\lambda + r - p(\lambda+\mu))x},$$

and so

$$\begin{aligned} |F^-(x,\eta_1,p) - F^-(x,\eta_2,p)| &\leq \frac{pr}{(\lambda + r - p(\lambda+\mu))x} |\eta_1 - \eta_2| \\ &\leq \frac{\sqrt{2B} r}{(r-\mu)x_1} |\eta_1 - \eta_2|^{1/2}. \end{aligned}$$

for all $x \in [x_1, x^*]$, $p \in [p_1, 1]$, $0 < \eta_1, \eta_2 \leq \frac{1}{r} H^{\max}(x,p)$.

(2) If $x\xi > \min\{xpL'(0), c'(0)\}$, we have instead

$$H_{\xi\xi}(x,\xi,p) \leq -\frac{1}{p} \min\left\{\frac{1}{pL''(u^*(x,\xi,p))}, \frac{x^2 p}{c''(v^*(x,\xi))}\right\},$$

thus, recalling that by assumption we have $L''(u) \geq \delta_0$ and $c''(v) \geq \delta_0$ for $0 < u < 1$ and $v \geq 0$, we obtain

$$-H_{\xi\xi}(x,\xi,p) \geq \frac{\min\{1, x_1^2 p_1\}}{\delta_0}.$$

By applying Lemma B.4 to $f(\cdot) = -\frac{1}{r} H(x,\cdot,p)$, we have

$$|F^-(x,\eta_1,p) - F^-(x,\eta_2,p)| \leq \sqrt{\frac{2r\delta_0}{\min\{1, x_1^2 p_1\}}} |\eta_2 - \eta_1|^{1/2}.$$

The proof is complete by choosing $C(x_1,p_1) \doteq \sqrt{\frac{2r\delta_0}{\min\{1, x_1^2 p_1\}}} + \frac{\sqrt{2B} r}{(r-\mu)x_1}$. □

In the next two subsections, we will provide a detail analysis on the existence of a solution to the system of Hamilton-Jacobi equation (4.8) which yields an equilibrium solution to the Debt Management Problem (4.1)-(4.3).

4.2. **Constant strategies.** We begin our analysis from the control strategies keeping the DTI constant in time, i.e., such that the corresponding solution $x(\cdot)$ of (4.1) is constant. In this case, there is no bankruptcy risk, i.e., $T_b = +\infty$.

Definition 4.7 (Constant strategies)**.** Let $\bar{x} > 0$ be given. We say that a pair $(\bar{u}, \bar{v}) \in [0,1[\times[0,+\infty[$ is a constant strategy for $\bar{x}$ if

$$\begin{cases} \left[\left(\dfrac{\lambda + r}{\bar{p}} - \lambda - \mu - \bar{v}\right)\bar{x} - \dfrac{\bar{u}}{\bar{p}}\right] = 0, \\ \\ \bar{p} = \dfrac{r+\lambda}{r+\lambda+\bar{v}}, \end{cases}$$

where the second relation comes from taking $T_b = +\infty$ in (4.3).

From these equations, if a couple $(\bar{u}, \bar{v}) \in [0,1[\times[0,+\infty[$ is a constant strategy then it holds $(r+\lambda)(r-\mu)\bar{x} = (r+\lambda+\bar{v})\bar{u}$. In this case, the borrower will never go bankrupt and thus the cost of this strategy in (4.2) is computed by

$$\begin{aligned} \frac{1}{r}\cdot\Big[L(\bar{u}) + c(\bar{v})\Big] &= \frac{1}{r}\cdot\left[L\left(\frac{(r+\lambda)(r-\mu)\bar{x}}{r+\lambda+\bar{v}}\right) + c\left(\bar{v}\right)\right] \\ &= \frac{1}{r}\cdot\left[L\left((r-\mu)\bar{x}\cdot\bar{p}\right) + c\left(\left(1-\frac{1}{\bar{p}}\right)(r+\lambda)\right)\right]. \end{aligned}$$

We notice that if $\bar{x}(r-\mu) > 1$, we must have $\bar{v} > 1$ and $\bar{p} < 1$, in particular if DTI is sufficiently large, every constant strategy needs to implement currency devaluation, with a consequently drop of p. A more precise estimate will be provided in Proposition 4.10.

We are now interested in the minimum cost of a strategy keeping the debt constant. To this aim, we first characterize the cost of a constant strategy in terms of the variables x, p.

Lemma 4.8. *Given any* $(x,p) \in]0,+\infty[\times]0,1]$*, we have*

$$H^{\max}(x,p) = \min\Big\{L(u) + c(v) : u \in [0,1],\ v \geq 0,\ u = \big[(\lambda+r) - (\lambda+\mu+v)p\big]\cdot x\Big\}. \tag{4.18}$$

Moreover, $(\hat{u}, \hat{v})$ *realizes the minimum in the right hand side of* (4.18) *if and only if*

$$\begin{cases} c(\hat{v}) + px\hat{v}\xi^\sharp(x,p) &= \displaystyle\min_{\zeta\geq 0}\Big\{px\xi^\sharp(x,p)\zeta + c(\zeta)\Big\}, \\ \\ L(\hat{u}) + \hat{u}\xi^\sharp(x,p) &= \displaystyle\min_{u\in[0,1]}\Big\{\xi^\sharp(x,p)u + L(u)\Big\}. \end{cases}$$

Proof. Set $F(v) := f(v) + g(\Lambda v)$ where $f(\zeta) = c(\zeta)$ for $\zeta \geq 0$ and $f(\zeta) = +\infty$ if $\zeta < 0$, $C(x,p) = \big[(\lambda+r) - (\lambda+\mu)p\big]\cdot x$, $g(\zeta) = L(C(x,p)+\zeta)$ if $C(x,p)+\zeta \in [0,1]$ and $g(\zeta) = +\infty$ if $C(x,p)+\zeta \notin [0,1]$, and $\Lambda = -xp$. By standard argument in convex analysis (see e.g. Theorem 4.2 and Remark 4.2 p. 60 of [7]), denoted by f°, g° the convex conjugates of f, g respectively, we have

$$\inf_{v\in\mathbb{R}} F(v) = \sup_{\nu\in\mathbb{R}}\ [-f^\circ(\Lambda\nu) - g^\circ(-\nu)]$$

$$
\begin{aligned}
&= \sup_{\nu\in\mathbb{R}} \left[\min_{\zeta\geq 0}\left\{c(\zeta)+xp\nu\zeta\right\} + \min_{C(x,p)+\zeta\in[0,1]}\left\{L(C+\zeta)+\nu\zeta\right\}\right]\\
&= \sup_{\nu\in\mathbb{R}} \left[\min_{\zeta\geq 0}\left\{c(\zeta)+xp\nu\zeta\right\} + \min_{u\in[0,1]}\left\{L(u)+\nu u\right\} - C\nu\right]\\
&= \sup_{\xi\in\mathbb{R}} \left[\min_{\zeta\geq 0}\left\{c(\zeta)-x\xi\zeta\right\} + \min_{u\in[0,1]}\left\{L(u)-u\cdot\frac{\xi}{p}\right\} + \frac{C(x,p)}{p}\cdot\xi\right]\\
&= \sup_{\xi\in\mathbb{R}} H(x,\xi,p) = H^{\max}(x,p).
\end{aligned}
$$

Moreover, since $\sup_{\xi\in\mathbb{R}} H(x,\xi,p)$ is attained only at $\xi=\xi^\sharp(x,p)$ according to the strict concavity of $\xi\mapsto H(x,\xi,p)$, $(\hat{u},\hat{v})$ realizes the minimum in the right hand side of (4.18) if and only if

$$
\begin{cases}
f(\hat{v})+f^\circ(\Lambda\xi^\sharp(x,p))-\Lambda\hat{v}\xi^\sharp(x,p)=0,\\
g(\Lambda\hat{v})+g^\circ(-\xi^\sharp(x,p))+\Lambda\hat{v}\xi^\sharp(x,p)=0,
\end{cases}
$$

which implies $\hat{v}\geq 0$, $C(x,p)-px\hat{v}\in[0,1]$, and

$$
\begin{cases}
c(\hat{v})+px\hat{v}\xi^\sharp(x,p)=\min\limits_{\zeta\geq 0}\ \{px\xi^\sharp(x,p)\zeta+c(\zeta)\},\\
L(C(x,p)-px\hat{v})-px\hat{v}\xi^\sharp(x,p)=\min\limits_{\nu\in\mathbb{R}}\ \left\{\xi^\sharp(x,p)\nu+L(C(x,p)+\nu)\right\}.
\end{cases}
$$

The second relation can be rewritten as

$$
L(\hat{u})+\hat{u}\xi^\sharp(x,p)=\min_{u\in[0,1]}\left\{\xi^\sharp(x,p)u+L(u)\right\}.
$$

□

Formula (4.18) allows us to give a simpler characterization of the minimum cost of a strategy keeping the debt-to-income ratio constant in time. Indeed, given $x\in[0,x^*]$, we select $(u(x),v(x))$ keeping the debt-to-income ratio constant in time. This defines uniquely a value $p=p(x)$ by Definition 4.7 and impose a relation between $u(x)$ and $v(x)$. Then we take the minimum over all the costs of such strategies, i.e., the right hand side of formula (4.18). This naturally leads to the following definition.

Definition 4.9 (Optimal cost for constant strategies). *Given $x\in[0,x^*]$, we define*

$$
W(x)=\frac{1}{r}\cdot H^{\max}\left(x,p_c(x)\right),
$$

where

$$
\begin{cases}
p_c(x)=\dfrac{r+\lambda}{r+\lambda+v_c(x)},\\[2ex]
v_c(x)=\operatorname*{argmin}\limits_{v\geq 0}\left[L\left(\dfrac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right)+c(v)\right].
\end{cases}
\tag{4.19}
$$

For every $x\in[0,x^*]$, $W(x)$ denotes the minimum cost of a strategy keeping the DTI ratio constant in time.

The next results proves that if the debt-to-income ratio is sufficiently small, the optimal strategy keeping it constant does not use the devaluation of currency.

Proposition 4.10 (Non-devaluating regime for optimal constant strategies)**.** *Let $x_c \geq 0$ be the unique solution of the following equation in x*

$$(r+\lambda)c'(0) = (r-\mu)xL'\left((r-\mu)x\right).$$

Then

- *for all $x \in [0, \min\{x_c, x^*\}]$ we have $W(x) = \frac{1}{r} \cdot L((r-\mu)x)$ and $p_c(x) = 1$,*
- *for all $x \in]\min\{x_c, x^*\}, x^*]$ we have*

$$W(x) = \frac{1}{r}\left[L\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v_c(x)}\right) + c(v_c(x))\right],$$

$$p_c(x) = \frac{r+\lambda}{r+\lambda+v_c(x)} < 1,$$

where $v_c(x) > 0$ solves the following equation in v

$$c'(v) = \frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v)^2} \cdot L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right).$$

- *for every $x \in]0, x^*[$ we have*

$$W'(x) = \frac{r-\mu}{r} p_c(x) L'(p_c(x)(r-\mu)x) < \xi^\sharp(x, p_c(x)). \tag{4.20}$$

Proof. Given $x \in]0, x^*[$, we define the convex function

$$F^x(v) \doteq \begin{cases} \frac{1}{r} \cdot \left[L\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right) + c(v)\right], & \text{if } v \geq 0,\ \frac{(r+\lambda)(r-\mu)x}{r+\lambda+v} \in [0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

We compute

$$\frac{d}{dv}F^x(v) = \frac{1}{r} \cdot \left[c'(v) - L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right)\frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v)^2}\right],$$

which is monotone increasing and satisfies $\lim_{v\to+\infty} \frac{d}{dv}F^x(v) = +\infty$,

$$\frac{d}{dv}F^x(v) \;\geq\; \frac{d}{dv}F^x(0) = \frac{1}{r} \cdot \left[c'(0) - L'\left((r-\mu)x\right)\frac{(r-\mu)x}{r+\lambda}\right].$$

Two cases may occur:

- If $\frac{d}{dv}F^x(0) \geq 0$, we have that $v = 0$ realizes the minimum of F on $[0, +\infty[$. This occours when $x \in [0, \min\{x_c, x^*\}]$ where x_c is the unique solution of

$$(r+\lambda)c'(0) = (r-\mu)xL'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda}\right),$$

  and it implies $W(x) = \frac{1}{r} \cdot L((r-\mu)x)$ and $p_c(x) = 1$.

- If we have $\min\{x_c, x^*\} < x \leq x^*$, then there exists a unique point $v_c(x) > 0$ such that $F'(v_c(x)) = 0$, and this point is characterized by

$$c'(v_c(x)) = \frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v_c(x))^2} \cdot L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v_c(x)}\right).$$

The remaining statements follows noticing that for $\min\{x_c, x^*\} < x \leq x^*$ we have

$$\begin{aligned} W'(x) &= \frac{\partial F^x}{\partial x}(v_c(x)) + \frac{\partial}{\partial v}F^x(v_c(x)) \cdot v_c'(x) = \frac{\partial F^x}{\partial x}(v_c(x)) \\ &= \frac{r-\mu}{r}p_c(x)L'(p_c(x)(r-\mu)x), \end{aligned}$$

and deriving the explicit expression of $W(x)$ for $[0, \min\{x_c, x^*\}]$ yields the same formula. Notice that, by (4.15), we have

$$\begin{aligned} \xi^\sharp(x, p_c(x)) &= p_c(x)L'\left(\left[(\lambda+r) - (\lambda+\mu+v^\sharp(x,p_c(x)))p_c(x)\right] \cdot x\right) \\ &= p_c(x)L'\left(\left[(\lambda+r) - (\lambda+\mu+v^\sharp(x,p_c(x))) \cdot \frac{\lambda+r}{\lambda+r+v^\sharp(x,p_c(x))}\right] \cdot x\right) \\ &= p_c(x)L'(p_c(x)(r-\mu)\cdot x) > W'(x), \end{aligned}$$

where we used the fact that L' is strictly increasing and, since the argument of L' must be nonnegative, we have

$$\frac{\lambda+r}{\lambda+\mu+v^\sharp(x,p_c(x))} \;\geq\; p_c(x).$$

The proof is complete. □

4.3. **Existence of an equilibrium solution.** In the subsection, we will establish an existence result of a equilibrium solution to the debt management problem (4.1)-(4.3). Before going to state our main theorem, we recall from Proposition 4.10 that v_c is the unique solution to

$$c'(v) = \frac{(r+\lambda)(r-\mu)x}{(r+\lambda+v)^2} \cdot L'\left(\frac{(r+\lambda)(r-\mu)x}{r+\lambda+v}\right),$$

and

$$p_c(x^*) = \frac{r+\lambda}{r+\lambda+v_c(x^*)} < 1,$$

$$W(x^*) = \frac{1}{r}\left[L\left(\frac{(r+\lambda)(r-\mu)x^*}{r+\lambda+v_c(x^*)}\right) + c(v_c(x^*))\right]. \tag{4.21}$$

Theorem 4.11. *Assume that the cost functions L and c satisfies the assumptions **(A1)-(A2)**, and moreover*

$$W(x^*) > B \qquad \text{and} \qquad \theta(x^*) \leq p_c(x^*). \tag{4.22}$$

Then the debt management problem (4.1)-(4.3) *admits an equilibrium solution* (u^*, v^*, p^*) *associated to Lipschitz continuous value functions V^* in feedback form such that p^* is decreasing, V^* is strictly increasing and*

$$V^*(x) \;\leq\; W^*(x) \qquad \text{for all } x \in [0, x^*].$$

Toward the proof of this theorem, we study basic properties of the backward solutions of the system of implicit ODEs (4.8). In fact, an equilibrium solution will be constructed by a suitable concatenation of backward solutions.

4.3.1. *Backward solutions.* We first define the backward solution to the system (4.8) starting from x^*.

Definition 4.12 (Backward solution for x^*). Let $x \mapsto (Z(x,x^*), q(x,x^*))$ be the backward solution of the system of ODEs

$$\begin{cases} Z'(x) &= F^-(x, Z(x), q(x)), \\ \\ q'(x) &= G^-(x, Z(x), q(x)), \end{cases} \qquad \text{with} \qquad \begin{cases} Z(x^*) &= B\,, \\ \\ q(x^*) &= \theta(x^*). \end{cases} \tag{4.23}$$

with $H_\xi(x, F^-(x, Z(x), q(x)), q(x)) \neq 0$.

The following Lemma states some basic properties of the backward solution. In particular, the backward solution $Z(\cdot, x^*)$, starting from B at x^* with $W(x^*) < B$, survives backward at least until the first intersection with the graph of $W(\cdot)$. Moreover, in this interval is monotone increasing and positive. In the same way, $q(\cdot, x^*)$ is always in $]0,1]$.

Proposition 4.13. *[Basic properties of the backward solution] Set*

$$x_W^* := \begin{cases} 0, & \text{if } Z(x,x^*) < W(x) \text{ for all } x \in]0,x^*[, \\ \sup\{x \in]0,x^*[:\ Z(x,x^*) \geq W(x)\}, & \text{otherwise}\,. \end{cases}$$

Assume that

$$W(x^*) > B \qquad \text{and} \qquad \theta(x^*) < \frac{r+\lambda}{r+\lambda+v^*(x^*, F^-(x^*, B, \theta(x^*)))}\,. \tag{4.24}$$

Denote by $I_{x^*} \subseteq [0,x^*]$ the maximal domain of the backward equation (4.23)*, define $y(x)$ to be the maximal solution of*

$$\begin{cases} \dfrac{dy}{dx}(x) = \dfrac{1}{H_\xi\left(x, Z'(x,x^*), q(x,x^*)\right)}, \\ \\ y(x^*) = 0, \end{cases}$$

and let J_{x^*} the intersection of its domain with $[0,x^*]$. Then

(1) $I_{x^*} \supseteq J_{x^*} \supseteq]x_W^*, x^*[$;
(2) $Z(\cdot, x^*)$ is strictly monotone increasing in $]x_W^*, x^*[$, and $Z(x,x^*) > 0$ for all $x \in]x_W^*, x^*]$;
(3) $q(x,x^*) \in]0,1]$ for all $x \in]x_W^*, x^*]$.

Proof.

1. We first claim that $q(\cdot, x^*)$ is non-increasing on $J_{x^*} \bigcap]x_W^*, x^*[$ and thus

$$q'(x,x^*) = \frac{[r+\lambda+v^*(x, Z'(x,x^*))]\cdot q(x,x^*) - (r+\lambda)}{H_\xi(x, Z'(x,x^*), q(x,x^*))} \ \leq\ 0, \text{ for all } x \in J_{x^*} \cap]x_W^*, x^*[\,. \tag{4.25}$$

By contradiction, assume that there exists $x_1 \in J_B \cap]x_{BW}, x^*[$ such that

$$q'(x_1,x^*) = \frac{[r+\lambda+v^*(x_1, Z'(x_1,x^*))]\cdot q(x_1,x^*) - (r+\lambda)}{H_\xi(x_1, Z'(x,x^*), q(x,x^*))} = 0, \text{ and } q''(x_1,x^*) < 0\,. \tag{4.26}$$

This yields

$$r+\lambda = [r+\lambda+v^*(x_1, Z'(x_1,x^*))]\cdot q(x_1,x^*) \text{ and } q(x_1,x^*) > 0.$$

Two cases are considered:

- if $x_1 Z'(x_1,x^*) \le c'(0)$ then, recalling the monotonicity of $Z'(\cdot,x^*)$, we have that $xV'(x,x^*) \le c'(0)$ for all $x \in J_{x^*} \cap]x_W^*, x^*[$ satisfying $x \le x_1$, and so
$$v^*(x, Z'(x,x^*)) = 0, \text{ for all } x \in J_{x^*} \cap]x_W^*, x^* [\text{with } x \le x_1.$$
Thus, $q(x_1,x^*) = 1$ and
$$q'(x,x^*) = \frac{[r+\lambda]\cdot[q(x,x^*)-1]}{H_\xi(x,Z'(x,x^*),q(x,x^*))} \quad \text{for all } x \in J_{x^*} \cap]x_W^*, x^*[\text{ with } x \le x_1.$$
This implies that $q(x,x^*) = 1$ for all $x \in J_{x^*} \cap]x_W^*, x^*[$ with $x \le x_1$. In particular, we have $q''(x_1,x^*) = 0$, which yields a contradiction.
- If $x_1 Z'(x_1,x^*) > c'(0)$ then
$$\frac{d}{dx}(v^*(x_1, Z'(x_1,x^*))) = \frac{Z''(x_1,x^*)x_1 + Z'(x_1,x^*)}{c''(x_1 Z'(x_1,x^*))} > 0.$$
From the first equation of (4.8) and (4.12), it holds
$$\begin{aligned} rZ'(x_1,x^*) = &\ H_x(x_1,Z',q) + H_\xi(x_1,Z',q)\cdot Z''(x_1,x^*) + H_p(x_1,Z,q)\cdot q'(x_1,x^*) \\ = &\ \Big[(\lambda+r) - q(x_1,x^*)(\lambda+\mu+v^*(x,Z'))\Big]\cdot\frac{Z'}{q} + H_\xi(x_1,Z',q)\cdot Z''(x_1,x^*) \\ = &\ (r-\mu)\cdot Z'(x_1,x^*) + H_\xi(x_1,Z',q)\cdot Z''(x_1,x^*). \end{aligned}$$
Observe that $Z'(x_1,x^*) > 0$ and $H_\xi(x_1,Z'(x_1,x^*),q(x_1,x^*)) > 0$, one obtains that
$$Z''(x_1,x^*) = \frac{\mu Z'(x_1,x^*)}{H_\xi(x_1,Z'(x_1,x^*),q(x_1,x^*))} > 0\,.$$
Taking the derivative respect to x in both sides of the second equation of (4.8), we have
$$\begin{aligned} &\big[r+\lambda+(v^*(x,Z'(x,x^*))\big]\cdot q'(x,x^*) + q(x,x^*)\cdot\frac{d}{dx}v^*(x,Z'(x,x^*)) \\ &\qquad = q''(x,x^*)H_\xi(x,Z'(x,x^*),q(x,x^*)) + q'(x,x^*)\frac{d}{dx}H_\xi(x,Z'(x,x^*),q(x,x^*))\,. \end{aligned}$$
Recalling (4.26), we obtain that
$$q''(x_1,x^*) = \frac{q(x_1,x^*)}{H_\xi(x_1,Z',q)}\cdot\frac{d}{dx}v^*(x_1,Z'(x_1,x^*)) > 0. \tag{4.27}$$
and it yields a contradiction.

Now assume that there exists $x_2 \in J_{x^*} \cap]x_W^*, x^*[$ such that $H_\xi(x_2, Z'(x_2,x^*), q(x_2,x^*)) = 0$. One has that
$$\xi^\sharp(x_2, q(x_2,x^*)) = Z'(x_2,x^*) \qquad \text{and} \qquad Z(x_2,x^*) = \frac{1}{r}\cdot H^{\max}(x_2,q(x_2,x^*)).$$
Moreover,
$$u^\sharp(x_2,q(x_2,x^*)) = \big[(\lambda+r) - (\lambda+\mu+v^\sharp(x_2,q(x_2,x^*)))q(x_2,x^*)\big]\cdot x_2\,.$$
On the other hand, since $q(x_2,x^*) \le \dfrac{r+\lambda}{r+\lambda+v^\sharp(x,Z'(x_2,x^*))}$, we estimate
$$\begin{aligned} H^{\max}&(x_2,q(x_2,x^*)) = L(u^\sharp(x_2,q(x_2,x^*))) + c(v^\sharp(x_2,q(x_2,x^*))) \\ &= L\Big(\big[(\lambda+r) - (\lambda+\mu+v^\sharp(x_2,q(x_2,x^*)))q(x_2,x^*)\big]\cdot x_2\Big) + c(v^\sharp(x_2,q(x_2,x^*))) \end{aligned}$$

$$\geq L\left(\frac{r+\lambda(r-\mu)x_2}{\lambda+\mu+v^\sharp(x_2,q(x_2,x^*))}\right)+c(v^\sharp(x_2,q(x_2,x^*)))$$
$$\geq H^{\max}(x_1,p_c(x_2)).$$

Thus,

$$Z(x_2,x^*)=\frac{1}{r}\cdot H^{\max}(x_2,q(x_2,x^*))\geq\frac{1}{r}\cdot H^{\max}(x_2,p_c(x_2))=W(x_2),$$

and it yields a contradiction.

2. By construction, $y(\cdot)$ is strictly monotone and invertible in $]x^*_W,x^*]$, let $x=x(y)$ be its inverse, from the inverse function theorem we get

$$\begin{cases}\dfrac{d}{dy}\,Z(x(y),x^*)=Z'(x(y),x^*)\cdot H_\xi\left(x(y),Z'(x(y),x^*),q(x(y),x^*)\right),\\[2ex]\dfrac{d}{dy}\,q(x(y),x^*)=q'(x(y),x^*)\cdot H_\xi\left(x(y),Z'(x(y),x^*),q(x(y),x^*)\right).\end{cases}$$

Since the map $\xi\mapsto H(x,\xi,q)$ is concave, it holds

$$H_\xi(x,0,q(x,x^*))\ \geq\ H_\xi(x,\xi,q(x,x^*))\ \geq\ H_\xi\left(x,Z'(x,x^*),q(x,x^*)\right),$$

for all $\xi\in[0,Z'(x,x^*)]$. We have

$$\begin{aligned}rZ(x(y),x^*)=&\ H\left(x(y),Z'(x(y),x^*),q(x(y),x^*)\right)\\&=\int_0^{Z'(x(y),x^*)}H_\xi(x,\xi,q(x(y),x^*))\,d\xi\\&\geq\ Z'(x(y),x^*)\cdot H_\xi(x,Z(x(y),x^*),q(x(y),x^*))=\frac{d}{dy}Z(x(y),x^*)\end{aligned}$$

and this yields $Z(x,x^*)\geq Be^{ry(x)}>0$ for all $x\in]x^*_W,x^*]$.

With a similar argument for $q(\cdot,x^*)$, we obtain

$$(r+\lambda+v^*(x(y),Z'(x(y),x^*))\cdot q(x(y),x^*)-(r+\lambda)=\frac{d}{dy}q(x(y),x^*)),$$

and so

$$(r+\lambda)(q(x(y),x^*)-1)\ \leq\ \frac{d}{dy}q(x(y),x^*)\ \leq\ (r+\lambda+v^*(x(y),Z'(x(y),x^*))\cdot q(x(y),x^*),$$

which in particular implies that for all $x\in I_{x^*}\cap[0,x^*]$

$$q(x,x^*)\ \leq\ 1,\qquad q(x,x^*)\ \geq\ \theta(x^*)\cdot e^{(r+\lambda+v^*(x,Z'(x,x^*))y(x)}>0,$$

and so $q(x,x^*)\in]0,1]$ for all $x\in]x^*_W,x^*]$. □

As far as the graph of $Z(\cdot,x^*)$ intersects the graph of $W(\cdot)$, we have that $Z(\cdot,x^*)$ is no longer optimal. We investigate now the local behavior of $Z(\cdot,x^*)$ and $W(\cdot)$ near to an intersection of their graphs.

Lemma 4.14 (Comparison between optimal constant strategy and backward solution)**.** *Let $I\subseteq]0,x^*[$ be an open interval, $(Z,q):I\to[0,+\infty[\times]0,1[$ be a backward solution, and $\bar{x}\in\bar{I}$. Assume that*

$$\lim_{\substack{x\to\bar{x}\\x\in I}}Z(x)=W(\bar{x}).$$

Then $p_c(\bar{x}) \geq \limsup_{\substack{x\to\bar{x}\\ x\in I}} q(x)$ *and* $W'(x) < F^-(x, W(x), p_c(x))$.

Proof. Let $\{x_j\}_{j\in\mathbb{N}} \subseteq I$ be a sequence converging to $\bar{x}$ and $q_{\bar{x}} \in [0,1]$ be such that $q_{\bar{x}} = \limsup_{x\to\bar{x}^+} q(x) = \lim_{j\to\infty} q(x_j)$. By assumption, we have

$$H^{\max}(x, p_c(x)) = \lim_{j\to+\infty} H\left(x_j, Z'(x_j), q(x_j)\right) \leq \lim_{j\to+\infty} H^{\max}(x_j, q(x_j)) = H^{\max}(\bar{x}, q_{\bar{x}}).$$

Recalling Lemma 4.2 (4), we have $p_c(\bar{x}) \geq q_{\bar{x}}$. By Proposition 4.10, we have $W'(\bar{x}) < \xi^\sharp(\bar{x}, p_c(\bar{x}))$, and so

$$H(\bar{x}, W'(\bar{x}), p_c(\bar{x})) < H^{\max}(\bar{x}, p_c(\bar{x})) = rW(\bar{x}),$$

thus, by applying the strictly increasing map $F^-(\bar{x}, \cdot, p_c(x))$ on both sides, we obtain $W'(x) < F^-(x, W(x), p_c(x))$. □

Since, the functions $F^-(x, Z, q)$ and $G^-(x, Z, q)$ are smooth for $H_\xi(x, Z, q) \neq 0$ but not only Hölder continuous with respect to Z near to the surface

$$\Sigma = \{(x, Z, q) \mid H_\xi(x, Z, q) = 0\} .$$

Thus, for any $x_0 \in [0, x^*)$, the definition of the solution of the Cauchy problem

$$\begin{cases} Z'(x) &= F^-(x, Z(x), q(x)), \\ q'(x) &= G^-(x, Z(x), q(x)), \end{cases} \qquad \text{with} \qquad \begin{cases} Z(x_0) &= W(x_0)\,, \\ q(x_0) &= p_c(x_0). \end{cases} \tag{4.28}$$

requires some care.

For any $\varepsilon > 0$, we denote by $Z_\varepsilon(\cdot, x_0), q_\varepsilon(\cdot, x_0)$ the backward solution to (4.28) with the terminal data

$$Z_\varepsilon(x_0, x_0) = W(y_0) - \varepsilon \qquad \text{and} \qquad q_\varepsilon(x_0, x_0) = p_c(x_0).$$

With the same argument in the proof of Proposition 4.13, this solution is uniquely defined on a maximal interval $[a_\varepsilon(x_0), x_0]$ such that $Z_\varepsilon(\cdot, x_0)$ is increasing, $q_\varepsilon(\cdot, x_0)$ is decreasing and

$$Z_\varepsilon(a_\varepsilon(x_0), x_0) = W(a_\varepsilon(x_0)), \qquad q_\varepsilon(a_\varepsilon(x_0), x_0) \leq p_c(a_\varepsilon(x_0)).$$

Let $x^\flat$ be the unique solution to the equation

$$c'(0) = x \cdot L'((r-\mu)x)\,. \tag{4.29}$$

It is clear that $0 < x^\flat < x_c$ where x_c is defined in Proposition 4.10 as the unique solution to the equaiton

$$(r+\lambda)c'(0) = (r-\mu)xL'\left((r-\mu)x\right).$$

Two cases are considered:

- **CASE 1:** For any $x_0 \in]0, x^\flat]$, we claim that

  $$a_\varepsilon(x_0) = 0, \qquad q_\varepsilon(x, x_0) = 1 \qquad \text{for all } x \in [0, x_0]\,,$$

  and $Z_\varepsilon(\cdot, x_0)$ solves backward the following ODE

  $$Z'(x) = F^-(x, Z(x), 1), \qquad Z(x_0) = W(x_0) - \varepsilon \tag{4.30}$$

  for $\varepsilon > 0$ sufficiently small. Indeed, let Z_1 be the unique backward solution of (4.30). From (4.15), it holds

  $$F^-(x, W(x), 1) = \xi^\sharp(x, 1) = L'((r-\mu)x) > \frac{r-\mu}{r} \cdot L'((r-\mu)x) = W'(x)$$

for all $x \in]0, x^\flat]$. As in [5], a contradiction argument yields

$$0 < Z_1(x) < W(x) \qquad \text{for all } x \in]0, x_0] \,.$$

Thus, Z_1 is well-defined on $[0, x_0]$ and $Z_1(0) = 0$. On the other hand, it holds

$$Z'(x_1) = F^-(x, Z(x), 1) \le \xi^\sharp(x, 1) = L'((r-\mu)x) \le L'((r-\mu)x^\flat)$$

for all $x \le x^\flat$ and (4.29) implies that

$$v^*(x, Z_1'(x)) = 0 \qquad \text{for all } x \in [0, x^\flat] \,.$$

Therefore, $(Z_1(x), 1)$ solves (4.28) and the uniqueness yields

$$Z_\varepsilon(x, x_0) = Z_1(x) \qquad \text{and} \qquad q_\varepsilon(x, x_0) = 1 \qquad \text{for all } x \in [0, x_0] \,.$$

Thanks to the monotone increasing property of the map $\xi \to F^-(x, \xi, 1)$, a pair $(Z(\cdot, x_0), q(\cdot, x_0))$ denoted by

$$q(x, x_0) = 1 \qquad \text{and} \qquad Z(x, x_0) = \sup_{\varepsilon > 0} Z_\varepsilon(x, x_0) \qquad \text{for all } x \in [0, x_0]$$

is the unique solution of (4.28). If the initial size of the debt is $\bar{x} \in [0, x_0]$ we think of $Z(\bar{x}, x_0)$ is as the expected cost of (4.4)-(4.5) with $p(\cdot, x_0) = 1$, $x(0) = x_0$ achieved by the feedback strategies

$$u(x, x_0) = \operatorname*{argmin}_{w \in [0,1]} \left\{ L(w) - Z'(x, x_0) \cdot w \right\}, \quad v(x, x_0) = 0 \tag{4.31}$$

for all $x \in [0, x_0]$. With this strategy, the debt has the asymptotic behavior $x(t) \to x_0$ as $t \to \infty$.

- **CASE 2:** For $x_0 \in (x^\flat, x^*_W]$, system of ODEs (4.28) does not admit a unique solution in general since it is not monotone, it. The following lemma will provide the existence result of (4.28) for all $x_0 \in (x^\flat, x^*_W]$.

Lemma 4.15. *There exists a constant $\delta_\flat > 0$ depending only on $x^\flat$ such that for any $x_0 \in \left(x^\flat, x^*_W\right)$, it holds*

$$x_0 - a_\varepsilon(x_0) \ge \delta_{x^\flat} \qquad \textit{for all } \varepsilon \in (0, \varepsilon_0)$$

for some $\varepsilon_0 > 0$ sufficiently small.

Proof. From (4.20) and (4.15), it holds

$$\inf_{x \in [x^\flat, x^*_W]} \left\{ \xi^\sharp(x, p_c(x)) - W'(x) \right\} = \delta_{1,\flat} > 0.$$

In particular, we have

$$F^-(x_0, W(x_0), p_c(x_0)) - W'(x_0) = \delta_{1,\flat}.$$

By continuity of the map $\eta \mapsto F^-(x_0, \eta, p_c(x_0))$ on $[0, W(x)]$, one can find a constant $\varepsilon_1 > 0$ sufficiently small such that

$$F^-(x_0, \eta, p_c(x_0)) \ge W'(x_0) + \frac{\delta_{1,\flat}}{2} \quad \text{for all } \xi \in [W(x_0) - \varepsilon_1, W(x_0)].$$

On the other hand, the continuity of W' yields

$$\delta_{2,\flat} = \sup \left\{ s \ge 0 \;\middle|\; W'(x_0 - \tau) < W'(x_0) + \frac{\delta_{1,\flat}}{4} \quad \text{for all } \tau \in [0, s] \right\} > 0.$$

For a fixed $\varepsilon \in (0, \varepsilon_1)$, denote by

$$x_1 \doteq \inf\left\{ s \in (0, x_0] \;\middle|\; F^-\big(x, Z_\varepsilon(x, x_0), q_\varepsilon(x, x_0)\big) > W'(x) \text{ for all } x \in (s, x_0] \right\}.$$

If $x_1 > x_0 - \delta_{2,\bar{x}}$ then it holds

$$F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_1, x_0)\big) = W'(x_1) \leq W'(x_0) + \frac{\delta_{1,\flat}}{4} \tag{4.32}$$

and there exists $x_2 \in (x_1, x_0]$ such that

$$F^-\big(x_2, Z_\varepsilon(x_2, x_0), q_\varepsilon(x_2, x_0)\big) = W'(x_0) + \frac{\delta_{1,\flat}}{2} \tag{4.33}$$

and

$$F^- \left(x, Z_\varepsilon(x, x_0), q_\varepsilon(x, x_0)\right) \leq W'(x_0) + \frac{\delta_{1,\flat}}{2} \qquad \text{for all } x \in [x_1, x_2]. \tag{4.34}$$

Recalling that $(x, \eta, p) \mapsto F^-(x, \eta, p)$ is defined by $H(x, F^-(x, \eta, p), p) = r\eta$, by the implicit function theorem, set $\xi = F^-(x, \eta, p)$, we have

$$\begin{aligned} \frac{\partial}{\partial p} F^-(x, \eta, p) &= -\frac{H_p(x, \xi, p)}{H_\xi(x, \xi, p)} \\ &= \frac{\xi}{p} \cdot \frac{u^*(x, \xi, p) - x(\lambda + r)}{u^*(x, \xi, p) - x(\lambda + r) + xp(\lambda + \mu + v^*(x, \xi))} \\ &= \left(1 + \frac{x(\lambda + \mu + v^*(x, \xi)}{H_\xi(x, \xi, p)}\right) \frac{\xi}{p} > \frac{F^-(x, \eta, p)}{p} > 0. \end{aligned}$$

Since $q_\varepsilon(\cdot, x_0)$ is decreasing, it holds

$$F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_1, x_0)\big) \geq F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_2, x_0)\big),$$

and (4.32)-(4.33) yield

$$F^-\big(x_2, Z_\varepsilon(x_2, x_0), q_\varepsilon(x_2, x_0)\big) - F^-\big(x_1, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_2, x_0)\big) \geq \frac{\delta_{1,\flat}}{4}.$$

On the other hand, from (4.12) one shows that the map $x \to F^-(x, \eta, p)$ is monotone decreasing and thus

$$F^-\big(x_2, Z_\varepsilon(x_2, x_0), q_\varepsilon(x_2, x_0)\big) - F^-\big(x_2, Z_\varepsilon(x_1, x_0), q_\varepsilon(x_2, x_0)\big) \geq \frac{\delta_{1,\flat}}{4}. \tag{4.35}$$

Observe that the map $\eta \to F^-(x, \eta, p)$ is Hölder continuous due to Lemma 4.6. More precisely, there exist a constant $C_{x^\flat} > 0$ such that

$$\big|F^-(x, \eta_2, p) - F^-(x, \eta_1, p)\big| \leq C_{x^\flat} \cdot \big|\eta_2 - \eta_1\big|^{\frac{1}{2}}$$

for all $\eta_1, \eta_2 \in (0, W(x)]$, $x \in [\bar{x}, x^*]$, $p \in [\theta(x^*), 1]$. From (4.35) it holds

$$|Z_\varepsilon(x_2, x_0) - Z_\varepsilon(x_1, x_0)| \geq \frac{\delta_{1,\flat}^2}{16 C_{x^\flat}^2}.$$

Recalling (4.34), we have

$$Z_\varepsilon'(x, x_0) = F^- \left(x, Z_\varepsilon(x, x_0), q_\varepsilon(x, x_0)\right) \leq W'(x_0) + \frac{\delta_{1,x^\flat}}{2} \qquad \text{for all } x \in [x_1, x_2]$$

and it yields

$$|x_2 - x_1| \geq \frac{\delta_{1,x^\flat}^2}{8C_{x^\flat}^2[2W'(x_0) + \delta_{1,x^\flat}]}.$$

Therefore,

$$x_0 - a_\varepsilon(x_0) \geq \delta_{x^\flat} \doteq \min \left\{\delta_{1,x^\flat}, \frac{\delta_{1,x^\flat}^2}{8C_{x^\flat}^2[2W'(x_0) + \delta_{1,x^\flat}]}\right\} > 0.$$

□

Remark 4.16. In general, the backward Cauchy problem (4.28) may admits more than one solution.

As a consequence of Lemma 4.15 , there exists a sequence $\{\varepsilon_n\}_{n\geq 0} \to 0+$ such that $\{(Z_{\varepsilon_n}(\cdot, x_0), q_{\varepsilon_n}(\cdot, x_0))\}_{n\geq 1}$ converges to $(Z(\cdot, x_0), q(\cdot, x_0))$ which is a solution of (4.28). With the same argument in the proof of Proposition 4.13, we can extend backward the solution $(Z(\cdot, x_0), q(\cdot, x_0))$ until $a(x_0)$ such that

$$\lim_{x\to a(x_0)+} Z(a(x_0), x_0) = W(a(x_0)),$$

and Lemma 4.14 yields $\lim_{x\to a(x_0)+} q(a(x_0), x_0) \leq p_c(a(x_0))$. If the initial size of the debt is $\bar{x} \in [a(x_0), x_0]$ we think of $Z(\bar{x}, x_0)$ is as the expected cost of (4.4)-(4.5) with $p(\cdot, x_0)$, $x(0) = x_0$ achieved by the feedback strategies

(4.36)
$$u(x, x_0) = \operatorname*{argmin}_{w\in[0,1]} \left\{L(w) - \frac{Z'(x, x_0)}{p(x, x_0)} \cdot w\right\}, \quad v(x, x_0) = \operatorname*{argmin}_{v\geq 0}\Big\{c(v) - vxZ'(x, x_0)\Big\}.$$

With this strategy, the debt has the asymptotic behavior $x(t) \to x_0$ as $t \to \infty$.

4.3.2. *Construction of an equilibrium solution.* We are now ready to construct an solution tothe system of Hamilton-Jacobi equation (4.8). By induction, we define a family of back solutions as follows:

$$x_1 \doteq x_W^*, \qquad (Z_1(x), q_1(x)) = (Z(x, x^*), q(x, x^*)) \qquad \text{for all } x \in [x_1, x^*]$$

and

$$x_{n+1} \doteq a(x_n), \qquad (Z(x, x_n), q(x, x_n)) \qquad \text{for all } x \in [x_{n+1}, x_n]\,.$$

From Case 1 and Lemma 4.15, there exists a natural number $N_0 < 1 + \dfrac{x^* - x^\flat}{\delta_{x^\flat}}$ such that our construction will be stop in N_0 step, i.e.,

$$x_{N_0} > 0, \qquad a(x_{N_0}) = 0 \qquad \text{and} \qquad \lim_{x\to a(x_{N_0})} Z(x, x_{N_0}) = 0\,.$$

We will show that a feedback equilibrium solution to the debt management problem is obtained as follows

(4.37)
$$(V^*(x), p^*(x)) = \begin{cases} (Z(x, x^*), q(x, x^*)) & \text{for all } x \in (x_W, x^*], \\ (Z(x, x_k), q(x, x_k)) & \text{for all } x \in (a(x_k), x_k], k \in \{1, 2, \dots, N_0\}\,. \end{cases}.$$

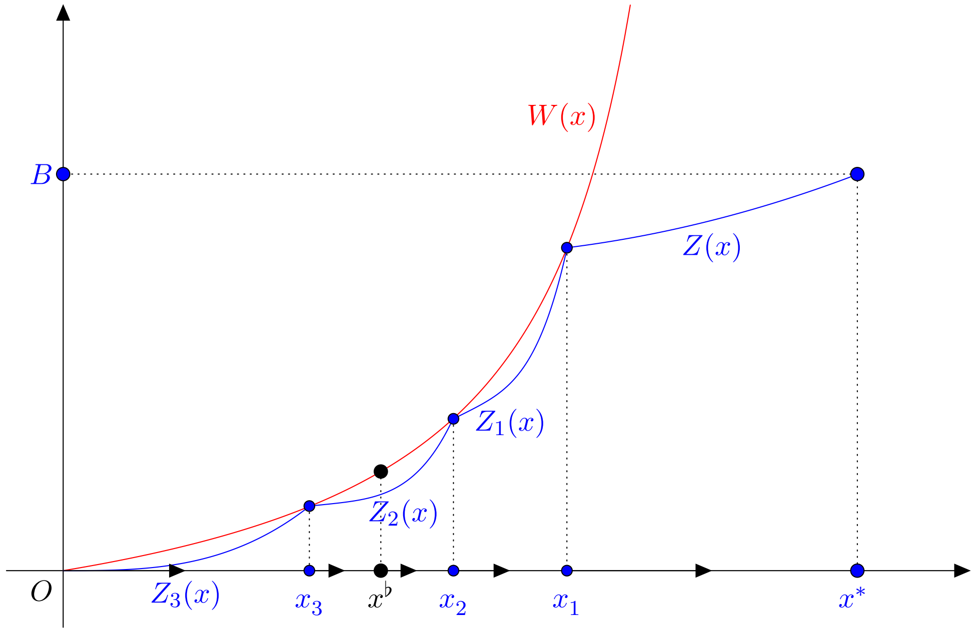

FIGURE 2. Construction of a solution: starting from (x^*, B) we solve backward the system until the first touch with the graph of W at $(x_1, W(x_1))$. Then we restart by solving backward the system with the new terminal conditions $(W(x_1), p_c(x_1))$, until the next touch with the graph of W at $(x_2, W(x_2))$ and so on. In a finite number of steps we reach the origin. If a touch occurs at $x_{n_0} < x^\flat$ then the backward solution from x_{n_0} reaches the origin with $q \equiv 1$. Given an initial value $\bar{x}$ of the DTI, if $0 \leq x_{n+1} < \bar{x} < x_n < x_1$ the the optimal strategy let the DTI increase asymptotically to x_n (no banktuptcy), while if $x_1 < \bar{x} < x^*$ then the optimal strategy let the DTI increase to x^*, thus providing bankruptcy in finite time.

$$u^*(x) = \underset{w\in[0,1]}{\operatorname{argmin}} \left\{ L(w) - \frac{(V^*)'(x)}{p^*(x)} \cdot w \right\}, \tag{4.38}$$

$$v^*(x) = \underset{v\geq 0}{\operatorname{argmin}} \left\{ c(v) - vx(V^*)'(x) \right\} . \tag{4.39}$$

Proof of Theorem 4.11. From the monotone increasing property of the maps $\xi \mapsto v^*(x^*, \xi)$, $\eta \mapsto F^-(x^*, \eta, \theta(x^*))$ and $p \mapsto F^-(x^*, W(x^*), p)$, we have

$$\begin{aligned}\theta(x^*) \cdot (r + \lambda + v^*(x^*, F^-(x^*, B, \theta(x^*))) \\ < p_c(x^*) \cdot (r + \lambda + v^*(x^*, F^-(x^*, W(x^*), p_c(x^*))) = r + \lambda\end{aligned}$$

and it yields (4.24). By Proposition 4.13 and Lemma 4.15, a pair $V^*(\cdot), p^*(\cdot)$ in (4.37) is well-defined on $[0, x^*]$. In the remaining steps, we show that V^*, p^*, u^*, v^* provide an equilibrium solution. Namely, they satisfy the properties (i)-(ii) in Definition 4.1.

1. To prove (i), let $V(\cdot)$ be the value function for the optimal control problem (4.4)-(4.5). For any initial value, $x(0) = x_0 \in [0, x^*]$, the feedback control u^* and v^* in (4.38)-(4.39) yields the cost $V^*(x_0)$. This implies

$$V(x_0) \ \le \ V^*(x_0).$$

To prove the converse inequality we need to show that, for any measurable control $u : [0,+\infty[\mapsto [0,1]$ and $v : [0,+\infty[\to [0,+\infty[$, calling $t\mapsto x(t)$ the solution to

$$\dot{x}(t) = \left(\frac{\lambda+r}{p^*(x(t))} - \lambda - \mu - v(t)\right) x(t) - \frac{u(t)}{p^*(x(t))}\,, \qquad x(0)=x_0\,, \tag{4.40}$$

one has

$$\int_0^{T_b} e^{-rt}[L(u(x(t))) + c(v(x(t)))]\,dt + e^{-rT_b}B \ge V^*(x_0), \tag{4.41}$$

where

$$T_b = \inf\left\{t\ge 0\,;\ x(t)=x^*\right\}$$

is the bankruptcy time (possibly with $T_b=+\infty$).

For $t\in[0,T_b]$, consider the absolutely continuous function

$$\phi^{u,v}(t) \doteq \int_0^t e^{-rs}\cdot[L(u(s))+c(v(s))]\ ds + e^{-rt}V^*(x(t)).$$

At any Lebesgue point t of $u(\cdot)$ and $v(\cdot)$, recalling that (V^*,p^*) solves the system (4.8), we compute

$$\begin{aligned}
\frac{d}{dt}\phi^{u,v}(t) &= e^{-rt}\cdot\Big[L(u(t)) + c(v(t)) - rV^*(x(t)) + (V^*)'(x(t))\cdot\dot{x}(t)\Big]\\
&= \ e^{-rt}\cdot\Big[L(u(t)) + c(v(t)) - rV^*(x(t))\\
&\qquad + (V^*)'(x(t))\left(\left(\frac{\lambda+r}{p^*(x(t))} - \lambda-\mu-v(t)\right)x(t) - \frac{u(t)}{p^*(x(t))}\right)\Big]\\
&\ge \ e^{-rt}\cdot\Big[\min_{\omega\in[0,1]}\left\{L(\omega) - \frac{(V^*)'(x(t))}{p^*(x(t))}\,\omega\right\} + \min_{\zeta\in[0+\infty[}\left\{c(\zeta) - (V^*)'(x(t))x(t)\,\zeta\right\}\\
&\qquad + \left(\frac{\lambda+r}{p^*(x(t))} - \lambda-\mu\right)x(t)(V^*)'(x(t)) - rV^*(x(t))\Big]\\
&= \ e^{-rt}\cdot\Big[H\left(x(t),(V^*)'(x(t)),p^*(x(t))\right) - rV^*(x(t))\Big] = 0.
\end{aligned}$$

Therefore,

$$V^*(x_0) = \phi^{u,v}(0) \le \lim_{t\to T_b-}\phi^{u,v}(t) = \int_0^{T_b} e^{-rt}\cdot[L(u(t))+c(v(t))]\ dt + e^{-rT_b}B,$$

proving (4.41).

2. It remains to check (ii). The case $x_0=0$ is trivial. Two main cases will be considered.

CASE 1: $x_0\in]x_1,x^*]$. Then $x(t)>x_1$ for all $t\in[0,T_b]$. This implies

$$\dot{x}(t) = H_\xi(x(t), Z(x(t),x^*), q(x(t),x^*))\,.$$

From the second equation in (4.8) it follows

$$\frac{d}{dt}\, p(x(t)) = p'(x(t))\dot{x}(t) = (r+\lambda+v^*(x(t)))p(x(t)) - (r+\lambda),$$

Therefore, for every $t \in [0, T_b]$ one has

$$p(x(0)) = p(t)\cdot \int_0^t e^{-(r+\lambda+v^*(x(\tau)))}\, d\tau + \int_0^t (r+\lambda)\int_0^\tau e^{-(r+\lambda+v^*(x(s)))}\, ds\, d\tau$$

Letting $t \to T_b$ we obtain

$$p(x_0) = \int_0^{T_b} (r+\lambda)\int_0^\tau e^{-(r+\lambda+v^*(x(s)))}\, ds\, d\tau + \theta(x^*)\cdot \int_0^{T_b} e^{-(r+\lambda+v^*(x(\tau)))}\, d\tau$$

proving (ii).

CASE 2: $x_0 \in [a(x_k), x_k[$ for $k \in \{1,2,...,N_0\}$.. In this case, $T_b = +\infty$ and $x(t) \in [a_{x_k}, x_k[$ such that

$$\lim_{t\to+\infty} x(t) = x_k\,.$$

With a similar computation, one has

$$p(x_0) = \theta(x^*)\cdot \int_0^\infty e^{-(r+\lambda+v^*(x(\tau)))}\, d\tau$$

proving (ii). □

4.4. **Dependence on x^*.** As in the stochastic case, we now study the behavior of the total cost for servicing when the maximum size x^* of the DTI, at which bankruptcy is declared, becomes very large.

Proposition 4.17. *Let $(V(x,x^*), p(x,x^*))$ be constructed in Theorem 4.11. The following holds:*

(i) if $\limsup_{s\to+\infty} \theta(s)s = R < +\infty$ then

$$\liminf_{x^*\to+\infty} V(x,x^*) \geq B\cdot\left(1-\frac{R}{x}\right)^{\frac{r}{r+\lambda}} \tag{4.42}$$

for all

$$x \geq \frac{1}{r-\mu}\cdot \max\left\{4, \frac{4B}{L'(0)}, \frac{4C_1B}{c'(0)}, 2C_1c^{-1}(rB)\right\}.$$

(ii) if $\lim_{s\to+\infty} \theta(s)s = +\infty$ then

$$\limsup_{x^*\to\infty} V(x,x^*) = 0 \qquad \text{for all } x\in[0,x^*[\tag{4.43}$$

Proof. **1.** We first provide an upper bound on $v(\cdot,x^*)$. For all $x \geq \frac{4}{r-\mu}$, from (4.8) and (4.10), we estimate

$$\begin{aligned} H(x,\xi,p) &\geq \min_{v\geq 0}\{c(v) - x\xi v\} + [(r-\mu)x - 1]\cdot\frac{\xi}{p} \\ &\geq \min_{v\geq 0}\{c(v) - x\xi v\} + \frac{(r-\mu)x}{2}\cdot\frac{\xi}{p} \doteq K(x,\xi,p) \end{aligned}$$

for all $\xi, p > 0$ and $x \geq \dfrac{2}{r-\mu}$. One computes

$$K_\xi(x,\xi,p) = \frac{(r-\mu)x}{2p} - xv_K$$

where

$$v_K = \begin{cases} 0, & \text{if } 0 \leq x\xi < c'(0), \\ (c')^{-1}(x\xi), & \text{if } x\xi \geq c'(0) > 0. \end{cases}$$

This implies that the maximum of K is achieved for $v_K = \frac{r-\mu}{2p}$ and its value is

$$\max_{\xi \geq 0} K(x,\xi,p) = K(x,\xi_K,p) = c\left(\frac{r-\mu}{2p}\right) \qquad \text{with} \qquad \xi_K = \frac{c'(v_K)}{x}.$$

Thus, the monotone increasing property of the map $\xi \to H(x,\xi,p(x,x^*))$ on the interval $\left[0, \xi^\sharp(x,p(x,x^*))\right]$ implies that

$$F^-(x,V(x,x^*),p(x,x^*)) < \xi_K \qquad \Longrightarrow \qquad v(x,x^*) \leq \frac{r-\mu}{2p(x,x^*)}. \tag{4.44}$$

provided that $c\left(\frac{r-\mu}{2p(x,x^*)}\right) \geq rB$. From (4.8)) and (4.10), one has

$$\begin{aligned} rB \geq& -xV'(x,x^*)v(x,x^*) + [(r-\mu)x - u(x,x^*)] \cdot \frac{V'(x,x^*)}{p(x,x^*)} \\ \geq& \left[\frac{(r-\mu)x}{2} - 1\right] \cdot \frac{V'(x,x^*)}{p(x,x^*)} \geq \frac{(r-\mu)x}{4} \cdot \frac{V'(x,x^*)}{p(x,x^*)}. \end{aligned}$$

Thus, if

$$p(x,x^*) \leq \min\left\{\frac{r-\mu}{2c^{-1}(rB)}, \frac{(r-\mu)c'(0)}{4B}\right\} \qquad \text{and} \qquad x \geq \max\left\{\frac{4}{r-\mu}, \frac{4B}{(r-\mu)L'(0)}\right\} \tag{4.45}$$

then

$$\frac{V'(x,x^*)}{p(x,x^*)} \leq \frac{4B}{(r-\mu)x} \leq L'(0) \qquad \Longrightarrow \qquad u(x,x^*) = 0. \tag{4.46}$$

and

$$V'(x,x^*)x \leq \frac{4B}{r-\mu} \cdot p(x,x^*) \leq c'(0) \qquad \Longrightarrow \qquad v(x,x^*) = 0. \tag{4.47}$$

In this case, from (4.8), (4.10) and (4.12), it holds

$$(r+\lambda)(p(x,x^*) - 1) = \left(\frac{\lambda + r}{p(x,x^*)} - \lambda - \mu\right) xp'(x,x^*).$$

Thus,

$$p(x,x^*) = \frac{\theta(x^*)x^*}{x} \cdot \left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}}$$

provided that (4.45) holds.

2. Assume that

$$\limsup_{s \in [0,+\infty)} \theta(s)s = R < +\infty,$$

we have

$$\sup_{s\in[0,+\infty)} \theta(s)s = C_1$$

for some $C_1 < +\infty$. Since $p(\cdot, x^*)$ is increasing, it holds

$$p(x,x^*) \le \frac{\theta(x^*)x^*}{x} \le \frac{C_1}{x} \qquad \text{if (4.45) holds}\,. \tag{4.48}$$

Denote by

$$M \doteq \frac{1}{r-\mu}\cdot \max\left\{4, \frac{4B}{L'(0)}, \frac{4C_1B}{c'(0)}, 2C_1c^{-1}(rB)\right\},$$

we then have

$$u(x,x^*) = v(x,x^*) = 0 \qquad \text{for all } x\in[M,x^*], x^*\ge M\,.$$

Recalling (4.8), (4.10) and (4.12), we have

$$\begin{cases} V'(x,x^*) &= \dfrac{rp}{[(\lambda+r)-(\lambda+\mu)p(x,x^*)]x}V\,, \\[2ex] p'(x,x^*) &= (\lambda+r)\cdot\dfrac{p(x,x^*)(p(x,x^*)-1)}{[(\lambda+r)-(\lambda+\mu)p(x,x^*)]\,x}\,. \end{cases} \tag{4.49}$$

for all $x\in[M,x^*]$ with $x^*\ge M$. Solving the above ODE (see in Section 5 of [5]), we obtain that

$$V(x,x^*) = B\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}}, \qquad p(x,x^*) = \frac{\theta(x^*)x^*}{x}\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}}$$

for all $x\ge[M,x^*]$. Thus,

$$\liminf_{x^*\to+\infty} V(x,x^*) \ge B\cdot\left(1-\frac{R}{x}\right)^{\frac{r}{r+\lambda}} \qquad \text{for all } x\ge M$$

and it yields (4.42).

3. We are now going to prove (ii). Assume that

$$\limsup_{s\to+\infty} \theta(s)s = +\infty\,. \tag{4.50}$$

Set

$$\gamma \doteq \min\left\{\frac{r-\mu}{2c^{-1}(rB)}, \frac{(r-\mu)c'(0)}{4B}\right\} \qquad \text{and} \qquad M_2 \doteq \max\left\{\frac{4}{r-\mu}, \frac{4B}{(r-\mu)L'(0)}\right\}.$$

For any $x^* > M_2$, denote by

$$\tau(x^*) \doteq \begin{cases} x^* & \text{if} \quad \theta(x^*)\ge\gamma\,, \\[2ex] \inf\left\{x\ge M_2 \;\middle|\; p(x,x^*)\le\gamma\right\} & \text{if} \quad \theta(x^*)<\gamma\,. \end{cases}$$

From (4.45)–(4.47), the decreasing property of p yields

$$p(x,x^*)\ge\gamma \qquad \text{for all } x\in[M_2,\tau(x^*)[\tag{4.51}$$

and

$$p(x,x^*)<\gamma \qquad \Longrightarrow \qquad u(x,x^*) = v(x,x^*) \qquad \text{for all } x\in[\tau(x^*),x^*]\,.$$

As in the step 2, for any $x \in [\tau(x^*), x^*]$, we have

$$V(x,x^*) = B \cdot \left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}}, \qquad p(x,x^*) = \frac{\theta(x^*)x^*}{x} \cdot \left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}}$$

This implies that

$$V(x,x^*) = B \cdot \left(\frac{p(x,x^*)x}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}} \le B \cdot \left(\frac{x}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}} \tag{4.52}$$

for all $x \in [\tau(x^*), x^*]$. On the other hand, for any $x \in [M_2, \tau(x^*)]$, from (4.8), (4.10) and (4.51), it holds

$$rV(x,x^*) \le \frac{r+\lambda}{p(x,x^*)} xV'(x,x^*) \le \frac{(r+\lambda)x}{\gamma} \cdot V'(x,x^*).$$

This implies that

$$V(x,x^*) \le V(\tau(x^*),x^*) \cdot \left(\frac{x}{\tau(x^*)}\right)^{\frac{r\gamma}{r+\lambda}} \le B \cdot \left(\frac{x}{\tau(x^*)}\right)^{\frac{r\gamma}{r+\lambda}} \tag{4.53}$$

for all $x \in [M_2, \tau(x^*)]$.

For any fix $x_0 \ge M_2$, we will prove that

$$\limsup_{x^* \to +\infty} V(x_0, x^*) = 0 \tag{4.54}$$

Two cases are considered:

- If $\limsup_{x^*\to+\infty} \tau(x^*) = +\infty$ then (4.53) yields
$$\lim_{x^*\to+\infty} V(x_0,x^*) \le \liminf_{x^*\to+\infty} B \cdot \left(\frac{x_0}{\tau(x^*)}\right)^{\frac{r\gamma}{r+\lambda}} = 0.$$
- If $\limsup_{x^*\to+\infty} \tau(x^*) < +\infty$ then
$$\tau(x^*) < M_3 \qquad \text{for all } x^* > 0$$
for some $M_3 > 0$. Recalling (4.52) and (4.50), we obtain that
$$\lim_{x^*\to\infty} V(x_0,x^*) \le \lim_{x^*\to\infty} V(x_0+M_3,x^*) \le \lim_{x^*\to\infty} B \cdot \left(\frac{x_0+M_3}{\theta(x^*)x^*}\right)^{\frac{r}{r-\mu}} = 0.$$

Thus, (4.54) holds and the increasing property of $V(\cdot,x^*)$ yields (4.43). □

The following result shows that for sufficiently large initial DTI and bankruptcy threshold and recovery fraction after bankruptcy, the optimal strategy for the borrower will use currency devaluation to deflate the DTI. For simplicity, let us consider x^* and B^* sufficiently large such that

$$x^* > \frac{L'(0)+Br}{L'(0)\cdot(r-\mu)} \qquad \text{and} \qquad B \ge \frac{2(r-\mu)c'(0)}{r}. \tag{4.55}$$

In this case, the following holds:

Proposition 4.18 (Devaluating strategies). *Let* $x \mapsto (V(x,x^*), p(x,x^*))$ *be an equilibrium solution of* (4.8) *with boundary conditions* (4.9). *If*

$$\theta(x^*)x^* > \frac{2(r+\lambda)c'(0)}{r-\mu} \cdot \left(\frac{1}{rB} + \frac{1}{L'(0)}\right) \tag{4.56}$$

then the function

$$v^*(x,x^*) = \operatorname*{argmin}_{\omega\geq 0}\left\{c(\omega) - \omega x V'(x,x^*)\right\}$$

is not identically zero.

Proof. Set $M := \dfrac{L'(0)+Br}{L'(0)\cdot(r-\mu)}$. Assume by a contradiction that $v^*(x,x^*)=0$ for all $x\in[M,x^*]$. In particular, we have

$$0 \leq xV'(x,x^*) \leq c'(0) \qquad x\in[M,x^*]. \tag{4.57}$$

The system (4.8) in $[M,x^*]$ reduces to

$$\begin{cases} rV(x) = \tilde{H}(x,V'(x),p(x)) \\ \\ (r+\lambda)(p(x)-1) = \tilde{H}_\xi(x,V'(x),p(x))\cdot p'(x) \end{cases} \tag{4.58}$$

with

$$\tilde{H}(x,\xi,p) \;=\; \min_{u\in[0,1]}\left\{L(u) - \frac{u}{p}\,\xi\right\} + \left(\frac{\lambda+r}{p} - \lambda - \mu\right) x\,\xi.$$

Since $r>\mu$ and $p\in[0,1]$, it holds

$$\tilde{H}(x,\xi,p) \;\geq\; -\frac{\xi}{p} + (\lambda + r - p(\lambda+\mu))\,x\,\frac{\xi}{p} \;\geq\; ((r-\mu)x - 1)\cdot\frac{\xi}{p}$$

and (4.58) yields

$$rB \;\geq\; rV(x,x^*) \;\geq\; ((r-\mu)x-1)\cdot\frac{V'(x,x^*)}{p(x,x^*)}.$$

Thus for $x\in[M,x^*]$ we obtain

$$\frac{V'(x,x^*)}{p(x,x^*)} \;\leq\; \frac{rB}{(r-\mu)x-1} \leq L'(0),$$

which immediately implies

$$u^*(x,x^*) \;:=\; \operatorname*{argmin}_{u\in[0,1]}\left\{L(u) - u\cdot\frac{V'(x,x^*)}{p(x,x^*)}\right\} \;=\; 0.$$

Hence, $(V(\cdot,x^*),p(\cdot,x^*))$ solves (4.8) on $[M,x^*]$ and

$$V(x,x^*) = B\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r}{r+\lambda}} \geq B\cdot\left(1 - \frac{r}{r+\lambda}\cdot p(x,x^*)\right)$$

$$p(x,x^*) = \frac{\theta(x^*)x^*}{x}\cdot\left(\frac{1-p(x,x^*)}{1-\theta(x^*)}\right)^{\frac{r-\mu}{r+\lambda}} \geq \frac{\theta(x^*)x^*}{x}\cdot\left(1 - \frac{r-\mu}{r+\lambda}\cdot p(x,x^*)\right)$$

for all $x\in[M,x^*]$. From the above inequality, one derives

$$p(x,x^*) \;\geq\; \frac{(r+\lambda)\theta(x^*)x^*}{(r+\lambda)x + (r-\mu)\theta(x^*)x^*}.$$

Thus, (4.57) and the first equation in (**??**) imply

$$\begin{aligned} c'(0) \geq & xV'(x,x^*) \;=\; rp(x,x^*)\cdot\frac{V(x,x^*)}{(\lambda+r)-(\lambda+\mu)p(x,x^*)} \\ \geq & \frac{rp(x,x^*)B}{r+\lambda}\cdot\frac{r+\lambda-rp(x,x^*)}{(\lambda+r)-(\lambda+\mu)p(x,x^*)} \geq \frac{rp(x,x^*)B}{r+\lambda} \end{aligned}$$

$$\geq \frac{rB\theta(x^*)x^*}{(r+\lambda)x + (r-\mu)\theta(x^*)x^*} \qquad \text{for all } x \in [M, x^*].$$

In particular, choose $x = M$ and recall (4.55), we get

$$M \geq \frac{rB - (r-\mu)c'(0)}{(r+\lambda)c'(0)} \cdot \theta(x^*)x^* \geq \frac{rB}{2(r+\lambda)c'(0)} \cdot \theta(x^*)x^*$$

and it contradicts to (4.56). □

Appendix A. Some results of convex analysis

We introduce now some concepts of convex analysis, referring the reader to [7] and [14] for a comprehensive introduction to the subject.

Definition A.1 (Convex conjugate and subdifferential)**.** We recall that the convex conjugate $F^\circ : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ of a map $F : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is the lower semicontinuous convex function defined by

$$F^\circ(z^*) = \sup_{z \in \mathbb{R}^d} \Big\{ \langle z^*, z\rangle - F(z) \Big\}.$$

Let $F : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper (i.e., not identically $+\infty$), convex, lower semicontinuous functions, $x \in \operatorname{dom} F := \{x \in \mathbb{R}^d : F(x) \in \mathbb{R}\}$. We define the *subdifferential in the sense of convex analysis* of F at x by setting

$$\partial F(x) := \{v_x \in \mathbb{R}^d : F(y) - F(x) \geq \langle v_x, y - x\rangle \text{ for all } y \in \mathbb{R}^d\}.$$

The following result provide a list of some properties of the subdifferential in the sense of convex analysis.

Lemma A.2 (Properties of the subdifferential)**.** *Let $F, G : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper (i.e., not identically $+\infty$), convex, lower semicontinuous functions,*

(1) If F is classically (Fréchet) differentiable at x, then $\partial F(x) = \{F'(x)\}$.
(2) $z^* \in \partial F(z)$ if and only if $z \in \partial F^\circ(z^*)$.
(3) $F(x_0) = \min_{x \in \mathbb{R}^d} F(x)$ if and only if $0 \in \partial F(x_0)$
(4) $z^* \in \partial F^\circ(z)$ if and only if $F(z) + F^\circ(z^*) = \langle z^*, z\rangle$. In this case $z^* \in \operatorname{dom} F^\circ$;
(5) $\lambda \geq 0$ we have $\partial(\lambda F)(z) = \lambda \partial F(z)$;
(6) if there exists $z \in \operatorname{dom}(F) \cap \operatorname{dom}(G)$ such that F is continuous at z then $\partial(F + G)(x) = \partial F(x) + \partial G(x)$ for all $x \in \operatorname{dom}(F) \cap \operatorname{dom}(G)$;
(7) let $\bar{y} \in \mathbb{R}^m$, $\Lambda : \mathbb{R}^m \to \mathbb{R}^d$ be a linear map, G be continuous and finite at $\Lambda(\bar{y})$; Then $\partial(G \circ \Lambda)(y) = \Lambda^T \partial G(\Lambda y)$ for all $y \in \mathbb{R}^m$, where $\Lambda^T : \mathbb{R}^d \to \mathbb{R}^m$ is the adjoint of Λ.

Appendix B. Properties of the Hamiltonian function

In this Appendix we collect some technical results related to the Hamiltonian function in the stochastic and in the deterministic case.

Lemma B.1. *Assume **(A1)-(A2)**. Then, denoted by L°, c° the convex conjugate (see Appendix A for the notation) of L and c, respectively, we have that $L^\circ, c^\circ : \mathbb{R} \to \mathbb{R}$ are continuously differentiable and*

$$(L^\circ)'(\rho) = \begin{cases} 0, & \text{if } \rho < L'(0), \\ \\ (L')^{-1}(\rho), & \text{if } \rho \geq L'(0), \end{cases} \qquad (c^\circ)'(\rho) = \begin{cases} 0, & \text{if } \rho < c'(0), \\ \\ (c')^{-1}(\rho), & \text{if } \rho \geq c'(0), \end{cases}$$

Moreover,

$$L^\circ(\rho) \le \max\{0,\rho\}, \qquad\qquad c^\circ(\rho) \le \max\{0, v_{\max}\rho\}$$

Proof. Recalling the assumptions **(A1)** $-$ **(A2)** on L, c, the equations

$$L^\circ(\rho_1) + L(u) = u\rho_1, \qquad\qquad c^\circ(\rho_2) + c(v) = v\rho_2,$$

admits as unique solutions

$$u(\rho_1) = \begin{cases} 0, & \text{if } \rho_1 < L'(0), \\ \\ (L')^{-1}(\rho_1), & \text{if } \rho_1 \ge L'(0) > 0, \end{cases} \qquad v(\rho_2) = \begin{cases} 0, & \text{if } \rho_2 < c'(0), \\ \\ (c')^{-1}(\rho_2), & \text{if } \rho_2 \ge c'(0) \ge 0. \end{cases}$$

The result now follows from Theorem 23.5, Theorem 25.1, and Theorem 26.3 in [14]. For the second part, set $I_C(s) = 0$ if $s \in C$ and 0 otherwise, since $L(u) \ge I_{[0,1]}(u)$ and $c(v) \ge I_{[0,v_{\max}]}(v)$, we have

$$L^\circ(\rho) \le I^\circ_{[0,1]}(\rho) = \max_{u\in[0,1]} \langle u,\rho\rangle = \max\{0,\rho\},$$
$$c^\circ(\rho) \le I^\circ_{[0,v_{\max}]}(\rho) = \max_{v\in[0,v_{\max}]} \langle v,\rho\rangle = \max\{0, v_{\max}\cdot\rho\}.$$

□

Lemma B.1 immediately implies

Lemma B.2. *Assume **(A1)-(A2)**, and let H be defined as in* (2.6)*. Then H is continuous differentiable and its gradient at points $(x,\xi,p) \in [0,+\infty[\times[0,+\infty[\times]0,1]$ can be expressed in terms of $u^*(\xi,p) := (L^\circ)'(\xi/p)$ and $v^*(x,\xi) := (c^\circ)'(x\xi)$ by*

$$\text{(B.1)}\qquad \begin{cases} H_x(x,\xi,p) = & \Big[(\lambda+r) - p(\lambda+\mu+v^*(x,\xi)-\sigma^2)\Big]\cdot\dfrac{\xi}{p}, \\ H_\xi(x,\xi,p) = & \dfrac{1}{p}\cdot\Big[x\big((\lambda+r) - p(\lambda+\mu+v^*(x,\xi)-\sigma^2)\big) - u^*(\xi,p)\Big], \\ H_p(x,\xi,p) = & (u^*(\xi,p) - x(\lambda+r))\cdot\dfrac{\xi}{p^2}, \end{cases}$$

Moreover,

$$\begin{cases} u^*(\xi,p) = & \operatorname*{argmin}_{u\in[0,1]} \left\{L(u) - u\,\dfrac{\xi}{p}\right\}, \\ \\ v^*(x,\xi) = & \operatorname*{argmin}_{v\ge 0} \{c(v) - vx\xi\}\,. \end{cases}$$

and for all $x > 0$, $0 < p \le 1$ we have also

$$\text{(B.2)}\qquad \nabla u^*(\xi,p) = \frac{(1, -L'(u^*(x,\xi,p)))}{pL''(u_*(x,\xi,p))}, \ \textit{if } \xi > pL'(0),$$
$$\nabla v^*(x,\xi) = \frac{(\xi,x)}{c''(v^*(x,\xi))}, \ \textit{if } x\xi > c'(0),$$
$$\lim_{\xi\to+\infty} v^*(x,\xi) = v_{\max},$$

Lemma B.3. *Let the assumptions **(A1)**-**(A2)** hold. Then*

(1) for all $\xi \geq 0$ and $p \in]0,1]$, the function H in (2.6) satisfies

$$
\begin{aligned}
H(x,\xi,p) &\leq \left(\frac{\lambda+r}{p} - (\lambda+\mu) + \sigma^2\right) x\xi;\\
H(x,\xi,p) &\geq \left(\frac{(\lambda+r)x-1}{p} + (\sigma^2-\lambda-\mu-v^*(x,\xi))x\right)\cdot\xi\\
&\geq \left(\frac{(\lambda+r)x-1}{p} + (\sigma^2-(\lambda+\mu)-v_{\max})x\right)\cdot\xi;\\
H_\xi(x,\xi,p) &\leq \left(\frac{\lambda+r}{p} - (\lambda+\mu) + \sigma^2\right) x;\\
H_\xi(x,\xi,p) &\geq \frac{(\lambda+r)x-1}{p} + (\sigma^2-(\lambda+\mu)-v^*(x,\xi))x\\
&\geq \frac{(\lambda+r)x-1}{p} + (\sigma^2-(\lambda+\mu)-v_{\max})x
\end{aligned}
$$

(2) for every $x,p>0$ the map $\xi \mapsto H(x,\xi,p)$ is concave down and satisfies

$$
H(x,0,p)=0, \qquad H_\xi(x,0,p) = \left(\frac{\lambda+r}{p} - (\lambda+\mu) + \sigma^2\right) x.
$$

Proof. The concavity of $\xi \mapsto H(x,\xi,p)$ for every $x,p>0$ is immediate from the definition of H in (2.6). The equalities in item (2) are immediate from Lemma B.1. The upper bound on $H(x,\xi,p)$ follows from the positivity of L° and c°. By concavity, the map $\xi \mapsto H_\xi(x,\xi,p)$ is monotone decreasing, thus $H_\xi(x,\xi,p) \leq H_\xi(x,0,p)$, which proves the upper bound on $H_\xi(x,\xi,p)$ together with item (2). The lower estimate for $H(x,\xi,p)$ comes from the second part of Lemma B.1, in particular from the upper estimate on $L^\circ(\cdot)$. The lower estimate for $H_\xi(x,\xi,p)$ comes from Lemma B.2, noticing that

$$
\lim_{\xi\to+\infty} u^*(\xi,p) = \lim_{\rho\to+\infty} (L')^{-1}(\rho) = 1, \lim_{\xi\to+\infty} v^*(x,\xi) = \lim_{\rho\to+\infty} (c')^{-1}(\rho) = v_{\max},
$$

and using the decreasing property of $\xi \mapsto H_\xi(x,\xi,p)$, i.e., the fact that

$$
\lim_{\zeta\to+\infty} H_\xi(x,\zeta,p) \leq H_\xi(x,\xi,p),
$$

for all $x \geq 0$, $p \in]0,1]$, $\xi \in \mathbb{R}$. □

Lemma B.4. *Assume that $f : I \to \mathbb{R}$ is a C^2 convex strictly increasing function defined on a real interval I, and satisfying $f'' \geq \delta > 0$. Then, denoted by g its inverse function, $g : f(I) \to I$, we have that g is 1/2-Hölder continuous.*

Proof. Indeed, let $x_1, x_2 \in f^{-1}(I)$ with $x_1 \leq x_2$, and set $y_1 = g(x_1)$ and $y_2 = g(x_2)$.

$$
\begin{aligned}
f(y_2) - f(y_1) &= \int_{y_1}^{y_2} f'(t)\,dt = \int_{y_1}^{y_2} [f'(t) - f'(y_1)]\,dt\\
&= f'(y_1)\cdot(y_2-y_1) + \int_{y_1}^{y_2}\int_{y_1}^{t} f''(s)\,ds\,dt\\
&\geq f'(y_1)\cdot(y_2-y_1) + \frac{\delta}{2}(y_2-y_1)^2 \geq \frac{\delta}{2}(y_2-y_1)^2,
\end{aligned}
$$

since f is strictly increasing, $f''(s) \geq \delta$, and $y_1 \leq y_2$. Thus if $x_2 \geq x_1$ we have

$$
|g(x_2) - g(x_1)| \leq \sqrt{\frac{2}{\delta}}|x_2 - x_1|^{1/2}.
$$

By switching the roles of x_2 and x_1, the same holds true if $x_1 \geq x_2$. □

Appendix C. List of symbols

$X = X(t)$	the total nominal value of the outstanding debt;
$Y = Y(t)$	the gross national product GDP measured in terms of the floating currency unit;
$U = U(t)$	the rate of payments that the borrower chooses to make to the lenders;
μ	average growth rate of the economy;
σ	the volatility;
λ	rate at which the borrower pays back the principal;
r	discount rate;
$x = x(t) = X(t)/Y(t)$	debt-to-GDP ratio (DTI);
$p = p(t)$	discounted bond price;
$\tilde{u} = \tilde{u}(t) = U(t)/Y(t)$	fraction of the total income allocated to reduce the debt;
$\tilde{v} = \tilde{v}(t)$	devaluation rate;
x^*	maximum DTI threshold;
θ	recovery fraction after bankruptcy;
B	bankruptcy cost;
$L = L(u)$	adversion toward austerity policy;
$c = c(v)$	social cost due to currency devaluation.

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