Quantitative isoperimetric inequalities for a class of nonconvex sets *

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Abstract

Quantitative versions (i.e., taking into account a suitable "distance" of a set from being a sphere) of the isoperimetric inequality are obtained, in the spirit of [17, 18], for a class of not necessarily convex sets called φ-convex sets. Our work is based on geometrical results on φ-convex sets, obtained using methods of both nonsmooth analysis and geometric measure theory.

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1 Introduction

The well known isoperimetric inequality (we make reference to the original paper by De Giorgi [12]) states that if K is a Borel set in $\mathbb{R}^n$, $n \geq 2$, with finite measure $|K|$ and finite perimeter $P(K)$, the ball with the same volume has smaller perimeter, i.e.,

$$n\omega_n^{\frac{1}{n}}|K|^{\frac{n-1}{n}} \leq P(K). \tag{1.1}$$

Moreover equality holds in (1.1) if and only if K is a sphere.

Several authors, starting from Bonnesen [2], have given *quantitative* versions of the isoperimetric inequality, in the sense that they have improved (1.1) by introducing a further term which is zero if and only if the inequality is an equality. This term is a kind of measure of the "distance" of K from being a sphere. For example, (1.1) may be refined to

$$n\omega_n^{\frac{1}{n}}|K|^{\frac{n-1}{n}}(1+a) \leq P(K),$$

a being a nonnegative number, zero if and only if K is a sphere, with a clear geometrical meaning.

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Among authors studying quantitative versions of (1.1) we quote [2, 21, 17, 19, 18]. Related results appear in [13, 7, 8, 16]. In particular, [17, 18] deal with *sharp* estimates of the correction term a. In [17], a is given using the *spherical deviation* $d(K)$ of K, which is actually the Hausdorff distance between a suitable normalization of K and the unit ball. The result in [17] is very precise, but requires strong assumptions on K. In particular, K needs to be "nearly spherical", in the sense that both its Hausdorff distance from the unit ball and the norm of the gradient of a suitable representation of the boundary must be small. The result in [17] applies mainly to the class of compact *convex* sets K with nonempty interior having "isoperimetric deficiency" $\Delta(K)$ (see Definition 2.2 below) small enough. The paper [18], instead, deals with *general* Borel sets with finite n-dimensional measure and finite perimeter. Of course the term a needs to be modified, as there is no geometric assumption on K. Yet, the authors succeed to give a sharp estimate of a in terms of the so called *Fraenkel asymmetry* of K (see Definition 2.4 below), which is essentially the Lebesgue measure of the symmetric difference between K and a suitable sphere. This solves a conjecture appearing in [19], which was the first paper considering quantitative isoperimetric inequalities for general sets in $\mathbb{R}^n$.

The research in geometric inequalities, in particular for convex bodies in $\mathbb{R}^n$, is very active (see, e.g., [20, 24, 25] and references therein). In this paper we try to make a step towards a relaxation of the strong assumption of convexity, yet keeping the main features of the results. We deal mainly with results on the line of [17] for a class of nonconvex sets called φ-convex sets. The concept of φ-convexity appears, for example, in connection with curvature measures (see [15]), with control theory (see [9, 10]), or with variational problems (see [5]), and shares several properties with convex set. φ-convex sets are defined (see Definition 2.5 below) through a suitable *external sphere condition with locally uniform radius*, proportional to $1/\varphi$. One of our isoperimetric inequality reads as follows (see Theorem 6.1 below for a more precise statement):

Theorem. *Let $\varphi \geq 0$ be given and let $K \subset \mathbb{R}^n$ be a compact φ-convex set with nonempty interior. Then there exist $d = d(|K|, \varphi_0, n)$, with $0 < d < 1$, $\eta = \eta(n) > 0$, and a continuous strictly increasing function f (with $f(0) = 0$) such that if $d(K) \leq d$ and $\Delta(K) \leq \eta$, then*

$$d(K) \leq f(\Delta(K)).$$

Explicit estimates for d, η are given; f is explicit as well, depends only on n, and is the same as in [17].

A more detailed summary of the ideas and the results now follows.

By observing that convex sets satisfy an external sphere condition with arbitrarily large radius, one may imagine that several features of convexity have a good counterpart for φ-convexity. For example, the smoothness of the distance function and – equivalently – the uniqueness of the metric projection hold *locally* for such sets, while hold *globally* for convex sets. Our results are based on several new geometric estimates a on compact φ-convex set K with nonempty interior, which give simple conditions guaranteeing that K is "nearly spherical". This permits to apply the general result of [17], proving a quantitative version of (1.1) of the same type of [17]. Following [17], we deal with a compact set K with nonempty interior, containing a ball of radius ρ and contained in a concentric ball of radius R. The techniques of convex analysis used in [17] of course cannot be used. However, suitable modifications for the φ-convex case are possible, and our results yield exactly those given in [17] if K is convex. More precisely, some compatibility conditions involving only φ, ρ and R are assumed. Such conditions vanish if the parameter φ is identically zero, i.e., if K is convex and the estimates become the convex ones. We ask, essentially, the parameter φ to be small enough, and ensure in turn that

1. K satisfies an internal cone condition with uniform width;
2. the barycentre of K belongs to K;
3. the convex hull of K is contained in the neighborhood of K where the distance to K is smooth.

Such results will be proved mainly in Section 3. Furthermore, in our proof of the isoperimetric inequality we make essential use of the extension of the classical Brunn-Minkowski theory to φ-convex sets. This is one of the main results in the seminal paper [15], where φ-convex sets appear for the first time (under the name of *sets with positive reach*) and are intensively studied.

Section 4 contains an estimate of the isoperimetric deficiency and of the spherical deviation of the convex hull of K in terms of the same quantities for K. Section 5 contains an upper bound for the spherical deviation of K involving its Fraenkel asymmetry. In general, spherical deviation and Fraenkel asymmetry are unrelated concepts, in the sense that for the same set one can be very large and the other very small. Under some conditions involving only ρ, R, and φ, we estimate the spherical deviation in terms of Fraenkel asymmetry. This result seems new also for convex sets.

Our isoperimetric inequalities are proved in Section 6. We prove first a result involving the spherical deficiency of K which generalizes [17]. Next, we compare our result with the estimate given in [18]. We show that results of the same type of ours can be deduced from [18] and conversely, for a class of φ-convex sets, but with less sharp exponents. Finally we deduce from the main result in [18] another inequality, valid for a different class of φ-convex sets. A

Finally, we remark that we do not write explicit formulas for the case $n = 2$, making reference for this case to [2, 17].

2 Preliminaries

2.1 Notations and basic results

In $\mathbb{R}^n$ the canonical basis will be denoted by e_i, $i = 1, \dots, n$, and the unit ball will be denoted by Ω, with boundary Σ. With $A \triangle B$ we mean the symmetric difference between the sets A and B. The n-dimensional Lebesgue measure of a set E will be denoted by $|E|$, while $\mathcal{H}^d(E)$ will indicate its d-dimensional Hausdorff measure.

Let $K \subseteq \mathbb{R}^n$ be closed. We denote by ∂K the topological boundary of K, and, for $x \in \mathbb{R}^n$,

$$\begin{aligned} d_K(x) &= \inf\{\|y-x\| \,:\, y \in K\} && \text{(the distance of } x \text{ from } K) \\ \pi_K(x) &= \{y \in K \,:\, \|y-x\| = d_K(x)\} && \text{(the metric projections of } x \text{ into } K). \end{aligned}$$

Moreover, we set

$$\operatorname{Unp}(K) = \{x \in \mathbb{R}^n \,:\, \pi_K(x) \ \text{ is a singleton}\}.$$

Definition 2.1 *Let $E \subset \mathbb{R}^n$ be measurable and let $\lambda > 0$. We set*

$$E(\lambda) = \{x \in \mathbb{R}^n \,:\, d_E(x) \le \lambda\}, \quad V(\lambda) = |E(\lambda)|, \quad P(\lambda) = \mathcal{H}^{n-1}(\partial E(\lambda)).$$

The *excess* of a set A over a set B in $\mathbb{R}^n$ is defined as

$$e(A,B) = \sup_{a \in A} d_B(a).$$

The convex hull of a set $E \subset \mathbb{R}^n$ is denoted by $\mathrm{co}E$ and the closed convex hull by $\overline{\mathrm{co}}E$. The following concept of normal vector will be used (see [11, Ch. 1] or [23, Ch. 6] also for all other definitions and statements of nonsmooth analysis). Let $x \in K$ and $v \in \mathbb{R}^n$. We say that v is a *proximal normal* to K at x (and we write $v \in N_K^P(x)$) if there exists $\sigma = \sigma(v,x) \ge 0$ such that

$$\langle v, y-x\rangle \le \sigma \|y-x\|^2 \quad \text{ for all } y \in K;$$

equivalently $v \in N_K^P(x)$ iff there exists $\lambda > 0$ such that $\pi_K(x+\lambda v) = \{x\}$.

We recall here only some basic definitions of geometric measure theory, making reference for other tools, such as density, reduced boundary and area and coarea formulas, to [14] or to [1]. The space $BV(\mathbb{R}^n)$ of functions with bounded variation is defined as the set of those $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for which there exists a sequence $\{f_h\} \subset \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $f_h \to f$ in $L^{n'}(\mathbb{R}^n)$ (where $1/n + 1/n' = 1$) and $\sup_h \int_{\mathbb{R}^n} \|\nabla f_h(x)\|\,dx < +\infty$. In this case the distributional derivative of f, Df, is a vector-valued Radon measure, and it is possible to construct f_h so that $\int_{\mathbb{R}^n} \|\nabla f_h(x)\|\,dx \to |Df|(\mathbb{R}^n) := \|Df\|$. A measurable set $E \subset \mathbb{R}^n$ is said to have *finite perimeter* in $\mathbb{R}^n$ if $\chi_E \in BV(\mathbb{R}^n)$. In this case the perimeter of E is defined as

$$P(E) = \|D\chi_E\|.$$

We now recall three concepts whose mutual relations are going to be analyzed in this paper.

Definition 2.2 *The isoperimetric deficiency of a set $K \subset \mathbb{R}^n$ with finite n-dimensional Lebesgue measure V and finite perimeter (in $\mathbb{R}^n$) S, is defined by*

$$\Delta(K) = \frac{S}{n\omega_n}\left(\frac{V}{\omega_n}\right)^{-\frac{n-1}{n}} - 1,$$

where ω_n denotes the n-dimensional Lebesgue measure of the unit ball Ω in $\mathbb{R}^n$.

The classical isoperimetric inequality states that $\Delta \geq 0$, and the equality holds if and only if K is (equivalent to) a sphere (see [12]).
We introduce now two functions which aim to evaluate the "distance from a sphere" of a set K. The first one is concerned with a metric estimate.

Definition 2.3 *The spherical deviation of a compact set $K \subset \mathbb{R}^n$ with n-dimensional Lebesgue measure $V > 0$ and barycentre $b(K) := \frac{1}{V}\int_K x\,dx$ is defined as*

$$d(K) = \min\left\{\alpha \geq 0 \,:\, (1-\alpha)_+\Omega \subseteq \left(\frac{\omega_n}{V}\right)^{1/n}(K - b(K)) \subseteq (1+\alpha)\Omega\right\}.$$

where $(1-\alpha)_+ = \max\{0, 1-\alpha\}$.

The following concept is a measure theoretic estimate of the deviation from a sphere. It is called *Fraenkel asymmetry.*

Definition 2.4 *Let $K \subset \mathbb{R}^n$. We set*

$$\lambda^*(K) = \min\left\{\frac{|K \triangle B(x,r)|}{|K|} : r^n\omega_n = |K|, x \in \mathbb{R}^n\right\}.$$

Of course, the above concepts are in general unrelated, in the sense that, for the same set, the second can be small and the first large.

2.2 φ-convex sets

We introduce now the class of sets which is the main object of our analysis.

Definition 2.5 *Let $K \subset R^n$ be closed and let $\varphi : K \to [0,\infty]$ be continuous. We say that K is φ-convex if for all $x, y \in K, v \in N_K(x)$, the inequality*

$$\langle v, y - x\rangle \leq \varphi(x)\,\|v\|\,\|x-y\|^2$$

holds. By φ_0-convexity we mean φ-convexity with $\varphi \equiv \varphi_0$.

Some properties of the distance from a φ-convex set K and the metric projection onto K are important features of φ-convexity.

Theorem 2.1 *Let $K \subset \mathbb{R}^n$ be a φ-convex set. Then there exists an open set $U \supset K$ such that*

(1) $d_K \in \mathcal{C}^{1,1}(U \setminus K)$ and $Dd_K(y) = \frac{y - \pi_K(y)}{d_K(y)}$ for every $y \in U \setminus K$;

(2) $Unp(K) \supset U$ and $\pi_K : U \to K$ is locally Lipschitz. In particular, if K is φ_0-convex (with $\varphi_0 > 0$), then $U \supset K_{\frac{1}{2\varphi_0}}$ and $\pi_K : K_{\frac{1}{4\varphi_0}} \to K$ is Lipschitz with Lipschitz ratio 2. More precisely, if $x \in K$ and $B(x,r) \subseteq \mathrm{Unp}(K)$ then, for all $0 < s < r$, π_K is Lipschitz in $B(x,r) \cap K_s$ with Lipschitz constant $r/(r-s)$. Moreover, $\pi_K : U \setminus K \to \partial K$ is onto.

Finally,

(3) K has finite perimeter in $\mathbb{R}^n$ (provided it is compact).

Proof. The proof of (1) and (2) can be found in [5, Proposition 2.6, 2.9, Remark 2.10] or in [15, §4]. The proof of (3) is in [9, §5]. $\square$

Remark 2.1 *Conditions (1) and (2) in Theorem 2.1 are actually equivalent to φ-convexity, as it is proved, e.g., in [15, §4].*

The following theorem is one of the the main results contained in the paper by Federer, where φ-convex sets were introduced and studied under the name of *sets with positive reach* (see [15, Theorem 5.6]).

Theorem 2.2 (Federer) *Let $K \subset \mathbb{R}^n$ be φ_0-convex. There exist unique numbers $W_0, \ldots, W_n$ such that, for all $\lambda \geq 0$ such that $2\varphi_0\lambda < 1$, one has*

$$
\begin{aligned}
V(\lambda) &= |\{x \in \mathbb{R}^n : d_K(x) \leq \lambda\}| = \sum_{i=0}^{n} \binom{n}{i} W_i \lambda^i, \\
P(\lambda) &= \mathcal{H}^{n-1}(\{x \in \mathbb{R}^n : d_K(x) = \lambda\}) = \sum_{i=1}^{n} i \binom{n}{i} W_i \lambda^{i-1}.
\end{aligned}
$$

The coefficients W_i are explicitly computable. In particular, $W_0 = |K|$, $nW_1 = \mathcal{H}^{n-1}(\partial K)$, $n(n-1)W_2 = \frac{d}{d\lambda}\mathcal{H}^{n-1}(\partial K(\lambda))|_{\lambda=0}$, and $W_n = \omega_n$.

Corollary 2.1 *The inequality*

$$W_1^2 \geq W_0 W_2$$

holds.

Proof. This is a consequence of the previous Theorem and of the extension of the Brunn-Minkowski inequality to arbitrary compact subsets of $\mathbb{R}^n$ (see [4, Theorem 8.1.1]). The proof for the convex case (see, e.g., the proof of formula (2), p. 98 in [3]) extends readily to the φ-convex case. $\square$

Finally, we recall a simple estimate on the measure of the boundary of a convex set.

Proposition 2.1 *Let $E \subset \mathbb{R}^n$ be convex and compact, and let $U \subset \mathbb{R}^n$ be measurable, bounded, and such that $E \subseteq U$. Then*

$$\mathcal{H}^{n-1}(\partial E) \leq \mathcal{H}^{n-1}(\partial U).$$

Proof. The metric projection of ∂U into ∂E is Lipschitz with ratio 1 and onto. The result then follows from the definition of Hausdorff measure. $\square$

3 Geometric results

The first results concern an estimate of the excess of the convex hull of a φ-convex set K over K.

Lemma 3.1 *Let K be φ_0-convex and let $x \in \mathrm{co}K$ be such that $2\varphi_0 d_K(x) < 1$. Then*

$$\|x - \pi_K(x)\| \leq \varphi_0 \sum_{i,j=1}^{n+1} t_i t_j \|x_i - x_j\|^2 ,$$

where $t_i \geq 0$, $\sum_{i=1}^{n+1} t_i = 1$, $x_i \in K$, and $x = \sum_{i=1}^{n+1} t_i x_i$.

Proof. By φ_0-convexity we have for each $i = 1, \dots, n+1$,

$$< x - \pi_K(x), x_i - \pi_K(x) > \leq \varphi_0 \|x - \pi_K(x)\| \|x_i - \pi_K(x)\|^2 ,$$

so that

$$< x - \pi_K(x), \sum_{i=1}^{n+1} t_i x_i - \pi_K(x) > \leq \varphi_0 \|x - \pi_K(x)\| \sum_{i=1}^{n+1} t_i \|x_i - \pi_K(x)\|^2 .$$

Recalling that $x = \sum_{i=1}^{n=1} t_i x_i$, we thus obtain

$$\|x - \pi_K(x)\| \leq \varphi_0 \sum_{i=1}^{n+1} t_i \|x_i - \pi_K(x)\|^2 . \tag{3.1}$$

Putting $I = \sum_{i=1}^{n+1} t_i \|x_i - x\|^2$, from an elementary computation taking into account the condition $\sum_{i-1}^{n+1} t_i(x - x_i) = 0$, we obtain, for all $v \in \mathbb{R}^n$,

$$\sum_{i=1}^{n+1} t_i \|x_i - v\|^2 = \|x - v\|^2 + I. \tag{3.2}$$

Now we compute I. Taking $v = x_j$ in (3.2), we have

$$\sum_{i=1}^{n+1} t_i \|x_i - x_j\|^2 = \|x - x_j\|^2 + I.$$

Thus we obtain both

$$t_j \sum_{i=1}^{n+1} t_i \|x_i - x_j\|^2 = t_j \|x - x_j\|^2 + t_j I$$

and

$$\sum_{j=1}^{n+1} \sum_{i=1}^{n+1} t_j t_i \|x_i - x_j\|^2 = \sum_{j=1}^{n+1} t_j \|x - x_j\|^2 + \sum_{j=1}^{n+1} t_j I.$$

From $\sum_{j=1}^{n+1} t_j = 1$ and $I = \sum_{j=1}^{n+1} t_j \|x - x_j\|^2$, we obtain

$$I = \frac{1}{2} \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} t_j t_i \|x_i - x_j\|^2 .$$

Using this expression in (3.2) with $\pi_K(x)$ in place of v, we obtain

$$\sum_{i=1}^{n+1} t_i \|x_i - \pi_K(x)\|^2 = \|x - \pi_K(x)\|^2 + \frac{1}{2} \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} t_j t_i \|x_i - x_j\|^2 .$$

Thus, recalling (3.1),

$$\|x - \pi_K(x)\| \leq \varphi_0 \|x - \pi_K(x)\|^2 + \frac{\varphi_0}{2} \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} t_j t_i \|x_i - x_j\|^2 .$$

Since $\varphi_0 \|x - \pi_K(x)\| = \varphi_0 d_K(x) < \frac{1}{2}$, the proof is concluded. □

Proposition 3.1 *Let $K \subset \mathbb{R}^n$ be φ_0-convex, and let the diameter of K be not larger than $2R$, $R > 0$. Assume that*

$$8\frac{n}{n+1}\varphi_0^2 R^2 < 1. \tag{3.3}$$

Then

$$\mathrm{co}K \subseteq \mathrm{Unp}(K). \tag{3.4}$$

Assume now that there exist $0 < \rho < R$ and a point $b \in K$ such that

$$B(b,\rho) \subset K \subset B(b,R). \tag{3.5}$$

Then

$$e(\mathrm{co}K, K) < 8\varphi_0(R^2 - \rho^2) := \lambda(\varphi_0, R, \rho). \tag{3.6}$$

Consequently, if

$$16\varphi_0^2(R^2 - \rho^2) < 1, \tag{3.7}$$

then (3.4) holds.

Proof. By Lemma 3.1, for every $x \in \mathrm{co}\, K$ the distance $d_K(x)$ satisfies the inequality

$$d_K(x) \leq 4\varphi_0 \frac{n}{n+1} R^2. \tag{3.8}$$

Recalling Theorem 2.1, the above inequality together with (3.3) imply (3.4).
To show (3.6), we take $x \in \mathrm{co}K$ such that $d_K(x) = e(\mathrm{co}K, K)$ and write

$$x = \sum_{i=1}^{n+1} t_i y_i,$$

where $y_i \in K$, $t_i > 0$ and $\sum_{i=1}^{n+1} t_i = 1$. Now, we construct x_i, $i = 1, \ldots, n+1$ such that

i) $x_i \in K$;

ii) $\|x - x_i\| \leq 2\sqrt{R^2 - \rho^2}$;

iii) $x = \sum_{i=1}^{n+1} s_i x_i$, with $s_i > 0$ and $\sum_{i=1}^{n+1} s_i = 1$.

Two cases may occur.
First case. The segment between y_i and x does not cut the closed ball $\overline{B}(b, \rho)$ We know that this segment is contained in $\overline{B}(b, R)$, so $\|x - y_i\| \leq 2\sqrt{R^2 - \rho^2}$. In this case,we choose $x_i = y_i$.
Second case. The segment between y_i and x cuts the ball $\overline{B}(b, \rho)$.
Then there exists $y_i' \in K$ such that $\|x - y_i'\| \leq R - \rho \leq 2\sqrt{R^2 - \rho^2}$ and $x - y_i = t_i'(x - y_i')$ with $t_i' > 0$. In this case, we choose $x_i = y_i'$.
By construction, x still belongs to the convex combination of the points $\{x_i : i = 1, \ldots, n+1\}$, i.e., we can write $x = \sum_{i=1}^{n+1} s_i x_i$, where $x_i \in K$, $s_i > 0$ and $\sum_{i=1}^{n+1} s_i = 1$.
From (3.1) and (3.2), with $\pi_K(x)$ in place of v and s_i in place of t_i, we obtain

$$d_K(x) = \|x - \pi_K(x)\| \leq \varphi_0 \|x - \pi_K(x)\|^2 + \varphi_0 \sum_{i=1}^{n+1} s_i \|x - x_i\|^2 .$$

Since obviously $\|x - \pi_K(x)\| \leq R - \rho$, we obtain from the previous inequality that $\|x - \pi_K(x)\| \leq 2\varphi_0 \sum_{i=1}^{n+1} s_i \|x - x_i\|^2$. From condition ii) we get finally that

$$\|x - \pi_K(x)\| < 8\varphi_0 \sum_{i=1}^{n+1} s_i (R^2 - \rho^2),$$

which concludes the proof of (3.6). The last statement follows again from Theorem 2.1.
□

The next results are concerned with some regularity properties of a compact φ_0-convex set K with nonempty interior.

Proposition 3.2 *Let $K \subset \mathbb{R}^n$ be φ_0-convex and let $b \in K$. Assume there exist $\rho, R > 0$ such that both (3.5) and*

$$\rho - \varphi_0(R^2 - \rho^2) \geq 0 \tag{3.9}$$

hold. Then K is starshaped with respect to b.
Assume now

$$\rho - \varphi_0(R^2 - \rho^2) := \delta > 0 \tag{3.10}$$

and let $x \in K$. Then K contains the circular cone with vertex x, height $\|x - b\| - \rho$, symmetry axis the segment joining x and b, and base radius

$$r = \sqrt{\frac{\|x - b\| - \rho}{\|x - b\| + \rho} \frac{\delta}{1 + \varphi_0 \rho}}. \tag{3.11}$$

Proof. Without loss of generality we may assume that $b = 0$. Therefore we wish first to prove that, for every $x \in K$ and all $t \in [0,1)$, $tx \in K$ (actually we will prove that $tx \in \mathrm{int}K$). To this aim, assume by contradiction that there exist $\bar{x}$ and $\bar{t} \in (0,1)$ such that $\bar{t}\bar{x} \in \partial K$. Since K is φ_0-convex, there exists $v \in N_K(\bar{t}\bar{x})$ with $\|v\| = 1$ such that

$$\overline{B}(\bar{t}\bar{x} + rv, r) \cap K = \{\bar{t}\bar{x}\} \quad \text{for all} \quad r < \frac{1}{2\varphi_0},$$

hence in particular

$$\left\| \bar{t}\bar{x} + \frac{v}{2\varphi_0} - \bar{x} \right\| \geq \frac{1}{2\varphi_0}.$$

Recalling assumption (3.5) we get also

$$\left\| \bar{t}\bar{x} + \frac{v}{2\varphi_0} \right\| \geq \frac{1}{2\varphi_0} + \rho.$$

Therefore,

$$\bar{t} \left\| \frac{v}{2\varphi_0} - (1-\bar{t})\bar{x} \right\|^2 + (1-\bar{t}) \left\| \frac{v}{2\varphi_0} + \bar{t}\bar{x} \right\|^2 \geq \bar{t} \left(\frac{1}{2\varphi_0} \right)^2 + (1-\bar{t}) \left(\frac{1}{2\varphi_0} + \rho \right)^2,$$

from which it follows

$$\bar{t}\, \|\bar{x}\|^2 \geq \frac{\rho}{\varphi_0} + \rho^2. \tag{3.12}$$

Recalling that $\|\bar{x}\| \leq R$ and $\bar{t} < 1$, we obtain from (3.12) that

$$\rho < \varphi_0(R^2 - \rho^2),$$

contradicting (3.9).
To prove the second part of the statement, set $\rho_1 = \rho - \delta$ and take a point y in the segment joining the origin with a tangent from x to the ball $B(0,\rho)$ such that $\|y\| = \delta$ and set $R_1 = \|x - y\|$. Observe that $R_1^2 = \|x\|^2 - \rho^2 + \rho_1^2$, so that by (3.10)

$$\rho_1 \geq \varphi_0(\|x\|^2 - \rho^2) = \varphi_0(R_1^2 - \rho_1^2). \tag{3.13}$$

Assume by contradiction that the segment joining x and y is not contained in the interior of K, i.e., there exists $0 < t_0 < 1$ such that $x_0 := (1-t_0)y + t_0 x$ belongs to ∂K. Let $v \in N_K(x_0)$, $\|v\| = 1$. Arguing as in the proof of the starshapedness we obtain the inequality

$$\varphi_0 R_1^2 > \varphi_0 \rho_1^2 + \rho_1,$$

which contradicts (3.13). Formula (3.11) can be obtained by elementary geometric calculations. $\square$

Remark 1. If $\|x\| = R$ and $b = 0$, then the base radius r of the above cone is

$$r = \sqrt{\frac{R-\rho}{R+\rho}}\, \frac{\rho - \varphi_0(R^2 - \rho^2)}{1 + \varphi_0\rho}.$$

Corollary 3.1 *Let $K \subset \mathbb{R}^n$ be φ_0-convex and satisfy (3.5) and (3.10). Then*

(i) the reduced boundary $\partial^* K$ coincides $\mathcal{H}^{n-1}$-a.e. with the topological boundary ∂K, so that $P(K) = \mathcal{H}^{n-1}(\partial K)$;

(ii) ∂K admits a global Lipschitz parametrization $w : \Sigma \longrightarrow \partial K$, $\xi \mapsto b + w(\xi)$, where Σ is the boundary of the unit ball and $w(\xi) = \xi u(\xi)$, $u : \Sigma \to [\rho, R]$.

Proof. (i) By Proposition 3.2, at every point $x \in \partial K$, the n-dimensional density of x with respect to K is positive. Consequently (see [9, Theorem 4.3]), the reduced boundary coincides $\mathcal{H}^{n-1}$-a.e. with the topological boundary.
(ii) Thanks to the starshapedness of K, the boundary of K admits a global parametrization, which, by the uniform internal cone condition proved in Proposition 3.2, is obviously Lipschitz. □

We give now a condition ensuring that the barycentre of a φ_0-convex set K belongs to K (Lemma 3.3 below). Of course the condition is automatically satisfied when K is convex. This result is motivated by the assumptions of Theorem 6.1 below, which require that the inscribed and circumscribed balls for K be centered at the barycentre of K. It will turn out that if φ_0 is small enough, this assumption is not restrictive.

Proposition 3.3 *Let K be φ_0-convex and such that $|K| > 0$. Then K has nonempty interior. Assume now that K satisfies (3.5), with $b = 0$. Then, if furthermore*

$$\varphi_0(R - \rho) < 1 \quad \text{and} \quad \frac{n-1}{n}\frac{\omega_{n-1}}{\omega_n}\left(\left(\frac{R}{\rho}\right)^2 - 1\right)^{\frac{n+3}{2}} \frac{\varphi_0^2 \rho^2}{(1+\varphi_0\rho)^2 - \varphi_0^2 R^2} < 1, \tag{3.14}$$

the barycentre $b(K)$ of K belongs to the interior of K.

Proof. To prove the first statement, assume by contradiction that $\mathrm{int}K = \emptyset$. Then a.e. point of K is on one hand a point with density 1, on the other hand a boundary point. But, thanks to the external sphere condition, every point in ∂K has density not larger than 1/2, a contradiction.
Assume now that K satisfies (3.5) with $b = 0$ and let by contradiction $b(K) \notin \mathrm{int}K$. Choose $y \in \pi_K(b(K)) \subset \partial K$. Then, since by assumption $B(0, \rho) \subseteq K$, by φ_0-convexity we have

$$B\left(y + \frac{v}{2\varphi_0}, \frac{1}{2\varphi_0}\right) \cap B(0, \rho) = \emptyset, \tag{3.15}$$

where $v = (b(K) - y)/\|b(K) - y\|$ if $b(K) \notin K$, and v is any unit vector in $N_K(y)$ if $b(K) \in \partial K$.
Let P be intersection of the ball $B\left(y + \frac{v}{2\varphi_0}, \frac{1}{2\varphi_0}\right)$ with the segment between 0 and $y + \frac{v}{2\varphi_0}$. Take the hyperplane H passing through the point P and perpendicular to the vector $y + \frac{v}{2\varphi_0}$. Recalling (3.15), this hyperplane splits the space into the half space H_1, which contains $B(0, \rho)$, and H_2, which contains $B(y + \frac{v}{2\varphi_0}, \frac{1}{2\varphi_0})$.

Putting $\rho_1 = \|P\|$ and $v_0 = \frac{y+\frac{v}{2\varphi_0}}{\left\|y+\frac{v}{2\varphi_0}\right\|}$, one can easily check that

$$H = \rho_1 v_0 + \{x \in \mathbb{R}^n | < x, v_0 >= 0\},$$

$$H_1 = \{x \in \mathbb{R}^n | < x, v_0 >< \rho_1\},$$

$$H_2 = \{x \in \mathbb{R}^n | < x, v_0 >\geq \rho_1\}.$$

Dividing the set K into three parts, $K_1 = B(0,\rho)$, $K_2 = (H_1 \cap K) \setminus B(0,\rho)$, and $K_3 = H_2 \cap K$, we obtain

$$< b(K), v_0 >= \frac{1}{|K|}\left(\int_{K_1} < x, v_0 > dx + \int_{K_2} < x, v_0 > dx + \int_{K_3} < x, v_0 > dx\right)$$

Observing that $\int_{K_1} < x, v_0 > dx = 0$ and $\int_{K_2} < x, v_0 > dx \leq \rho_1 |K_2|$, we obtain also

$$< b(K), v_0 >\leq \frac{1}{|K|}\left(\rho_1|K_2| + \int_{K_3} < x, v_0 > dx\right). \tag{3.16}$$

Since $K \subseteq B(0,R)$, we see that $K \cap H_2 \subseteq (B(0,R) \cap H_2) \setminus B(y + \frac{v}{2\varphi_0}, \frac{1}{2\varphi_0})$. Now observe that the set $(B(0,R) \cap H_2) \setminus B(y + \frac{v}{2\varphi_0}, \frac{1}{2\varphi_0})$ is contained in the set C obtained by making a 2π-rotation around the axis through 0 and P of the triangle PQ_1Q_2 (see fig. 3). Observe that the base length of this triangle is $\sqrt{R^2 - \rho_1^2}$, while its height is $h := \frac{\varphi_0(R^2-\rho_1^2)}{\sqrt{(1+\varphi_0\rho_1)^2 - \varphi_0^2 R^2}}$. Observe also that C is contained in H_2. Thus,

$$|K \cap H_2| < |C| = \frac{(n-1)\omega_{n-1}}{n}\left(R^2 - \rho_1^2\right)^{\frac{n+1}{2}} \frac{\varphi_0}{\sqrt{(1+\varphi_0\rho_1)^2 - \varphi_0^2 R^2}}.$$

Moreover, $\|x - \pi_H(x)\| \leq h$ for all $x \in C$, whence $< x, v_0 >\leq \rho_1 + h$. Therefore

$$\int_{K_3} < x, v_0 > dx \leq (\rho_1 + h)|H_2 \cap K| \leq \rho_1 |H_2 \cap K| + h|C|.$$

Combining with (3.16) we obtain

$$< b(K), v_0 >\leq \frac{1}{|K|}\left(\rho_1|K_2| + \rho_1|K_3| + h|C|\right).$$

Thus

$$< b(K), v_0 >\leq \frac{1}{|K|}\left(\rho_1|K_2| + \rho_1|K_3| + \frac{(n-1)\omega_{n-1}}{n}\left(R^2 - \rho_1^2\right)^{\frac{n+3}{2}} \frac{\varphi_0^2}{(1+\varphi_0\rho_1)^2 - \varphi_0^2 R^2}\right)$$

By construction $\rho \leq \rho_1$. Using (3.14) we therefore obtain

$$\frac{(n-1)\omega_{n-1}}{n}\left(R^2 - \rho_1^2\right)^{\frac{n+3}{2}} \frac{\varphi_0^2}{(1+\varphi_0\rho_1)^2 - \varphi_0^2 R^2} < \omega_n \rho^{n+1} \leq \rho_1 |K_1|.$$

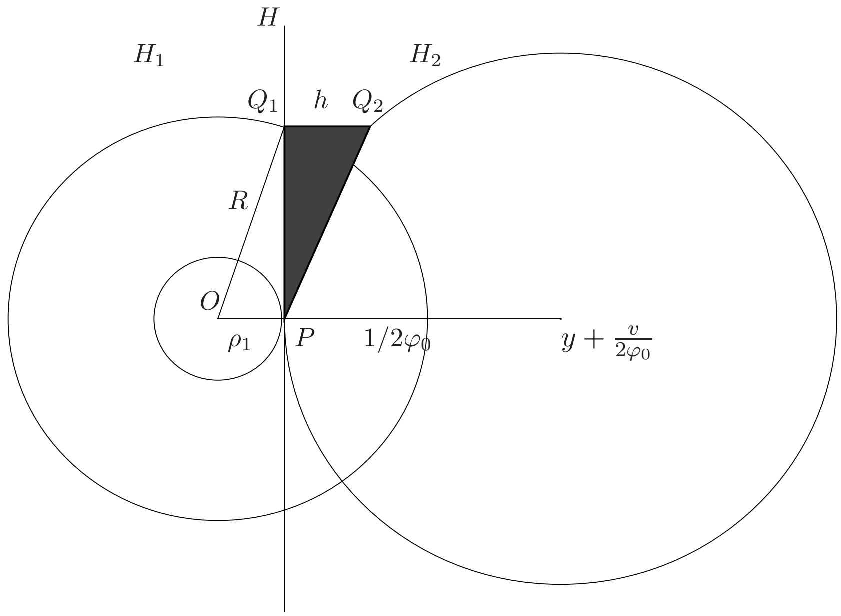

Figure 1: Construction of the set C.

Therefore,

$$< b(K), v_0 >< \frac{1}{|K|}(\rho_1|K_2| + \rho_1|K_3| + \rho_1|K_1|) = \rho_1,$$

which is absurd because $b(K)$ belongs to H_2 by construction. The proof is concluded.$\square$

We are now ready to state a result concerning the relations between the shape of K and ρ, R and φ_0.

Lemma 3.2 *Let $K \subset \mathbb{R}^n$ be a φ_0-convex set satisfying (3.5) and (3.10). Let u be the map defined in (ii) of Corollary 3.1. Then, for $\mathcal{H}^{n-1}$-a.e. $\xi \in \Sigma$,*

$$\|\nabla u(\xi)\| \le \frac{R\sqrt{(R^2-\rho^2)(1+\varphi_0(R+\rho))(1-\varphi_0(R-\rho))}}{\rho+\varphi_0(\rho^2-R^2)}. \tag{3.17}$$

Proof. Without loss of generality assume $b = 0$. Let $x \in \partial K$ and let $v \in N_K(x)$, $\|v\| = 1$. Denote by ϑ the angle between x and v. By the starshapedness, $|\vartheta| \le \pi/2$. We claim that $|\vartheta|$ is bounded away from $\pi/2$, uniformly with respect to x.
By our assumptions, $\rho \le \|x\| \le R$, and

$$\left\| x + \frac{v}{2\varphi_0} \right\| \ge \frac{1}{2\varphi_0} + \rho.$$

Therefore we have

$$\cos\vartheta \ge \frac{\rho + \varphi_0(\rho^2 - \|x\|^2)}{\|x\|} \ge \frac{\rho+\varphi_0(\rho^2-R^2)}{R} = \frac{\delta}{R},$$

hence completing the proof of the claim. It then follows that

$$\tan\vartheta \leq \frac{\sqrt{(R^2-\rho^2)(1+\varphi_0(R+\rho))(1-\varphi_0(R-\rho))}}{\rho+\varphi_0(\rho^2-R^2)}. \tag{3.18}$$

In order to estimate $\|\nabla w(\xi)\|$ for a.e. $\xi \in \Sigma$, let ξ be such that $\nabla w(\xi)$ exists and let $s \mapsto y(s)$ be a $\mathcal{C}^1$-curve in Σ such that $y(0)=\xi$, $\|\dot{y}(0)\|=1$. Set $w(\xi)=\xi u(\xi)$, with $u:\Sigma\to[\rho,R]$ a Lipschitz function. Since $\|w(y(s))\|^2 = u(y(s))^2$, we obtain that

$$\langle w(\xi), \frac{d}{ds}w(y(s))_{|s=0}\rangle = u(\xi)\frac{d}{ds}u(y(s))_{|s=0}. \tag{3.19}$$

Moreover,

$$\frac{d}{ds}w(y(s))_{|s=0} = \dot{y}(0)u(\xi) + y(0)\frac{d}{ds}u(y(s))_{|s=0},$$

so that, recalling that $\dot{y}(0)$ is orthogonal to $y(0)$, we obtain

$$\left\|\frac{d}{ds}w(y(s))_{|s=0}\right\| = \sqrt{u^2(\xi)+\frac{d}{ds}u(y(s))^2_{|s=0}}.$$

Set ϑ_1 to be the angle between $w(\xi)$ and $\frac{d}{ds}w(y(s))_{|s=0}$. Recalling (3.19),

$$\frac{|\frac{d}{ds}u(y(s))_{|s=0}|}{u(\xi)} = \cot\vartheta_1 \leq \tan\vartheta.$$

We therefore obtain that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \partial K$

$$\frac{\|\nabla u(\xi)\|}{u(\xi)} \leq \tan\vartheta,$$

which, by (3.18), proves (3.17). □

Remark. Under the normalization condition used in [17], i.e., with $\rho = 1-d(K)$ and $R = 1+d(K)$, formula (3.17) becomes

$$\|\nabla u\|_\infty \leq 2\sqrt{d(K)}\frac{(1+d(K))\sqrt{(1+2\varphi_0)(1-2\varphi_0 d(K))}}{1-(1+4\varphi_0)d(K)}. \tag{3.20}$$

The above inequality reduces to the main estimate in Lemma 2.2 of [17] when $\varphi_0 = 0$, i.e., K is convex.

Observe moreover that the right-hand side $g = g(\varphi_0, d(K))$ of (3.20) is strictly increasing as a function of $d(K) \in [0, 1/(1+4\varphi_0))$.

Finally, observe that g is increasing also with respect to φ_0. Therefore we have

$$g(\varphi_0,d) \leq g(1,d) := g_0(d) = 2\sqrt{d}\frac{(1+d)\sqrt{3(1-2d)}}{1-5d}.$$

For future use, let a be such that

$$d \leq \frac{a}{2} \quad \text{implies} \quad g_0(d) \leq 1/2, \tag{3.21}$$

and observe that

$$a < \frac{1}{24}. \tag{3.22}$$

4 Estimates for the convex hull

The main result of this section is Proposition 4.1, where the isoperimetric deficiency and the spherical deviation of the convex hull of a φ_0-convex set K, are estimated using the corresponding quantities for K. This will permit in the next section to prove one of our main results using an adaptation of the argument of Theorem 2.3 in [17]. Some estimates can be improved (in particular, the upper bound on Δ can be avoided).
We begin with a technical lemma.

Lemma 4.1 *Let $K \subset \mathbb{R}^n$ be compact, φ_0-convex, and such that there exist $b \in K$ and $\rho > 0$ with*

$$B(b, \rho) \subseteq K.$$

Then $\Delta(K(\lambda))$ is nonincreasing with respect to λ belonging to the interval $[0, t]$, provided $2\varphi_0 t < 1$. Assume furthermore that

$$\Delta(K) \leq 1$$

and set, for $\lambda \geq 0$, $T(\lambda) := \frac{V(\lambda)}{V(0)}$, where $V(0) = |K|$. Then, for all $0 \leq \lambda \leq \min\{\frac{\rho}{2n}, \frac{1}{2\varphi_0}\}$ the estimate

$$T(\lambda) \leq \frac{\rho}{\rho - 2n\lambda} \tag{4.1}$$

holds true.

Proof. We begin with some preliminary estimates on $T(\lambda)$.
According to Theorem 2.2, the function $V(\lambda)$ is continuously differentiable with respect to λ. So we can apply the mean value theorem and find $\lambda_1 \in (0, \lambda)$ such that

$$T(\lambda) = 1 + \frac{V(\lambda) - V(0)}{V(0)} = 1 + \lambda \frac{P(\lambda_1)}{V(0)}.$$

Observe now that it is possible to write the above expression as

$$T(\lambda) = 1 + n\lambda \frac{\omega_n^{1/n}}{V(0)^{1/n}} \frac{P(\lambda_1)}{n\omega_n^{1/n} V(\lambda)^{\frac{n-1}{n}}} T(\lambda)^{\frac{n-1}{n}}.$$

From $V(\lambda) \geq V(\lambda_1) \geq V(0)$ and $\left(\frac{\omega_n}{V(0)}\right)^{1/n} \leq \frac{1}{\rho}$, one can easily deduce from the previous expression that

$$T(\lambda) \leq 1 + n\lambda \frac{1}{\rho}(\Delta(K(\lambda_1)) + 1) T(\lambda)^{\frac{n-1}{n}}. \tag{4.2}$$

We now observe that the function $\Delta(K(\cdot))$ is nonincreasing in the interval $[0, t]$, provided $2\varphi_0 t < 1$. In fact, the logarithmic derivative of $(1 + \Delta(\lambda))^n$ at $\lambda = 0$ is (see [17, proof of Lemma 2.5] and Corollary 2.1)

$$n(n-1)\frac{W_0 W_2 - W_1^2}{W_0 W_1} \leq 0.$$

Observe that $K(\lambda+\mu) = K(\lambda)(\mu)$ for all $\mu \geq 0$ and $K(\lambda)$ is φ_0'-convex for a suitable φ_0', provided $2\varphi_0\lambda < 1$ (see [15, Corollary 4.9]). Thus the derivative of the isoperimetric deficiency is nonpositive for all $\lambda \geq 0$ such that $2\varphi_0\lambda < 1$. Therefore, for such λ, $\Delta(K(\lambda_1))$ is bounded from above by $\Delta(K)$. If $\Delta(K) \leq 1$, (4.2) becomes (4.1). $\square$

The following estimate is crucial in order to obtain a sharp quantitative isoperimetric inequality valid for a small isoperimetric deficiency $\Delta(K)$. Therefore, although a more general statement can be obtained, for simplicity we prove it only for $\Delta(K) \leq 1$.

Proposition 4.1 *Let K be a φ_0-convex set satisfying (3.5) with $b = b(K)$, the barycentre of K, and assume that*

$$\rho - 16n\varphi_0(R^2-\rho^2) > 0. \tag{4.3}$$

Then

(i) $\Delta(\mathrm{co}K) \leq \Delta(K) + \varphi_0 \dfrac{32n(R^2-\rho^2)}{\rho - 16n\varphi_0(R^2-\rho^2)}$, *provided* $\Delta(K) \leq 1$;

(ii) $d(K) \leq d(\mathrm{co}K) + \varphi_0 \dfrac{16n(R^2-\rho^2)}{\rho - 16n\varphi_0(R^2-\rho^2)} \left(\dfrac{2R}{\rho} + \dfrac{1+d(\mathrm{co}K)}{n} \right)$, *provided* $d(K) \leq 1$.

Proof of part (i). Set $\mathcal{H}^{n-1}(\partial\, \mathrm{co}K) := P(\mathrm{co}K)$. By definition we have both

$$\Delta(K) = \frac{1}{n\omega_n^{1/n}} \frac{P(K)}{|K|^{\frac{n-1}{n}}} - 1 \quad \text{and} \quad \Delta(\mathrm{co}K) = \frac{1}{n\omega_n^{1/n}} \frac{P(\mathrm{co}K)}{|\mathrm{co}K|^{\frac{n-1}{n}}} - 1.$$

From Propositions 3.1 and 2.1 we obtain that $P(\mathrm{co}K) \leq P(\lambda(\varphi_0, R, \rho))$, with $\lambda(\varphi_0, R, \rho) = 8\varphi_0(R^2-\rho^2)$.
Since $|\mathrm{co}K| \geq |K|$, we have

$$\Delta(\mathrm{co}K) \leq \frac{1}{n\omega_n^{1/n}} \frac{P(\lambda(\varphi_0, R, \rho)) - P(K)}{|K|^{\frac{n-1}{n}}} + \Delta(K). \tag{4.4}$$

Now, observing that $K(\lambda)$ is φ_1-convex, with $\frac{1}{2\varphi_1} = \frac{1}{2\varphi_0} - \lambda$, for λ small enough, we apply Theorem 2.2 and develop $P(\lambda)$ around $\lambda(\varphi_0, R, \rho)$. For a suitable $0 < \lambda_1 < \lambda(\varphi_0, R, \rho)$, we obtain therefore from (4.4) the inequality

$$\Delta(\mathrm{co}K) \leq \frac{\lambda(\varphi_0, R, \rho)}{n\omega_n^{1/n}} \frac{P'(\lambda_1)}{|K|^{\frac{n-1}{n}}} + \Delta(K),$$

where P' denotes the derivative of $P(\lambda)$. Using the Minkowski inequality (i.e., Corollary 2.1) we have $P'(\lambda_1) \leq \frac{n-1}{n} \frac{P^2(\lambda_1)}{V(\lambda_1)}$, from which it follows

$$\Delta(\mathrm{co}K) \leq \frac{\lambda(\varphi_0, R, \rho)(n-1)}{n^2\omega_n^{1/n}} \frac{P^2(\lambda_1)}{V(\lambda_1)} \frac{1}{|K|^{\frac{n-1}{n}}} + \Delta(K).$$

Thus

$$\Delta(\mathrm{co}K) \leq \lambda(\varphi_0, R, \rho)(n-1)\omega_n^{1/n} \left(\frac{1}{n\omega_n^{1/n}} \frac{P(\lambda_1)}{V(\lambda_1)^{\frac{n-1}{n}}} \right)^2 \frac{V(\lambda_1)^{\frac{n-2}{n}}}{|K|^{\frac{n-1}{n}}} + \Delta(K),$$

which is equivalent to

$$\Delta(\mathrm{co}K) \le \lambda(\varphi_0, R, \rho)(n-1)\omega_n^{1/n} \left(\Delta(K(\lambda_1)) + 1\right)^2 \frac{V(\lambda_1)^{\frac{n-2}{n}}}{|K|^{\frac{n-1}{n}}} + \Delta(K).$$

From Lemma 4.1, $\Delta(K) \ge \Delta(K(\lambda_1))$. Therefore we obtain

$$\Delta(\mathrm{co}K) \le \lambda(\varphi_0, R, \rho)(n-1)\omega_n^{1/n}(\Delta(K) + 1)^2 \frac{V(\lambda_1)^{\frac{n-2}{n}}}{|K|^{\frac{n-1}{n}}} + \Delta(K).$$

Assuming $\Delta(K) \le 1$ it then follows that

$$\Delta(\mathrm{co}K) \le 4\lambda(\varphi_0, R, \rho)(n-1)\frac{\omega_n^{1/n}}{|K|^{1/n}} \left(\frac{V(\lambda_1)}{|K|}\right)^{\frac{n-2}{n}} + \Delta(K).$$

Since $B(b, \rho) \subseteq K$, we obtain $\frac{\omega_n^{1/n}}{|K|^{1/n}} \le \frac{1}{\rho}$, so that

$$\Delta(\mathrm{co}K) \le \frac{4\lambda(\varphi_0, R, \rho)(n-1)}{\rho} \left(\frac{V(K(\lambda_1))}{|K|}\right)^{\frac{n-2}{n}} + \Delta(K),$$

which implies

$$\Delta(\mathrm{co}K) \le \frac{4\lambda(\varphi_0, R, \rho)(n-1)}{\rho} \frac{V(\lambda(\varphi_0, R, \rho))}{|K|} + \Delta(K). \tag{4.5}$$

Set now $T := \frac{V(\lambda(\varphi_0,R,\rho))}{|K|}$. Recalling Lemma 4.1, we have

$$T \le \frac{\rho}{\rho - 2n\lambda(\varphi_0, R, \rho)}. \tag{4.6}$$

Combining with (4.5) we obtain

$$\Delta(\mathrm{co}K) \le \Delta(K) + 4(n-1)\lambda(\varphi_0, R, \rho)\frac{1}{\rho - 2n\lambda(\varphi_0, R, \rho)}. \tag{4.7}$$

Recalling that $\lambda(\varphi_0, R, \rho) = 8\varphi_0(R^2 - \rho^2)$, (4.7) implies

$$\Delta(\mathrm{co}K) \le \Delta(K) + \varphi_0 \frac{32n(R^2 - \rho^2)}{\rho - 16n\varphi_0(R^2 - \rho^2)},$$

which concludes the proof of (i).
Proof of part (ii). Denoting by $b(K)$ (resp. $b(\mathrm{co}K)$) the barycentre of K (resp. of

co$K$), we compute

$$\begin{aligned}
\|b(\mathrm{co}K)-b(K)\| &= \left\|\frac{1}{|\mathrm{co}K|}\int_{\mathrm{co}K} x\,dx - \frac{1}{|K|}\int_K x\,dx\right\| \\
&= \left\|\frac{1}{|\mathrm{co}K|}\int_{\mathrm{co}K} (x-b(K))\,dx - \frac{1}{|K|}\int_K (x-b(K))\,dx\right\| \\
&\leq \frac{1}{|\mathrm{co}K|}\left\|\int_{\mathrm{co}K\setminus K}(x-b(K))\,dx\right\| + \frac{|\mathrm{co}K|-|K|}{|\mathrm{co}K||K|}\left\|\int_K (x-b(K))\,dx\right\| \\
&\leq 2R\frac{|\mathrm{co}K|-|K|}{|\mathrm{co}K|} \leq 2R\frac{V(\lambda(\varphi_0,R,\rho))-|K|}{|K|} \\
&= 2R(T-1). \qquad (4.8)
\end{aligned}$$

Consequently, recalling (4.6) we obtain

$$\|b(\mathrm{co}K)-b(K)\| \leq \frac{32nR\varphi_0(R^2-\rho^2)}{\rho-16n\varphi_0(R^2-\rho^2)}. \qquad (4.9)$$

By definition of spherical deviation, we have

$$\left(\frac{\omega_n}{|\mathrm{co}K|}\right)^{-\frac{1}{n}}(1-d(\mathrm{co}K))_+\Omega \subseteq \mathrm{co}K - b(\mathrm{co}K) \subseteq \left(\frac{\omega_n}{|\mathrm{co}K|}\right)^{-\frac{1}{n}}(1+d(\mathrm{co}K))\Omega,$$

from which it follows

$$\begin{aligned}
&\left[\left(\frac{\omega_n}{|\mathrm{co}K|}\right)^{-\frac{1}{n}}(1-d(\mathrm{co}K))_+ - \|b(\mathrm{co}K)-b(K)\|\right]_+\Omega \\
&\qquad\subseteq \quad \mathrm{co}K - b(K) \subseteq \left[\left(\frac{\omega_n}{|\mathrm{co}K|}\right)^{-\frac{1}{n}}(1+d(\mathrm{co}K)) + \|b(\mathrm{co}K)-b(K)\|\right]\Omega.
\end{aligned}$$

Since $K \subseteq \mathrm{co}K \subseteq K + e(\mathrm{co}K, K)\Omega$ the above inclusions imply

$$(1-\alpha_1)_+\Omega \subseteq (K-b(K))\left(\frac{\omega_n}{|K|}\right)^{1/n} \subseteq (1+\alpha_2)\Omega,$$

where, recalling (4.8), (3.6)

$$\begin{aligned}
\alpha_1 &\leq d(\mathrm{co}K) + \frac{2R(T-1)}{\rho} + 8\varphi_0\frac{R^2-\rho^2}{\rho}, \qquad \text{if } d(\mathrm{co}K)\leq 1 \\
\alpha_1 &= 1 \qquad \text{if } d(\mathrm{co}K) > 1, \\
\alpha_2 &\leq d(\mathrm{co}K) + \frac{2R(T-1)}{\rho} + (1+d(\mathrm{co}K))\left(\left(\frac{|\mathrm{co}K|}{|K|}\right)^{1/n} - 1\right) \\
&\leq d(\mathrm{co}K) + \frac{2R(T-1)}{\rho} + \frac{16\varphi_0(R^2-\rho^2)(1+d(\mathrm{co}K))}{\rho-16n\varphi_0(R^2-\rho^2)}.
\end{aligned}$$

Consequently, recalling (4.9),

$$d(K) \leq d(\text{co}K) + \varphi_0 \frac{16n(R^2 - \rho^2)}{\rho - 16n\varphi_0(R^2 - \rho^2)} \left(\frac{2R}{\rho} + \frac{1 + d(\text{co}K)}{n} \right).$$

The proof is concluded. $\square$

We conclude the section with an estimate of the isoperimetric deficiency involving only R, ρ, n.

Proposition 4.2 *Let $K \subset \mathbb{R}^n$ be φ_0-convex and satisfy (3.5) together with (4.3), and (3.7). Then*

$$\Delta(K) \leq \frac{1}{\left(1 - 16\varphi_0^2(R^2 - \rho^2)\right)^{n-1}} \left(\frac{R}{\rho}\right)^{n-1} - 1.$$

Proof. From (3.6) we have $co(K) \subseteq K(\lambda(\varphi_0, R, \rho))$, where $\lambda = 8\varphi_0(R^2 - \rho^2)$. Recalling Proposition 2.1, $\pi_K : \text{co}K \to \partial K$ is single-valued and Lipschitz with ratio $L = \frac{1}{1 - 2\varphi_0\lambda(\varphi_0, R, \rho)}$. Thus

$$\mathcal{H}^{n-1}(\pi_K(\partial \text{co}K)) \leq L^{n-1}\mathcal{H}^{n-1}(\partial \text{co}K).$$

Using the above considerations, we obtain

$$\mathcal{H}^{n-1}(\partial K) \leq L^{n-1}\mathcal{H}^{n-1}(\partial \text{co}K).$$

Now, we can estimate the isoperimetric deficiency of K:

$$\Delta(K) = \frac{\mathcal{H}^{n-1}(\partial K)}{n\omega_n^{1/n}|K|^{\frac{n-1}{n}}} - 1 \leq L^{n-1}\frac{\mathcal{H}^{n-1}(\partial \text{co}K)}{n\omega_n^{1/n}|K|^{\frac{n-1}{n}}} - 1.$$

We know that K contains a ball with radius ρ and co$K$ is contained in a ball with radius R. We therefore obtain, recalling Proposition 2.1,

$$\Delta(K) \leq L^{n-1}\left(\frac{R}{\rho}\right)^{n-1} - 1.$$

The proof is concluded. $\square$

5 Comparison between spherical deviation and Fraenkel asymmetry

It is easy to construct examples, even of compact φ-convex sets with nonempty interior, where Fraenkel asymmetry is small, while spherical deviation is very large (add a suitable thorn to a ball). However, assuming some compatibility conditions involving φ_0, ρ, R it is possible to give an upper bound to the spherical deviation using Fraenkel asymmetry.

Proposition 5.1 *Let K be a φ_0-convex set such that $B(0,\rho) \subset K \subset B(0,R)$, where 0 is the barycenter of K, R is the smallest radius of a ball centered at 0 containing K, and ρ the largest radius of a concentric ball contained in K. Let $\sigma > 0$ and assume $\Delta(K) \geq \sigma$. Let $c_0 > 0$ be given and assume*

$$\begin{aligned} 16\varphi_0^2(R^2-\rho^2) &\leq 1 - \frac{1+c_0}{(1+\sigma)^{\frac{1}{n-1}}}, && (5.1)\\ \varphi_0|K|^{1/n} &\leq \omega_n^{1/n}, && (5.2)\\ \rho - \varphi_0(R^2-\rho^2) &\geq \delta R. && (5.3) \end{aligned}$$

where $\delta > 0$. Then there exists a positive constant C, depending only on n, c_0, δ, such that

$$d(K)^{\frac{n+1}{2}} \leq C\lambda^*(K).$$

The proof is based on the following technical lemma.

Lemma 5.1 *Let K be a φ_0-convex set satisfying (3.5), where R is the smallest radius of a ball containing K and ρ is the largest radius of a ball contained in K (both centered at b). Assume also that $\Delta(K) \geq \sigma > 0$, and let (5.1), (5.2), (5.3) hold. Then there exist positive real numbers C_1, C_2, C_3, depending only on n, δ, c_0, such that the following properties are satisfied:*

(i) $r \geq (1+C_1)\rho$;

(ii) $R \geq (1+C_2)r$;

(iii) $|K \setminus B(b,r)| \geq C_3 r^n$.

Proof of Lemma 5.1. Without loss of generality, suppose that $b = 0$ and let $|K| = r^n\,\omega_n$. From (5.1) and Proposition 4.2 we obtain that

$$\frac{R}{\rho} \geq (1+\sigma)^{\frac{1}{n-1}}(1 - 16\varphi_0^2(R^2-\rho^2)) \geq 1 + c_0. \tag{5.4}$$

Now we are going to prove (i) and (ii). First, we wish to compare r with ρ. To this aim, we recall that by Proposition 3.2, K contains the cone D of height $R-\rho$ and base radius

$$r_0 = \frac{\sqrt{R-\rho}(\rho - \varphi_0(R^2-\rho^2))}{\sqrt{R+\rho}(1+\varphi_0\rho)} = (R-\rho)\frac{\rho - \varphi_0(R^2-\rho^2)}{\sqrt{R^2-\rho^2}(1+\varphi_0\rho)}.$$

Recalling (5.2), we obtain that $\varphi_0\rho < 1$. Therefore, by (5.3) we have

$$r_0 \geq \frac{\delta}{2}(R-\rho), \tag{5.5}$$

from which we obtain

$$|K| = r^n\omega_n \geq |B(0,\rho)| + |D| = \rho^n\omega_n + \frac{\omega_{n-1}\delta^{n-1}}{n2^{n-1}}(R-\rho)^n.$$

Now (i) follows from the above inequality and (5.4).
Next, we wish to compare r with R. Since ρ is the largest radius of a ball contained in K, there exists a point $y \in \partial B(0,\rho) \cap \partial K$. From the definition of φ_0-convexity, it is easy to see that $w = \frac{y}{||y||}$ is the unique unit outer normal vector of K at y, so that $B(y + \frac{w}{2\varphi_0}, \frac{1}{2\varphi_0}) \cap K = \emptyset$. Therefore, recalling (5.3),

$$\begin{aligned} |K| &\leq |B(0,R)| - \left| B(0,R) \cap B\left(y + \frac{w}{2\varphi_0}, \frac{1}{2\varphi_0}\right) \right| \\ &\leq |B(0,R)| - \left| B\left(y + \frac{R-\rho}{2} w, \frac{R-\rho}{2}\right) \right|, \end{aligned}$$

from which we obtain

$$\omega_n r^n \leq \omega_n R^n - \frac{\omega_n}{2^n}(R-\rho)^n. \tag{5.6}$$

Now, recalling (i), $R - \rho \geq \frac{C_1}{1+C_1} r$, so that (5.6) yields (ii).
Finally, we prove (iii). From Proposition 3.2 we obtain that K contains a cone E with $(n-1)$-spherical base centered at some point b_1 such that $\|b_1\| = \rho$, radius r_0 and height $R-\rho$. Thus, the set $K \setminus B(0,r)$ contains the cone E_1 with height $R-r$ and radius $\frac{R-r}{R-\rho} r_0$. Therefore, recalling (5.5),

$$|K \setminus B(0,r)| \geq (R-r) \frac{\omega_{n-1}\delta^{n-1}}{n 2^{n-1}} \left(\frac{R-r}{R-\rho}\right)^{n-1} (R-\rho)^{n-1}.$$

Using (ii) we obtain (iii). □

Proof of Proposition 5.1. We divide the proof into two cases:
First case, $R \geq 4r$.
From Proposition 3.2 we obtain that K contains the cone E with height $R-\rho$ and base radius $r_0 = \frac{\sqrt{R-\rho}(\rho - \varphi_0(R^2-\rho^2))}{\sqrt{R+\rho}(1+\varphi_0\rho)}$. By considering a strip of width $2r$ and using (5.5) one can check that

$$|K \triangle B(x,r)| \geq |K \setminus B(x,r)| \geq |E_1|,$$

where E_1 is a cone with height $R - \rho - 2r \geq \frac{1}{3}(R-r)$ and base radius

$$r_3 = \frac{R-\rho-2r}{R-\rho} r_0 \geq \frac{\delta}{6}(R-r).$$

Therefore,

$$|K \triangle B(x,r)| \geq \frac{\omega_{n-1}\delta^{n-1}}{3n 6^{n-1}} (R-r)^n,$$

from which it follows

$$\frac{|K \triangle B(x,r)|}{|K|} \geq \frac{\omega_{n-1}\delta^{n-1}}{3n\omega_n 6^{n-1}} \left(\frac{R-r}{r}\right)^n.$$

From $R \geq 4r$, we obtain that $d(K) = \max\left\{\frac{r-\rho}{r}, \frac{R-r}{r}\right\} = \frac{R-r}{r}$. Therefore, we get

$$\frac{|K \triangle B(x,r)|}{|K|} \geq \frac{\omega_{n-1}\delta^{n-1}}{3n\omega_n 6^{n-1}} d(K)^n \geq \frac{\omega_{n-1}\delta^{n-1}}{3n\omega_n 6^{n-1}} d(K)^{\frac{n+1}{2}}$$

for all $x \in R^n$.

Second case, $R \le 4r$**.**

Observing that $d(K) = \max\left\{\frac{r-\rho}{r}, \frac{R-r}{r}\right\} \le 3$, we only need to prove that there exists a constant C_2 satisfying $|K \triangle B(x,r)| \ge C_2 r^n$ for all $x \in \mathbb{R}^n$.

Let x be in $\mathbb{R}^n$: three cases may occur.

First case. The point x belongs to ∂K. From the definition of φ_0-convexity, there exists a unit vector $v \in N_K(x)$ such that $B(x + \frac{v}{2\varphi_0}, \frac{1}{2\varphi_0}) \cap K = \emptyset$. Therefore, recalling (5.2),

$$|B(x,r)\triangle K| \ge |B(x,r) \setminus K| \ge \left| B(x,r) \cap B\left(x + \frac{v}{2\varphi_0}, \frac{1}{2\varphi_0}\right)\right| \ge \left|B\left(x + \frac{r}{2}v, \frac{r}{2}\right)\right|.$$

Consequently

$$|B(x,r)\triangle K| \ge \frac{\omega_n}{2^n} r^n.$$

Second case. The point x belongs to the interior of $\mathbb{R}^n \setminus K$. Let ρ_1 be the largest radius ξ such that $B(x,\xi) \subset \mathbb{R}^n \setminus K$. We have two cases:

If $\rho_1 \ge r$ then $|B(x,r)\triangle K| \ge |B(x,r)| = \omega_n r^n$.

If $\rho_1 < r$ then there exists a point $x_1 \in \partial B(x,\rho_1) \cap \partial K$. By the same argument as above, from the definition of φ_0-convexity, there exists a unit vector $v_1 \in N_K(x_1)$ such that $B(x_1 + \frac{v_1}{2\varphi_0}, \frac{1}{2\varphi_0}) \cap K = \emptyset$. One can easily check that $B(x,r) \setminus K \supseteq B(x,\rho_1) \cup B(y, \frac{r-\rho_1}{2})$, where y belongs to the segment between x and $x_1 + \frac{v_1}{2\varphi_0}$. Therefore,

$$|B(x,r) \setminus K| \ge \frac{1}{2}\left(|B(x,\rho_1)| + \left|B\left(x, \frac{r-\rho_1}{2}\right)\right|\right) = \frac{\omega_n}{2}\left(\rho_1^n + \left(\frac{r-\rho_1}{2}\right)^n\right),$$

from which we obtain

$$|B(x,r)\triangle K| \ge \frac{\omega_n}{2^{2n}} r^n.$$

Third case. The point x belongs to the interior of K. Let ρ_2 be the largest radius ξ such that $B(x,\xi) \subseteq K$ and let R_1 be the smallest radius such that $B(x,\xi) \supseteq K$. We have two cases:

If $\rho_2 \le \rho$ then there exists a point $x_2 \in \partial B(x,\rho_2) \cap \partial K$. From the definition of φ_0-convexity, $v_2 = \frac{x_2 - x}{||x_2 - x||} \in N_K(x_2)$ and $B(x_2 + \frac{v_2}{2\varphi_0}, \frac{1}{2\varphi_0}) \cap K = \emptyset$. Therefore,

$$|B(x,r) \setminus K| \ge \left|B(x,r) \cap B\left(x_2 + \frac{v_2}{2\varphi_0}, \frac{1}{2\varphi_0}\right)\right| \ge \left|B\left(x_2 + \frac{r-\rho_2}{2} v_2, \frac{r-\rho_2}{2}\right)\right|,$$

whence

$$|B(x,r)\triangle K| \ge \frac{\omega_n}{2^n}(r-\rho_2)^n \ge \frac{\omega_n}{2^n}(r-\rho)^n.$$

Using (i) in Lemma 5.1, we therefore obtain

$$|B(x,r)\triangle K| \ge \frac{\omega_n}{2^n}\left(\frac{C_1}{1+C_1}\right)^n r^n.$$

Otherwise, $\rho_2 \geq \rho$. We have two cases:
First case, $R_1 \leq R$: since $\rho \leq \rho_2$ and $R_1 \leq R$, the assumptions of Lemma 5.1 are satisfied. Therefore, using (iii) in Lemma 5.1 with x in place of b, we obtain

$$|B(x,r)\triangle K| \geq |K \setminus B(x,r)| \geq C_3 r^n.$$

Second case, $R_1 \geq R$: then there exists a point $z \in K$ such that $\|z - x\| = R$. According with Proposition 3.2, K contains a cone F with $(n-1)$-spherical base centered at some point x_1 satisfying $\|x - x_1\| = \rho$, radius $r_0 \geq \frac{\delta}{2}(R-\rho)$ and height $R - \rho$. Therefore,

$$|K \setminus B(x,r)| \geq \frac{\omega_{n-1}\delta^{n-1}}{n2^{n-1}}(R-r)\left(\frac{R-r}{R-\rho}\right)^{n-1}(R-\rho)^{n-1}.$$

Using (i) and (ii) in Lemma 5.1 we see that

$$|B(x,r)\triangle K| \geq C_2 r^n.$$

The proof is concluded. □

The next result provides, for a particular case, sharp inequalities between spherical deviation and Fraenkel asymmetry. It will be used in Remark 6.2 below.

Proposition 5.2 *Let $0 < \rho < R$ be such that $R \leq 2\rho$. Let $\varphi_0 > 0$ be such that $a := \frac{1}{2\varphi_0} \geq \rho$ and consider the φ_0-convex set*

$$K = \overline{B}(0,R) \setminus [B((a+\rho)e_1, a) \cup B(-(a+\rho)e_1, a))].$$

Then there exist positive constants $C_1 = C_1(n)$, $C_2 = C_2(n)$ such that

$$C_1(n)d(K)^{\frac{n+1}{2}} \leq \lambda^*(K) \leq C_2(n)d(K)^{\frac{n+1}{2}}.$$

Proof. We make some preliminary remarks and then estimate the measure of the symmetric difference between K and $B(x,r)$.
The barycenter of K, $b(K)$, is 0. Let r be such that $|K| = r^n\omega_n$. Then, by definition, the spherical deviation of K satisfies $d(K) = \max\left\{\frac{r-\rho}{r}, \frac{R-r}{r}\right\}$. Moreover, we have that $|K| \leq |B(0,R)| - 2|D_1|$, where D_1 is the union of two cones with the same $(n-1)$-spherical basis of radius $r_1 = \sqrt{\frac{R^2-\rho^2}{2(a+\rho)}\left(2a - \frac{R^2-\rho^2}{2(a+\rho)}\right)}$, one opposite to the other, with height, respectively, $h_1 = (R^2-\rho^2)/(2(a+\rho))$, $h_2 = R - (\rho + h_1)$. Our choice of a and the condition $R \leq 2\rho$ imply that

$$\frac{1}{2}\sqrt{R^2-\rho^2} \leq r_1 \leq \sqrt{R^2-\rho^2}. \tag{5.7}$$

As a consequence,

$$|K| \leq |B(0,R)| - 2\frac{\omega_{n-1}}{n}(R-\rho)\left(\frac{1}{2}\sqrt{R^2-\rho^2}\right)^{n-1}$$

or, equivalently,

$$|K| \leq |B(0,R)| - \frac{\omega_{n-1}}{n2^{n-2}}(R-\rho)^{\frac{n+1}{2}}(R+\rho)^{\frac{n-1}{2}}.$$

Notice now that K is symmetric w.r.t. the origin, so that $|K \triangle B(x,r)| = |K \triangle B(-x,r)|$. Moreover, since $B(x,r) \cap B(-x,r) \subseteq B(0,r)$, we have that $|K \triangle B(x,r)| \geq \frac{1}{2}|K \setminus B(0,r)|$ for all x. From the properties of the set K we obtain finally that

$$|K \setminus B(0,r)| \geq \frac{1}{2}(|B(0,R)| - |K|).$$

The above remarks permit to estimate the measure of the symmetric difference between K and $B(x,r)$ for all $x \in R^n$:

$$|K \triangle B(x,r)| \geq \frac{1}{4}(|B(0,R)| - |K|) \geq \frac{\omega_{n-1}}{n2^n}(R-\rho)^{\frac{n+1}{2}}(R+\rho)^{\frac{n-1}{2}}.$$

Using the definition of Fraenkel asymmetry, we have that

$$\lambda^*(K) \geq \frac{\omega_{n-1}}{n2^n\omega_n}(\frac{R-\rho}{r})^{\frac{n+1}{2}}(\frac{R+\rho}{r})^{\frac{n-1}{2}}.$$

Since $d(K) \leq \frac{R-\rho}{r}$, we obtain one side of the desired inequality:

$$\frac{\omega_{n-1}}{n2^n\omega_n}d(K)^{\frac{n+1}{2}} \leq \lambda^*(K).$$

The other side is similar. Indeed, we have $|K| \geq |B(0,R)| - 2|D|$ where D is a cylinder with height $R-\rho$ and radius $r_1 = \sqrt{\frac{R^2-\rho^2}{2(a+\rho)}\left(2a - \frac{R^2-\rho^2}{2(a+\rho)}\right)}$. Therefore $|K \triangle B(0,r)| \leq |B(0,R)| - |K| + 2|D| \leq 4|D|$, so that recalling (5.7) we obtain

$$|K \triangle B(0,r)| \leq 4\omega_{n-1}(R-\rho)^{\frac{n+1}{2}}(R+\rho)^{\frac{n-1}{2}}.$$

By definition of Fraenkel asymmetry, we have

$$\lambda^*(K) \leq \frac{4\omega_{n-1}}{\omega_n}\left(\frac{R-\rho}{r}\right)^{\frac{n+1}{2}}\left(\frac{R+\rho}{r}\right)^{\frac{n-1}{2}},$$

while from $d(K) = \max\left\{\frac{r-\rho}{r}, \frac{R-r}{r}\right\}$ we obtain $d(K) \geq \frac{R-\rho}{2r}$. Consequently,

$$\lambda^*(K) \leq 2^{\frac{n+5}{2}}3^{\frac{n-1}{2}}\frac{\omega_{n-1}}{\omega_n}d(K)^{\frac{n+1}{2}}.$$

The proof is concluded. □

We conclude the section with an easy general converse estimate.

Proposition 5.3 *Let $K \subset \mathbb{R}^n$ be compact with $|K| > 0$ and $d(K) < 1$. Then*

$$\lambda^*(K) \leq n2^n d(K). \tag{5.8}$$

Proof. Since spherical deviation and Fraenkel asymmetry are scale invariant, there is no loss of generality in assuming that $|K| = \omega_n$ (and $b(K) = 0$). Hence, from the definition of Fraenkel asymmetry (Definition 2.4), we have

$$\lambda^*(K) \leq \frac{|K \triangle \Omega|}{|K|} = \frac{|K \triangle \Omega|}{\omega_n}.$$

On the other hand, from the definition of spherical deviation $d(K)$ (Definition 2.3) we get

$$(1 - d(K))\Omega \subseteq K \subseteq (1 + d(K))\Omega.$$

Thus we obtain

$$|K \triangle \Omega| \leq |(1 - d(K))\Omega \triangle (1 + d(K))\Omega|,$$

from which it follows

$$|K \triangle \Omega| \leq \omega_n((1 + d(K))^n - (1 - d(K))^n).$$

Thus, $\lambda^*(K) \leq 2n(1+s)^{n-1}d(K)$ for some $s \in (-d(K), d(K))$, which implies (5.8). $\square$

6 Isoperimetric inequalities

In this section we prove some quantitative isoperimetric inequalities for φ_0-convex sets. We treat first the case where the isoperimetric deficiency is small enough (Theorem 6.1), through suitable modifications of the technique due to Fuglede [17]. According to [17], the exponent appearing in the estimate is optimal for Δ small. Next we treat the complementary case (Theorem 6.2), i.e., Δ larger than a given number, as a corollary of the main result of [18]. The result is similar to Theorem 6.1, but the exponent is different.

Theorem 6.1 *Let $\varphi_0 \geq 0$ be given and let $K \subset \mathbb{R}^n$ be a compact φ_0-convex set with nonempty interior. Then there exist $0 < d_0 < 1$, $\eta > 0$ and a continuous strictly increasing function $f : [0, 1] \to [0, +\infty)$ (with $f(0) = 0$) such that if $d(K) \leq d_0$ and $\Delta(K) \leq \eta/2$, then*

$$d(K) \leq f(\Delta(K)).$$

Explicit formulas for f and η as well as explicit estimates for d_0 appear in Remark 6.1 just after the proof.

Proof. We wish to apply Theorem 1.2 in [17], by assuming that the inequalities (6.6), (6.7), and (6.8) hold. Without loss of generality assume also that $b(K) = 0$. By setting

$$\rho = 1 - d(K), \qquad R = 1 + d(K),$$

we observe that

$$B(0, \rho) \subseteq \left(\frac{|K|}{\omega_n}\right)^{-1/n} K \subseteq B(0, R),$$

so that formula (6.6) becomes (4.3). Therefore, we can apply Proposition 4.1, obtaining by (6.7) and (6.8) that

$$\begin{aligned} \Delta(\mathrm{co}K) &\leq \Delta(K) + \eta/2, & (6.1)\\ d(K) &\leq d(\mathrm{co}K) + a/2. & (6.2) \end{aligned}$$

Therefore, if

$$\Delta(K) \leq \frac{\eta}{2} := \sigma$$

by (i) in Proposition 4.1 and (6.1) it holds

$$\Delta(\mathrm{co}K) < \eta.$$

The proof of Theorem 2.3 in [17] (see formula (39)) shows then that

$$d(\mathrm{co}K) < \frac{a}{2}.$$

Recalling (3.22), by (ii) in Proposition 4.1 and (6.2) it follows that

$$d(K) < a.$$

Using (3.21), we are ready to apply Theorem 1.2 in [17], which gives

$$d(K) \leq f(\Delta(K)),$$

where f is as in (6.3), (6.4). □

Remark 6.1 *Explicit formulas for η and f and estimates for d_0.*
The function f is defined as follows:
if $n = 3$

$$f(\Delta) = C\left(\Delta \log \frac{1}{\Delta}\right)^{\frac{1}{2}}, \qquad \Delta \in [0, \frac{1}{e}]; \tag{6.3}$$

if $n \geq 4$

$$f(\Delta) = C\,\Delta^{\frac{2}{n+1}}, \qquad \Delta \geq 0, \tag{6.4}$$

where the constant C is explicitly given (depending only on n) by [17, Remark 1.5 and pages 630, 631]. Recalling the number a defined in (3.21), $\eta = \eta(n)$ is defined by the property that

$$f(\eta) = a/2. \tag{6.5}$$

Finally, d_0 is any positive number satisfying the following inequalities:

$$\begin{aligned} \varphi_0\, 64n \left(\frac{|K|}{\omega_n}\right)^{1/n} &< \frac{1-d_0}{d_0}, & (6.6)\\ \varphi_0\, 64n(\eta+4) \left(\frac{|K|}{\omega_n}\right)^{1/n} &< \eta\frac{1-d_0}{d_0}, & (6.7)\\ \varphi_0\, \frac{128(1+d_0)(2n+1-d_0)}{1-d_0-64n\varphi_0 d_0} &< a\frac{1-d_0}{d_0}. & (6.8) \end{aligned}$$

Remark 6.2 *The exponent appearing in the inequalities stated in Proposition 5.2 shows that estimates of the type (6.3) and (6.4), for $\Delta(K)$ small, cannot be obtained by applying Theorem 1.1 in [18]. In fact, the set K appearing in the statement of Proposition 5.2 satisfies the assumptions of Theorem 6.1 if ρ is suitably close to R, so that (for $n \geq 4$) we obtain that $d(K) \leq C\,\Delta(K)^{\frac{2}{n+1}}$, while from Proposition 5.2 and Theorem 1.1 in [18] we obtain $d(K) \leq n2^n C\,\Delta(K)^{\frac{1}{n+1}}$.*

As a corollary of Theorem 6.1, we prove now a result of the same nature of Theorem 1.1 in [18], where the spherical deviation $d(K)$ is substituted by the measure theoretic concept of Fraenkel asymmetry. The exponents appearing in our estimate are far from being sharp as those appearing in the result obtained in [18], which is optimal in the most general class of sets with finite perimeter and finite Lebesgue measure. However our proof, valid for the restricted class of sets considered in Theorem 6.1, is very simple.

Corollary 6.1 *Let $K \subset \mathbb{R}^n$ be a φ_0-convex set satisfying the inequalities of Remark 6.1. Then there exists a constant $T(n)$ such that*

$$\lambda^*(K) \leq n2^n f(\Delta(K)),$$

where f is as in (6.3), (6.4).

Proof. Recalling (5.8) we obtain

$$\lambda^*(K) \leq n2^n\, d(K),$$

while, by Theorem 6.1, $d(K) \leq f(\Delta(K))$. □

We now prove another quantitative isoperimetric inequality, as a corollary of Theorem 1.1 in [18]. Differently from Theorem 6.1, it is valid for $\Delta(K)$ large enough.

Theorem 6.2 *Let K be satisfying the assumptions of Proposition 5.1. Then there exists a constant $C_4 = C_4(n, c_0, R_0, \rho_0) > 0$ such that*

$$d(K) \leq C_4 \Delta(K)^{\frac{1}{n+1}}. \tag{6.9}$$

Proof. It is an immediate corollary of Theorem 1.1 in [18] and of Proposition 5.1. □

The last statement of this section puts together Theorems 6.1 and 6.2.

Theorem 6.3 *Let K be a φ_0-convex set and let $\sigma > 0$. Assume that if $\Delta(K) \leq \sigma$ then the inequalities in Remark 6.1 are satisfied, while if $\Delta(K) > \sigma$ then the assumptions of Proposition 5.1 hold. Then there exists a continuous strictly increasing function $f : [0, +\infty) \longrightarrow [0, +\infty)$ such that*

$$d(K) \leq f(\Delta(K)). \tag{6.10}$$

More precisely, $f(\Delta)$ is given by the right-hand sides of (6.3) and (6.4) if $\Delta(K)$ is small enough, and by the right-hand side of (6.9) if $\Delta(K)$ is large enough.

Proof. It is an immediate corollary of Theorems 6.1 and 6.2. □

References

[1] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Science Publications, Clarendon Press, Oxford (2000).

[2] T. Bonnesen, Über das isoperimetrische Defizit ebener Figuren, (German) *Math. Ann.* **91** (1924), 252-268.

[3] T. Bonnesen, W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho (1987).

[4] Yu. D. Burago, V. A. Zalgaller, *Geometric inequalities*, Springer, Berlin (1988).

[5] A. Canino, On p-convex sets and geodesics, *J. Differential Equations* **75** (1988), 118-157.

[6] A. Canino, Local properties of geodesics on p-convex sets, *Ann. Mat. Pura Appl.* **(4) 159** (1991), 17-44.

[7] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, The sharp Sobolev inequality in quantitative form (2007), preprint.

[8] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, On the isoperimetric deficit in the Gauss space (2008), preprint.

[9] G. Colombo, A. Marigonda, Differentiability properties for a class of non-convex functions, *Calc. Var.* **25** (2006), 1-31.

[10] G. Colombo, A. Marigonda, P. R. Wolenski, Some new regularity properties for the minimal time function, *SIAM J. Control* **44** (2006), 2285-2299.

[11] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York (1998).

[12] E. De Giorgi, Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita, (Italian) *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat., Sez. I* **8** (1958), 33-44; also (in English) in De Giorgi, Selected papers, L. Ambrosio, G. Dal Maso, M. Forti, M. Miranda, S. Spagnolo (Eds.), Springer, Berlin (2006).

[13] L. Esposito, N. Fusco, C. Trombetti, A quantitative version of the isoperimetric inequality: the anisotropic case *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **5** 4 (2005), 619-651.

[14] L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton (1992).

[15] H. Federer, Curvature Measures, *Trans. Amer. Math. Soc.* **93** (1959), 418–491.

[16] S. Filippas, V. Maz'ya, A. Tertikas, Critical Hardy-Sobolev inequalities, *J. Math. Pures Appl.* **87** (2007), 37-56.

[17] B. Fuglede, Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbb{R}^n$, *Trans. Am. Math. Soc.* **314** (1989), 619-638.

[18] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative isoperimetric inequality, *Ann. of Math.*, in print.

[19] R. R. Hall, A quantitative isoperimetric inequality in n-dimensional space, *J. Reine Angew. Math.* **428** (1992), 161-176.

[20] V. D. Milman, G. Schechtman (Eds.) *Geometric aspects of functional analysis. Papers from the Israel Seminar (GAFA) held 2004–2005.* Lecture Notes in Mathematics **1910**, Springer, Berlin (2007).

[21] R. Osserman, Bonnesen-style isoperimetric inequalities, *Amer. Math. Monthly* **86** (1979), 1-29.

[22] R. A. Poliquin, R. T. Rockafellar, L. Thibault, Local differentiability of distance functions, *Trans. Amer. Math. Soc.* **352** (2000), 5231-5249.

[23] R. T. Rockafellar, R. J.-B. Wets, *Variational Analysis*, Springer, Berlin (1998).

[24] R. Schneider, *Convex bodies: the Brunn-Minkowski Theory*, Enc. of Math. and its Appl. **44**, Cambridge University Press, Cambridge (1993).

[25] G. Xiong, W.-S. Cheung, D.-Y. Li, Bounds for inclusion measures of convex bodies, *Adv. Appl. Math.* **41** (2008), 584-598.