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The regularity of the minimum time function via nonsmooth analysis and geometric measure theory

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I dedicate this thesis
to my beloved grandfather.

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Contents

Chapter 1

Introduction

The minimum time problem is classical in control theory. Given a nonempty closed target $\mathcal{S}$ and a control system

$$\begin{cases} \dot{y}(t) = f(t, y(t), u(t)) & a.e. \\ u(t) \in \mathcal{U} & a.e. \\ y(0) = x, \end{cases} \tag{1.0.1}$$

where the function $f : \mathbb{R} \times \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^N$ is smooth enough and the control set $\mathcal{U}$ is a compact nonempty subset of $\mathbb{R}^M$, for each admissible control $u(\cdot) \in \mathcal{U}_{ad}$, i.e. $u(\cdot)$ is measurable and takes value in $\mathcal{U}$, there exists a unique solution $y^{x,u}(\cdot)$ of (1.0.1) which is the trajectory starting from x under the control $u(\cdot)$. The minimum time needed to steer x to $\mathcal{S}$, regarded as a function of x, is called the minimum time function and is denoted by

$$T_{\mathcal{S}}(x) := \inf \{\theta_{\mathcal{S}}(x, u) \mid u(\cdot) \in \mathcal{U}_{\mathrm{ad}}\},$$

where $\theta_{\mathcal{S}}(x, u) := \inf \{t \geq 0 \mid y^{x,u}(t) \in \mathcal{S}\}$. In general, $T_{\mathcal{S}} \in [0, \infty]$. The controllable set $\mathcal{R}$ consists of all points $x \in \mathbb{R}^N$ such that $T_{\mathcal{S}}(x)$ is finite. The regularity of the minimum time function is related on one hand to the controllability properties of system (1.0.1), on the other one to the regularity of the target and of the dynamics, together with suitable relations between them.

Such topics were studied by several authors (see, e.g., [12, 13, 17, 18, 21, 22, 26, 32, 71] and reference therein) under different viewpoints. In particular, it is well known that in general the minimum time function T is not everywhere differentiable. It is also well known that suitable controllability conditions imply the Hölder continuity of T (see, e.g., [12, Chapter IV] and references therein). However, the latter fact does not provide information on differentiability. In a 1995 paper (see [21] and also Chapter 8 in the book [22]), Cannarsa and Sinestrari found a connection between the control system and the target which actually implies the semiconcavity (or the semiconvexity) of T. Semiconcave functions are – essentially – $\mathcal{C}^2$-perturbations of concave functions and therefore inherit several regularity properties from convexity. Several features of semiconcavity were thoroughly studied (see Chapters 3, 4, 5 in [22] and references therein), thus providing a rich set of information on the structure of the minimum time function and suggesting semiconcavity/semiconvexity as a good regularity class for such value functions. The main result in [21] shows that if the target satisfies a *uniform internal ball condition* (see Definition 2.2.2 below) and the control system is smooth enough, then T is semiconcave, provided a strong *controllability assumption*, called Petrov condition, holds. A partially symmetric result, contained in [21], states that if the target is convex and the control system is linear, then T is semiconvex, provided, again, Petrov condition holds. The latter requires that the minimized Hamiltonian at all boundary points of $\mathcal{S}$, computed along unit normal vectors, be bounded away from zero locally uniformly, i.e., for all $R > 0$ there exists $\mu > 0$ such that for all $x \in \text{bdry}\mathcal{S} \cap B(0, R)$,

$$\min_{u \in \mathcal{U}} \langle f(x,u), \zeta \rangle < -\mu, \qquad \text{for all } \zeta \in N_{\mathcal{S}}(x),\ \|\zeta\| = 1. \tag{1.0.2}$$

It is well known that Petrov condition is equivalent to the local Lipschitz continuity of T (see, e.g., [22, Section 8.2]).

In an entirely different setting, a class of sets which includes both convex and $\mathcal{C}^2$-sets was studied independently by several authors (including Federer [42], Canino [15], Clarke, Stern and Wolenski [29], Poliquin, Rockafellar and Thibault [58]) under different names, for example *sets with positive reach* [42], *φ-convex sets* [15], *proximally smooth sets* [29], and *prox-regular sets* [58]. Such sets, which in this thesis will be called sets with positive reach, are characterized by a strong external sphere condition (see Definition 2.2.1 below): every normal vector must be realized by a locally uniform ball. By observing that a convex set satisfies the same type of external sphere condition with an arbitrarily large radius, it is natural to expect that sets with positive reach enjoy locally several properties that convex sets enjoy globally. In particular, this holds for the metric projection, which is unique in a neighborhood of a set with positive reach K. This fact is used in proving all the regularity properties which are satisfied by sets with positive reach (see, e.g., [42, Section 4]). Semiconcave functions and sets with positive reach, through the hypograph, are linked together (see, e.g., Theorem 5.2 in [29], where semiconvex functions are called *lower–$\mathcal{C}^2$*): a locally Lipschitz function is semiconcave if and only if its hypograph has positive reach. Of course an entirely symmetric characterization for semiconvex functions can be expressed using the epigraph. Trying to generalize to functions whose hypo/epigraph has positive reach some regularity properties enjoyed by semiconcave/convexity functions was therefore a natural challenge. Some results on this line were obtained in [30, 31], including the a.e. twice differentiability (see Theorem 2.2.2 below) together some results on the structure of singularities.

In several control problems, controllability assumptions weaker than Petrov condition hold, and therefore the minimum time function is not locally Lipschitz. A natural question therefore is trying to understand whether the structure of the minimum time function remains unchanged if in the above setting the controllability assumptions are weakened. In other words it is natural to investigate whether the hypograph/epigraph of T has positive reach if T is supposed to be only continuous.

This thesis has been inspired by the above question. It is divided into two parts. The first one is devoted to the analysis of the minimum time function. The second one is motivated by the first part, and contains results on the regularity of merely continuous functions.

Part I: On the structure of the minimum time function

This part is dedicated to two types of regularity of the minimum time function T. More precisely, we will study in Chapter 3 semiconcavity type results for T. We first assume that the nonlinear control system is (essentially) $\mathcal{C}^2$, the target $\mathcal{S}$ satisfies an internal sphere condition, and T is continuous, and study the hypograph of T in the complement of $\mathcal{S}$. Since the internal sphere property is closed with respect to the union operator, one can see intuitively that the reachable set $\mathcal{R}^t$, which is the set of points reachable from $\mathcal{S}$ in time less than t, inherits such property from $\mathcal{S}$. By combining this fact and the Hamiltonian function, a regularity result on the hypograph of T can be obtained. The corresponding theorem is as follows:

Theorem 1.0.1 *Under the above assumptions, the hypograph of T satisfies an external sphere condition.*

From this theorem, we obtain that if T is Lipschitz then T is semiconcave (see [54]). However, here the situation is more complicated than in the Lipschitz case: the main results depend on the pointedness of the normal cone to the hypograph. Indeed, from a representation of generalized supergradient of T, we prove that

Theorem 1.0.2 *Together with the above assumptions, if the normal cone to the hypograph is pointed in the complement of $\mathcal{S}$, then the hypograph of T has positive reach.*

In the last section of this chapter, we also prove Theorem 1.0.1 for a class of differential inclusions taken from [23]. Moreover, in the spirit of [18] we finally extend this result to arbitrary target $\mathcal{S}$.

The next chapter is devoted to semiconvexity type results for T. For a linear control system and a convex target, the result is contained in [32]. However, for a nonlinear control system, one has to face the difficulty that the convexity (even the external sphere property) of the reachable set $\mathcal{R}^t$

can be easily broken after any small time t (see example 4.3 in [21]). This is quite natural since the union of convex sets usually has inner corners or even cusps. Moreover, the "external normal regularity" of the reachable set is also related to the uniqueness of the optimal trajectory from a point to the target $\mathcal{S}$. Hence, finding a class of nonlinear control systems such that the convexity (or the external sphere property) of $\mathcal{R}^t$ still holds up to small time t is a natural problem.

This chapter is first devoted to a result of this type. Indeed, we assume that the target is the origin and the linear part of the nonlinear control system at the origin is *normal* (see the definition in Theorem 1.0.3 below) and study the reachable set $\mathcal{R}^t$. More precisely, for $t > 0$ small the normality of the linearization together with a further condition on the Taylor development at 0 of the nonlinear control system yields the strict convexity of the reachable set $\mathcal{R}^t_L$ corresponding to the linear nonautonomous systems which are obtained by linearizing the nonlinear control system along the optimal trajectories. Therefore, it is reasonable to conjecture that the convexity of the reachable set, for $t > 0$ small, still holds also for a suitable nonlinear control system. For this approach, the main preliminary result is as follows:

Theorem 1.0.3 *Consider the linear control system*

$$\dot{x} = Ax + Bu, \tag{1.0.3}$$

where $A \in \mathbb{M}_{N\times N}$, $B \in \mathbb{M}_{N\times M}$, $M \leq N$ *and* $u = (u_1, u_2, ..., u_M) \in \mathbb{R}^M$, $|u_j| \leq 1$ *for* $j = 1, 2, ..., M$.

Assume that (1.0.3) is normal, i.e., for every column b_j, $j = 1, 2, ..., M$ *of* B,

$$\text{rank } [b_j, Ab_j, ..., A^{N-1}b_j] = N.$$

Then for all $T > 0$ *there exists a constant* $\gamma > 0$, *depending only on* N, M, A, B, T *such that for all* $x, y \in \mathcal{R}^T$, *for all* $\zeta \in N_{\mathcal{R}^T}(x)$, *it holds*

$$\langle \zeta, y - x\rangle \leq -\gamma \ \|\zeta\| \ \|y - x\|^N. \tag{1.0.4}$$

However, the exponent of strict convexity of $\mathcal{R}^t_L$ is N as in (1.0.4), while the exponent of the perturbative term appearing from the linearization is 2. Therefore, this approach is effective only in the two dimensional case (see Theorem 4.4.1). On the basis of the preceding result, we will prove that the epigraph of T in a neighborhood of 0 has positive reach (see Theorem 4.5.2). This will require proving that all points close enough to the origin are indeed optimal.

Part II: The regularity of a class of continuous functions

Since verifying that a set has positive reach is often demanding, finding sufficient conditions for this property appears of some interest. In [53], a class of sets which are characterized by an external sphere condition (at each point on the boundary, there exists *one* proximal normal vector realized by a locally uniform ball) is considered. The authors proved that if a set satisfies this condition and is *wedged* (this concept was introduced by Rockafellar in [59]) then it has positive reach. This results was later generalized in [54] by the same authors to investigate the relationships among functions whose hypograph satisfies an external sphere condition, the functions with positive reach hypograph and semiconcave functions. Wedgedness of a set C is equivalent to the pointedness of the Clarke normal cone to C, i.e. the normal cone does not contain lines (see [28] and [64]). Moreover, the pointedness assumption for the proximal normal cone to the hypograph of the minimum time function T appears pivotal in our result [20] (mentioned in the first part) for computing generalized gradients of T and then for proving that the hypograph of T has positive reach.

Several counterexamples (see. e.g, [53]), though, show that the external sphere condition is in general weaker than positive reach. In particular, in Example 2 in Section 6 of Chapter 3, we constructed a minimum time function with a constant dynamics and a $C^{1,1}$ target such that its hypograph satisfies an external sphere condition but has not positive reach everywhere. On the other hand, the pointedness assumption for the normal cone to the hypograph of a continuous function is hard to verify since it is related to the representation formula for its generalized supergradient (this problem is studied in [33]). Therefore, the problem of understanding whether some concavity features are preserved under the external sphere condition appears natural. In Chapter 5 an answer to this question is provided. Our main result reads -essentially- as follows

Theorem 1.0.4 *Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that the hypograph of f satisfies the weak external sphere condition. Then there exists a closed set Γ with zero Lebesgue measure such that the hypograph of the restricted function $f_{\Omega\setminus\Gamma}$ has positive reach.*

Consequently, a function satisfying the assumption of the above theorem enjoys several regularity properties inherited by functions whose hypograph has positive reach. Therefore, using Theorem 1.0.1 and Theorem 1.0.4 the pointedness assumption of the hypograph of T in Chapter 3 is removed and the a.e. twice differentiability of T for a class of differential inclusions is also obtained.

In general, however, sets with null Lebesgue measure can be very irregular and possess almost no structure. A natural question is then that of investigating the properties of the singular set $\Sigma(f)$ for special classes of a.e. differentiable functions f.

When f is convex or concave, the properties of $\Sigma(f)$ were first investigated in [39] and then developed in [70], [69], [67], [68], [3] and [4]. The basic approach in such papers is that of estimating the size of $\Sigma(f)$. We mention here a result which is essentially due to L. Zajíček and was later extended to semiconcave functions by G. Alberti, L. Ambrosio and P. Cannarsa [2]. By $\partial^F f(x)$ we denote here the Fréchet supergradient of f at x.

Theorem 1.0.5 ([2]) *Let f be locally semiconcave. Then, for any $k = 1, \ldots, N$ the singular set $\Sigma^k(f) := \{x \in \Omega \mid \dim \partial^F f(x) = k\}$ is countably $(N-k)$-rectifiable. In particular, $\Sigma(f)$ is countably $(N-1)$-rectifiable and $\Sigma^N(u)$ is at most countable.*

The result also holds for the case of a continuous function f which has the hypograph with positive reach (see in [30]). Therefore, at the end we will study in Chapter 6 the rectifiability of the zero Lebesgue measure set Γ. The corresponding result is Theorem 1.0.4.

Chapter 2

Premilinary

2.1 Nonsmooth analysis and geometric measure theory

2.1.1 Nonsmooth analysis

We quickly review in this subsection some basic concepts from nonsmooth analysis. Standard references are in [28, 48, 64].

Let $x \in Q$ and $v \in \mathbb{R}^N$. We say that v is a *proximal normal* to Q at x (and we will denote this fact by $v \in N_Q^P(x)$) if there exists $\sigma = \sigma(v, x) \geq 0$ such that

$$\langle v, y - x\rangle \leq \sigma \|y - x\|^2 \quad \text{for all } y \in Q; \tag{2.1.1}$$

equivalently $v \in N_Q^P(x)$ if and only if there exists $\lambda > 0$ such that $\pi_Q(x + \lambda v) = \{x\}$. We say that the proximal normal v *is realized by a ball of radius* $\rho > 0$ if ρ is the supremum of all λ such that $\pi_Q(x + \lambda v) = \{x\}$. In this case the best constant σ such that (2.1.1) holds true is $\|v\|/(2\rho)$. The following further concepts of normal vectors will be used (see [28, Chapter I] and [64, Chapter VI]). Let $x \in Q$ and $v \in \mathbb{R}^N$. We say that:

1. v is a *Fréchet normal* (or Bouligand normal) to K at x ($v \in N_Q^F(x)$) if
$$\limsup_{Q \ni y \to x} \langle v, \frac{y - x}{\|y - x\|}\rangle \leq 0;$$
2. v is a *limiting, or Mordukhovich, normal* to Q at x ($v \in N_Q^L(x)$) if
$$v \in \{w \mid w = \lim w_n,\ w_n \in N_Q^P(x_n),\ x_n \to x\}$$
and is a *Clarke normal* ($v \in N_Q^C(x)$) if $v \in \overline{\text{co}} N_Q^L(x)$. It is well known that $N_Q^P(x)$ is convex.

Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. By using $\text{epi}(f) := \{(x, \xi) \mid \xi \geq f(x)\}$ and $\text{hypo}(f) := \{(x, \xi) \mid \xi \leq f(x)\}$ one can define some concepts of generalized differential for f at $x \in \mathrm{dom}(f) = \{x \in \mathbb{R}^N |\, f(x) < +\infty\}$. Let $x \in \mathrm{dom}(f)$, $v \in \mathbb{R}^N$. We say that:

1. v is a *proximal subgradient* of f at x ($v \in \partial_P f(x)$) if $(v, -1) \in N^P_{\mathrm{epi}(f)}(x, f(x))$; equivalently (see [28, Theorem 1.2.5]), $v \in \partial_P f(x)$ iff there exist σ, $\eta > 0$ such that for all $y \in B(x, \eta) \cap \mathrm{dom}\,(f)$, it holds
$$f(y) \geq f(x) + \langle v, y - x\rangle - \sigma \|y - x\|^2\,; \tag{2.1.2}$$

2. v is a *proximal supergradient* of f at x ($v \in \partial^P f(x)$) if $(-v, 1) \in N^P_{hypo(f)}(x, f(x))$; equivalently $v \in \partial^P f(x)$ iff $-v \in \partial_P(-f)(x)$, i.e., iff there exist σ, $\eta > 0$ such that for all $y \in B(x, \eta) \cap \mathrm{dom}\,(f)$, it holds
$$f(y) \leq f(x) + \langle v, y - x\rangle + \sigma \|y - x\|^2\,; \tag{2.1.3}$$

3. v is a *Fréchet subgradient* of f at x ($v \in \partial_F f(x)$) if $(v, -1) \in N^F_{epi(f)}(x, f(x))$, i.e.,
$$\liminf_{y \to x} \frac{f(y) - f(x) - \langle v, y - x\rangle}{\|y - x\|} \geq 0;$$

4. v is a *Fréchet supergradient* of f at x ($v \in \partial^F f(x)$) if $(-v, 1) \in N^F_{hypo(f)}(x, f(x))$;

5. v is a *limiting subgradient* of f at x ($v \in \partial_L f(x)$) if $(v, -1) \in N^L_{epi(f)}(x, f(x))$.

6. v is a *limiting supergradient* of f at x ($v \in \partial^L f(x)$) if $(-v, 1) \in N^L_{hypo(f)}(x, f(x))$.

7. v is a *Clarke generalized gradient* of f at x ($v \in \partial f(x)$) if $(v, -1) \in N^C_{\mathrm{epi}(f)}(x, f(x))$. We recall that *if f is Lipschitz continuous in a neighborhood of x*, then $v \in \partial f(x)$ if and only if $v \in \overline{\mathrm{co}}\{\zeta |\, \zeta = \lim Df(x_i), x_i \in \mathrm{dom}(Df), x_i \to x\}$ (see [28, Theorem 8.1]).

It follows readily from the definitions that the inclusions
$$N^P_Q(x) \subseteq N^F_Q(x) \subseteq N^L_Q(x) \subseteq N^C_Q(x)$$
hold, together with their analogues for the sub- and supergradient. Moreover, if a vector v belongs to both the Fréchet sub- and supergradient of f at x, then f is Fréchet differentiable at x and $Df(x) = v$.

For a not necessarily Lipschitz function f, the *horizon subgradient* $\partial_\infty f$ plays an important role. This is defined as
$$\partial_\infty f(x) = \{v \in \mathbb{R}^N \mid (v, 0) \in N^C_{\mathrm{epi}(f)}(x, f(x))\},$$

and is clearly a closed convex cone. In particular, if f is not locally Lipschitz in a neighborhood of x, then $\partial f(x)$ may be represented using ∂_∞, namely (see [48, Prop. 2.6] or [64, Theorem 8.49])

$$\partial f(x) = \text{cl}\,(\text{co}\,\partial_L f(x) + \text{co}\,\partial_\infty f(x)) \,. \tag{2.1.4}$$

Finally, we also consider a notion of *proximal* horizon supergradient, namely the convex cone

$$\partial^\infty f(x) = \{v \in \mathbb{R}^N \mid (-v, 0) \in N^P_{hypo(f)}(x, f(x))\}.$$

2.1.2 Geometric measure theory

We just introduce in this subsection some definitions needed our results. For basic concepts of geometric measure theory we refer to [6, 40].

Let $k \geq 0$ and $A \subset \mathbb{R}^N$ be fixed. The *k-dimensional Hausdorff measure* of A is defined as

$$\mathcal{H}^k(A) := \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(A) = \sup_{\delta > 0} \mathcal{H}^k_\delta(A)$$

where for any $\delta > 0$ we set

$$\mathcal{H}^k_\delta(A) := \inf \left\{ \sum_{i \in I} (\text{diam}\, A_i)^k \mid A \subset \bigcup_{i \in I} A_i,\ \text{diam}\, A_i < \delta \right\}.$$

The *Hausdorff dimension* of A is

$$\mathcal{H}\text{-dim}(A) := \inf\{k \geq 0 \mid \mathcal{H}^k(A) = 0\} = \sup\{k \geq 0 \mid \mathcal{H}^k(A) = \infty\}\,.$$

It is well known (see e.g. [43, 50]) that $\mathcal{H}^k$ is a Borel measure on $\mathbb{R}^N$; $\mathcal{H}^0$ is the counting measure. Moreover, if $k \in \mathbb{N}$ and S is a k-dimensional Lipschitz surface, then the surface measure of S coincides with $\frac{2^k}{\omega_k}\mathcal{H}^k \llcorner S$.

Let $k \in \mathbb{N}$; we say that $A \subset \mathbb{R}^N$ is *countably k-rectifiable* if

$$A \subset \mathcal{N} \cup \bigcup_{i=1}^{\infty} S_i$$

where S_i are suitable Lipschitz k-dimensional surfaces and $\mathcal{N}$ is a $\mathcal{H}^k$-negligible set. We say that A is *k-rectifiable* if it is countably k-rectifiable and $\mathcal{H}^k(A) < \infty$.

Any countably k-rectifiable set A satisfies $\mathcal{H}\text{-dim}(A) = k$. It is well known that, if $f : A \subset \mathbb{R}^k \to \mathbb{R}^N$ is Lipschitz continuous, then $f(A)$ is countably k-rectifiable; if A is bounded, then $f(A)$ is k-rectifiable.

In what follows, given $A \subset \mathbb{R}^N$ we define its ϵ-neighborhood $(A)_\epsilon$ by

$$(A)_\epsilon := \{x \in \mathbb{R}^N \mid \text{there exists } y \in A \text{ such that } \|x - y\| < \epsilon\}\,.$$

Let $\mathcal{K}$ denote the set of closed subsets of $S^{N-1} \subset \mathbb{R}^N$; for $A, B \in \mathcal{K}$ we introduce the *Hausdorff distance* $d_{\mathcal{H}}(A, B)$ by

$$d_{\mathcal{H}}(A, B) = \inf\{\epsilon > 0 \mid A \subset (B)_\epsilon \text{ and } B \subset (A)_\epsilon\}.$$

It turns out (see e.g. [7]) that $(\mathcal{K}, d_{\mathcal{H}})$ is a complete compact metric space.

2.2 Positive reach and external sphere condition

2.2.1 Positive reach

The concept of reach originates from the unique nearest point property. More precisely, the reach of a subset Q of $\mathbb{R}^N$ is the largest r (possibly ∞) such that if $x \in \mathbb{R}^N$ and the distance , $d_Q(x)$, from x to Q smaller than r, then Q contains a unique point, $\pi_Q(x)$, nearest to x.

Let $Q \subseteq \mathbb{R}^N$ be closed. We denote by ∂Q the topological boundary of Q, and, for $x \in \mathbb{R}^N$,

$$\begin{aligned} d_Q(x) &= \inf\{\|y - x\| \mid y \in Q\} \qquad \text{(the distance of } x \text{ from } Q) \\ \pi_Q(x) &= \{y \in Q \mid \|y - x\| = d_Q(x)\} \text{(the metric projections of } x \text{ into } Q). \end{aligned}$$

Moreover, we set

$$\mathrm{Unp}(Q) = \{x \in \mathbb{R}^N \ : \ \pi_Q(x) \ \text{ is a singleton}\}.$$

If $x \in Q$ then

$$reach(x, Q) = \sup\{r \mid B(x, r) \subset \mathrm{Unp}(Q)\}$$

where $B(x, r) = \{y \mid \|y - x\| < r\}$.
Also

$$reach(Q) = \inf\{reach(Q, x) \mid x \in Q\}.$$

Remark 2.2.1 *The function $reach(Q, \cdot)$ is continuous in Q. Moreover if $reach(Q) > 0$ then Q is closed.*

If $reach(Q) > 0$ we say that Q has positive reach. Moreover, a set with positive reach can be alternatively defined as follow (see in [15, 38]).

Definition 2.2.1 *Let $Q \subset R^N$ be closed. We say that Q has positive reach if there exists a continuous function $\varphi : Q \to [0, \infty)$ be continuous such that for all $x, y \in Q, v \in N_Q^P(x)$, the inequality*

$$\langle v, y - x\rangle \le \varphi(x) \|v\| \|x - y\|^2$$

holds, i.e. $v \in N_Q^P(x)$ is realized by a ball of radius $\frac{1}{2\varphi(x)}$.

Remark 2.2.2 *If a set Q has positive reach then the function $\varphi(\cdot)$ in the definition (2.2.1) can be replaced by $\frac{1}{2\ reach(\cdot,Q)}$.*

Proof. The proof can be found in [42, Theorem 4.8 (7)]. □

It is therefore clear that every closed and convex set has positive reach, with $reach(Q) = \infty$, and every closed set with a $\mathcal{C}^{1,1}$-boundary has positive reach, with $reach(Q) = L/2$, where L is the Lipschitz constant of a suitable parametrization of ∂Q. Some properties of the distance from a set with positive reach Q and the metric projection onto Q are important features of this class of sets.

Theorem 2.2.1 *Let $Q \subset \mathbb{R}^N$ be a set with positive reach. Then there exists an open set $U \supset K$ such that*

(1) $d_Q \in \mathcal{C}^{1,1}(U \setminus Q)$ and $Dd_Q(y) = \frac{y-\pi_Q(y)}{d_K(y)}$ for every $y \in U \setminus Q$;

(2) $\pi_Q : U \to Q$ is a locally Lipschitz single-valued map. In particular, the function $\pi_Q : \{x \in \mathbb{R}^N \mid d(x,Q) < \frac{reach(Q)}{2}\} \to Q$ is Lipschitz with Lipschitz ratio 2.

Moreover,

(3) Q has finite perimeter in $\mathbb{R}^N$ (provided it is compact);

(4) for every $x \in Q$, $N_Q^P(x) = N_Q^C(x)$;

(5) the set valued map $N_Q^P(\cdot)$ has closed graph in $\partial Q \times \mathbb{R}^N$.

Proof. The proof of (1) and (2) can be found in [15, Proposition 2.6, 2.9, Remark 2.10] or in [42, §4]. The proof of (3) is in [30, §5], while (4) and (5) can be found in several papers, including [58]. □

Remark 2.2.3 *Conditions (1) and (2) in Theorem 2.2.1 are actually equivalent to positive reach, as it is proved, e.g., in [42, §4]. Examples of finite dimensional sets with positive reach can be found, e.g., in [42].*

We also give here Lemma 3.1 in [35] which concerns an estimate of the excess of the convex hull of a set with positive reach Q over Q .

Lemma 2.2.1 *Let Q be a set with positive reach and let $x \in \mathrm{co}Q$ and $d_Q(x) < reach(Q)$. Then*

$$\|x - \pi_Q(x)\| \leq \frac{1}{2\ reach(Q)} \sum_{i,j=1}^{N+1} t_i t_j \|x_i - x_j\|^2,$$

where $t_i \geq 0$, $\sum_{i=1}^{N+1} t_i = 1$, $x_i \in Q$, and $x = \sum_{i=1}^{N+1} t_i x_i$.

Proof. From Remark 2.2.2, we have for each $i = 1, \ldots, N+1$,

$$< x - \pi_Q(x), x_i - \pi_Q(x) > \leq \frac{1}{2\; reach(Q)} \|x - \pi_Q(x)\| \, \|x_i - \pi_Q(x)\|^2 ,$$

so that

$$< x - \pi_Q(x), \sum_{i=1}^{N+1} t_i x_i - \pi_Q(x) > \leq \frac{\|x - \pi_Q(x)\|}{2\; reach(Q)} \sum_{i=1}^{N+1} t_i \|x_i - \pi_Q(x)\|^2 .$$

Recalling that $x = \sum_{i=1}^{N+1} t_i x_i$, we thus obtain

$$\|x - \pi_Q(x)\| \leq \frac{1}{2\; reach(Q)} \sum_{i=1}^{N+1} t_i \|x_i - \pi_Q(x)\|^2 . \tag{2.2.1}$$

Putting $I = \sum_{i=1}^{N+1} t_i \|x_i - x\|^2$, from an elementary computation taking into account the condition $\sum_{i-1}^{N+1} t_i(x - x_i) = 0$, we obtain, for all $v \in \mathbb{R}^N$,

$$\sum_{i=1}^{N+1} t_i \|x_i - v\|^2 = \|x - v\|^2 + I. \tag{2.2.2}$$

Now we compute I. Taking $v = x_j$ in (2.2.2), we have

$$\sum_{i=1}^{N+1} t_i \|x_i - x_j\|^2 = \|x - x_j\|^2 + I.$$

Thus we obtain both

$$t_j \sum_{i=1}^{N+1} t_i \|x_i - x_j\|^2 = t_j \|x - x_j\|^2 + t_j I$$

and

$$\sum_{j=1}^{N+1} \sum_{i=1}^{N+1} t_j t_i \|x_i - x_j\|^2 = \sum_{j=1}^{N+1} t_j \|x - x_j\|^2 + \sum_{j=1}^{N+1} t_j I.$$

From $\sum_{j=1}^{N+1} t_j = 1$ and $I = \sum_{j=1}^{N+1} t_j \|x - x_j\|^2$, we obtain

$$I = \frac{1}{2} \sum_{j=1}^{N+1} \sum_{i=1}^{N+1} t_j t_i \|x_i - x_j\|^2 .$$

Using this expression in (2.2.2) with $\pi_Q(x)$ in place of v, we obtain

$$\sum_{i=1}^{N+1} t_i \|x_i - \pi_Q(x)\|^2 = \|x - \pi_Q(x)\|^2 + \frac{1}{2} \sum_{j=1}^{N+1} \sum_{i=1}^{N+1} t_j t_i \|x_i - x_j\|^2 .$$

Thus

$$\|x - \pi_Q(x)\| \leq \frac{1}{2\, reach(Q)} \Big(\|x - \pi_Q(x)\|^2 + \frac{1}{2} \sum_{j=1}^{N+1} \sum_{i=1}^{N+1} t_j t_i \|x_i - x_j\|^2 \Big).$$

Since $\|x - \pi_Q(x)\| = d_Q(x) < reach(Q)$, the proof is concluded. □

In both optimal control and partial differential equations theory, *semiconcave functions* play an important role (see, e.g., [12, 22]). Let $\Omega \subset \mathbb{R}^N$ be open: a function $f : \Omega \longrightarrow \mathbb{R}$ is said to be semiconcave if for every $x \in \Omega$ and every $\delta > 0$ there exists a constant $C > 0$ such that

$$f(x) - C\|x\|^2 \text{ is concave in } B(x, \delta).$$

Semiconcave functions are therefore locally Lipschitz. Moreover, thanks to Theorem 5.2 in [29], the hypograph of such functions has positive reach. More in general, upper semicontinuous functions which have hypograph with positive reach (or l.s.c. functions which have epigraph with positive reach) enjoy several of the regularity properties, except Lipschitz continuity, that semiconcave functions satisfy. Such functions identify the class which we want to show that our minimum time belongs to. To this aim, we state a result which collects the main properties.

Theorem 2.2.2 *Let $\Omega \subset \mathbb{R}^N$ be open, and let $f : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be proper, upper semicontinuous, and such that hypo(f) has positive reach. Then there exists a sequence of sets $\Omega_h \subseteq \Omega$ such that Ω_h is compact in* dom(f) *and*

(1) the union of Ω_h covers $\mathcal{L}^N$-almost all dom(f)*;*

(2) for all $x \in \bigcup_h \Omega_h$ there exist $\delta = \delta(x) > 0$, $L = L(x) > 0$ such that f is Lipschitz on $B(x, \delta)$ with ratio L, and hence semiconcave on $B(x, \delta)$.

Consequently,

(3) f is a.e. Fréchet differentiable and admits a second order Taylor expansion around a.e. point of its domain.

Moreover, the set of points where the graph of f is nonsmooth has small Hausdorff dimension. More precisely, for every $k = 1, \ldots, N$, the set

$$\{x \in \operatorname{int} \operatorname{dom}(f) \mid \operatorname{Dim}(\partial^P f(x)) \textit{ is } \geq k\}$$

is countably $\mathcal{H}^{N-k}$-rectifiable.

This result is essentially Theorem 5.1 in [30].

2.2.2 External sphere condition

Definition 2.2.2 *Let $Q \subset \mathbb{R}^N$ be closed and let $\theta(\cdot) : \partial Q \to (0, \infty]$. We say that Q satisfies the $\theta(\cdot)$-external sphere condition if and only if for every $x \in \partial Q$, there exists a vector $v_x \neq 0$ such that $v_x \in N_Q^P(x)$ is realized by a ball of radius $\theta(x)$, i.e.,*

$$\left\langle \frac{v_x}{\|v_x\|}, y - x \right\rangle \leq \frac{1}{2\theta(x)} \|y - x\|^2.$$

for all $y \in Q$.

Moreover, denote Q' by the closure of the complement of Q, we also say that the set Q satisfies the $\theta(\cdot)$-*internal sphere condition* if Q' satisfies the $\theta(\cdot)$-external sphere condition.

In general, a set which satisfies an $\theta(\cdot)$-external sphere condition doesn't have positive reach (see. e.g, [53]). However, under a wedgedness assumption these two concepts are equivalent in [53]. The wedgedness assumption was first introduced by Rockafellar in [59].

Let $C \subset \mathbb{R}^N$ be a cone (i.e., if $x \in C$ and $\lambda \geq 0$, then $\lambda x \in C$). We say that C is *wedged* if $C \cap (-C) = \{0\}$. In [60, Corollary 18.7.1, p. 169] it is proved that

if C is closed, convex, and wedged,
then it is the closed convex hull of its exposed rays. (2.2.3)

We recall (see [60, p.163]) that an exposed ray $\mathbb{R}^+ v$ of a convex cone C is defined by the property that there exists a linear functional h which is zero on it and is such that if $h(p) = 0$ and $p \in C$ then $p \in \mathbb{R}^+ v$.

Theorem 2.2.3 *Let Q satisfy the $\theta(\cdot)$-external sphere condition. Assume that the Clarke normal cone $N_Q^C(x)$ is wedged at every $x \in \partial Q$ then Q has positive reach.*

To end up this subsection, we expose here the relationships among functions whose hypograph satisfies an external sphere condition, the functions with positive reach hypograph and semiconcave functions (see in [54]).

Theorem 2.2.4 *Let $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ be Lipschitz. Then f is semiconcave if and only if the hypograph of f satisfies a $\theta(\cdot)$-external sphere condition.*

Proof. The proof is based on Theorem 2.2.3 under remark that Clarke normal cones to hypo(f) at every point on graph(f) are wedged. □

2.3 Control theory

2.3.1 Control systems

We just consider here autonomous control systems, namely, $f(x, u)$ does not depend on t; this is done for the sake of simplicity since the results we present can be easily extended to the nonautonomous case. Standard references are in [12, 22, 14].

Definition 2.3.1 *A control system consists of a pair $(f,\mathcal{U})$, where $\mathcal{U} \subset \mathbb{R}^N$ is a closed set and $f : \mathbb{R}^N \times \mathcal{U}$ is a continuous function. The set $\mathcal{U}$ is called the control set, while f is called the dynamics of the system. The state equation associated with the system is*

$$\begin{cases} \dot{y}(t) &= f(y(t), u(t)), \quad t \in [0, +\infty) \ a.e. \\ u(\cdot) &\in \mathcal{U}_{ad}, \\ y(0) &= x, \end{cases} \tag{2.3.1}$$

where $\mathcal{U}_{\text{ad}}$ the set of admissible controls, i.e., the measurable functions $u : \mathbb{R} \to \mathbb{R}^m$, such that $u(t) \in \mathcal{U}$ a.e.

Two basic assumptions of the control systems are

(H1) The control set $\mathcal{U}$ is nonempty and compact.

(H2) The function f satisfies:

$$\|f(y, u) - f(x, u)\| \leq L\|y - x\| \quad \forall x, y \in \mathbb{R}^N, \forall u \in \mathcal{U},$$

for a positive constant L.

Under the assumption (H2), for any $u(\cdot) \in \mathcal{U}_{\text{ad}}$, there is a unique Carathéodory solution of (2.3.1) denoted by $y^{x,u}(\cdot)$. The solution $y^{x,u}(\cdot)$ is called the trajectory starting from x associated with the control $u(\cdot)$. The attainable set $\mathcal{A}(T)$ from x in time T is thus defined by

$$\mathcal{A}^T(x) \ = \ \{y^{(u,x)}(t) \mid t \leq T, u(\cdot) \in \mathcal{U}_{ad}\} \tag{2.3.2}$$

Observe that assumption (H1) and (H2), together with the continuity of f, imply

$$\|f(x, u)\| \leq C + L\|x\|, \quad x \in \mathbb{R}^N, u \in \mathcal{U},$$

where $C = \max\{\|f(0, u)\| \mid u \in \mathcal{U}\}$. Therefore, the set $\mathcal{A}^T(x)$ is bounded for all $x \in \mathbb{R}^N$ and $T < \infty$. An upper bound of norm of points in the attainable set $\mathcal{A}^T(x)$ can be found in the Appendix.

The set $\mathcal{A}^T(x)$ is in general not closed. However, by standard results,

Remark 2.3.1 *$\mathcal{A}^T(x)$ is compact if $f(z,\mathcal{U})$ is convex for every $z \in \mathbb{R}^N$.*

Proof. The proof is based on Filippov's Lemma and the compactness property for the trajectories for the control system (see Theorem 7.1.5 and Theorem 7.1.6 in [22]). □

2.3.2 Minimum time function

Together with the system 2.3.1, we consider a closed nonempty set $\mathcal{S} \subset \mathbb{R}^N$, which is called the target.

We first set $T(x) = 0$ for all $x \in \mathcal{S}$. For a fixed $x \in \mathbb{R}^N \setminus \mathcal{S}$, we define

$$\theta(x, u) := \min \{t \geq 0 \mid y^{x,u}(t) \in \mathcal{S}\}.$$

Of course, $\theta(x, u) \in (0, +\infty]$, and $\theta(x, u)$ is the time taken for the trajectory $y^{x,u}(\cdot)$ to reach $\mathcal{S}$, provided $\theta(x, u) < +\infty$. The *minimum time* $T(x)$ to reach $\mathcal{S}$ from x is defined by

$$T(x) := \inf \{\theta(x, u) \mid u(\cdot) \in \mathcal{U}_{\text{ad}}\}. \tag{2.3.3}$$

Equivalently,

$$T(x) := \inf \{t \geq 0 \mid \mathcal{A}^t(x) \cap \mathcal{S} \neq \varnothing\}.$$

Remark 2.3.2 *If $f(z,\mathcal{U})$ is convex for every $z \in \mathbb{R}^N$ then*

$$T(x) := \min \{\theta(x, u) \mid u(\cdot) \in \mathcal{U}_{\text{ad}}\}. \tag{2.3.4}$$

Proof. This is a consequence of Remark 2.3.1. □

A minimizing control in (2.3.4), say $\bar{u}(\cdot)$, is called an *optimal control*. The trajectory $y^{x,\bar{u}}(\cdot)$ associated with $\bar{u}(\dot{})$ is called an *optimal trajectory*.

We finally give in this subsection a result which one can see intuitively from the definition of the minimum time function:

Theorem 2.3.1 *(Dynamic Programming Principle) Assume that the control system satisfies (H1) and (H2). For $x \in \mathbb{R}^N \backslash \mathcal{S}$, and for $0 < t \leq T(x)$, we have*

$$T(x) = t + \inf \{T(y) \mid y \in \mathcal{A}^t(x)\}. \tag{2.3.5}$$

Equivalently, for all $u(\cdot)$ if we set $x(\cdot) = y^{x,u}(\cdot)$ then the function $t \longmapsto t + T(x(t))$ is increasing in $[0, T(x)]$.

Moreover, if $x(\cdot)$ is an optimal trajectory then $t \longmapsto t + T(x(t))$ is constant in $[0, T(x)]$, i.e.,

$$T(x(t)) = t - s + T(x(s)) \quad \text{for } 0 \leq s \leq t \leq T(x).$$

Proof. The proof can be found in several books, e.g.,[12, 22, 14]. □

2.3.3 Controllability and continuity

Continuity properties of the minimal time function is a widely studied topic, mainly in connection with controllability. In this subsection, we will give shortly some definitions and some basic results which are concerned with our works. For the references we prefer to quote [12, 22].

We first introduce the notations

$$\begin{aligned} \mathcal{R}(t) &= \{x \in \mathbb{R}^N \mid T(x) < t\}, \quad t > 0, \\ \mathcal{R} &= \bigcup_{t>0} \mathcal{R}(t) = \{x \in \mathbb{R}^N \mid T(x) < \infty\}, \end{aligned}$$

where the letter $\mathcal{R}$ stands for *reachable*: $\mathcal{R}(t)$ is the set of points which can reach to the target $\mathcal{S}$ with the control dynamics in time less than t, namely, for all $x \in \mathcal{R}(t)$

$$\mathcal{A}^t(x) \cap \mathcal{S} \neq \varnothing.$$

The set $\mathcal{R}$ is also called the *controllable set*.

Definition 2.3.2 *The system $(f,\mathcal{U})$ is small-time controllable on $\mathcal{S}$ (briefly STC$\mathcal{S}$) if $\mathcal{S} \subseteq \operatorname{int}\mathcal{R}(t)$ for all $t > 0$. If $\mathcal{S} = \{0\}$ this property is called small-time local controllability (STLC).*

Note that STLC is equivalent to the continuity of T in 0 if $\mathcal{S} = \{0\}$, since $T(0) = 0$ by definition. The next Proposition extends this observation to the general case.

Proposition 2.3.1 *Assume that the control system $(f,\mathcal{U})$ satisfies (H1), (H2) and the target $\mathcal{S}$ is compact. Then the following statements are equivalent:*

(i) the system $(f,\mathcal{U})$ is STC$\mathcal{S}$;

(ii) T is continuous in x for all $x \in \partial\mathcal{S}$;

(iii) there exists $\delta > 0$ and $\omega_T : [0,\delta] \to [0,\infty[$ such that $\lim_{s\to 0} \omega_T(s) = 0$ and $T(x) \leq \omega_T(d_{\mathcal{S}}(x))$ for all $x \in B(\mathcal{S},\delta) = \{z \in \mathbb{R}^N \mid d_{\mathcal{S}}(z) < \delta\}$.

Proof. The proof is in [12, Proposition 1.2, Chapter IV] . □

Remark 2.3.3 *Under assumption in Proposition 2.3.1, $T(x) > 0$ if and only if $x \notin \mathcal{S}$.*

Some consequence of STC$\mathcal{S}$ which is in [12, Chapter IV].

Proposition 2.3.2 *Under assumptions in Proposition 2.3.1, if the system $(f,\mathcal{U})$ is STC$\mathcal{S}$ then:*

(i) the controllable set $\mathcal{R}$ is open;

(ii) T is continuous in $\mathcal{R}$;

(iii) $\lim_{x\to x_0} T(x) = +\infty$ for any $x_0 \in \partial\mathcal{R}$.

We now introduce a special controllability condition which implies the Lipschitz continuity of the minimum time function.

Definition 2.3.3 *We say that the control system* $(f,\mathcal{U})$ *and the target* $\mathcal{S}$ *satisfy the Petrov condition if, for any* $R > 0$*, there exists* $\mu > 0$ *such that*

$$\min_{u\in\mathcal{U}} \langle f(x,u), \nu\rangle \leq -\mu\|\nu\|, \quad \forall x \in \partial\mathcal{S} \cap B(0,R), \nu \in N_S^P(x). \tag{2.3.6}$$

Theorem 2.3.2 *Under assumptions (H0),(H1) and the compactness of* $\mathcal{S}$*, if the control system* $(f,\mathcal{U})$ *then:*

(i) for any $R > 0$*, there exist* $\delta, k > 0$ *such that*

$$B(\mathcal{S},\delta) \cap B(0,R) \subset \mathcal{R} \ \text{ and } \ T(x) \leq kd_{\mathcal{K}}(x), \ x \in \mathcal{S},\delta) \cap B(0,R);$$

(ii) the minimum time function T *is locally Lipschitz continuous on* $\mathcal{R}$*.*

Proof. Standard reference for the proof is in [22, Chapter VIII]. □

We finally give in the subsection a theorem in which the minimum time function is just continuous under the weak Petrov condition.

Theorem 2.3.3 *Under the assumptions (H1), (H2) and the compactness of the target* $\mathcal{S}$*, if the control system satisfies the weak Petrov condition, i.e., there exist* $\delta > 0$ *and a continuous nondecreasing function* $\mu : [0,] \to [0,+\infty)$ *with the properties*

(a) $\mu(0) = 0$, $\mu(\rho) > 0$ *for* $\rho > 0$ *and* $\int_0^\delta \frac{d\rho}{\mu(\rho)} d\rho < +\infty$*;*

(b) for all $x \in B(\mathcal{S},\delta)\backslash\mathcal{S}$*, there exists* $\bar{s} \in \pi_{\mathcal{S}(x)}$ *such that*

$$\min_{u\in\mathcal{U}} \langle f(x,u), x - \bar{s}\rangle \leq -\mu(d_{\mathcal{S}}(x))d_{\mathcal{S}}(x).$$

Then the control system $(f,\mathcal{S})$ *is SCTS and* T *is continuous in* $\mathcal{R}$*.*

Proof. There are various versions, obtained with different methods, which can be found, e.g., in [21, 46, 49, 55, 66]. □

2.4 Differential inclusions

We shall give here some basic definitions and theorems which are needed in Subsection 3.7. For the reference, we refer to [10].
Consider differential inclusions of the form

$$\begin{cases} \dot{x}(t) \in F(x(t)), \\ x(0) = x_0, \end{cases} \tag{2.4.1}$$

where $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is a multifunction, and $x_0 \in \mathbb{R}^N$ is the starting point.

Definition 2.4.1 *Let* $y^{x_0}(\cdot) : (a,b) \to \mathbb{R}^N$ *where* $a < 0 < b$ *be such that* $y^{x_0}(0) = x_0$. *We say that* $y^{x_0}(\cdot)$ *is a solution of (2.4.1) if* $y^{x_0}(\cdot)$ *is absolutely continuous and*

$$\dot{y}^{x_0}(t) \in F(y^{x_0}(t)) \quad a.e.\ t \in (a,b).$$

The solution $y^{x_0}(\cdot)$ *is usually called a trajectory starting from* x_0 *and associated with (2.4.1).*

There are several results about the existence of solution to the differential inclusion (2.4.1). For the reference, we refer to Chapter 2 and Chapter 3 in [10].
The attainable $\mathcal{A}^T(x_0)$ from x_0 in time T is now denoted by

$$\mathcal{A}^T(x_0) = \{y^{x_0}(t)\ |0 \leq t \leq T\}.$$

We will end this subsection with a theorem which gives the existence of a minimal time T. In the following statement, we prefer to consider assumptions which will be used in Section 3.7.

The multifunction $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is Lipschitz with respect to the Hausdorff distance if there exists a constant $M > 0$ such that

$$d_{\mathcal{H}}(F(x, F(y))) \leq M\|y - x\|.$$

Theorem 2.4.1 *Assume that* F *is Lipschitz with respect to the Hausdorff distance and* $F(x)$ *is nonempty, convex, and compact for each* $x \in \mathbb{R}^n$. *If there exists a constant* $M_2 > 0$ *so that* $\max\{\|v\| \mid v \in F(x)\} \leq M_2(1 + \|x\|)$ *then for all* $T > 0$ *the attainable set* $\mathcal{A}^T(x_0)$ *is compact.*

Proof. One can find an original version and the proof in Section 2, Chapter 2 [10]. □

Part I

On the structure of the minimum time function

Chapter 3

Semiconcavity type results

We will first study in this Chapter a minimum time problem with a nonlinear smooth dynamics and a target satisfying an internal sphere condition. Under the assumptions that the minimum time T be continuous and the proximal normal cone to the hypograph of T be wedged, we show that $hypo(T)$ has positive reach. Consequently, T satisfies the list of properties in Theorem (2.2.2). In particular, T is a.e. twice differentiable.

The result is based on an analysis of how proximal normals (to the complement of the target) are transported by the adjoint flow, which in turn permits a representation of the generalized gradient of T in terms of suitable adjoint vectors (Theorems 3.2.1 and 3.2.2). Here the wedgedness assumption plays a major role: actually *exposed rays* of the normal cone to the hypograph are special normals, as they can be approximated by normals at differentiability points of T (Lemma 3.3.7). Moreover, wedgedness is used in Theorem 3.2.3 in order to obtain a uniform estimate for radii of the balls realizing proximal normals to the hypograph. We show also through an example (Example 2 in Section 3.6) that if the normal cone is not wedged, then Theorem 3.2.3 may fail. However, an external sphere condition to the hypograph of T still holds (see Proposition 3.2.1). An analysis of this general case will be discussed in Chapter 5 and Chapter 6 , where topological and measure theoretic results on the set where the normal cone is not wedged are given.

Moreover, on the basis in Chapter 5 and Chapter 6, we also study the minimum time function for a class of differential inclusions. For such class, under an internal sphere condition on the target $\mathcal{S}$ the hypograph of T still satisfies an external sphere condition. The proof will be based on the Hamiltonian function and Pontryagin's maximum principle. At the end of this chapter, we will partially extend our result to an arbitrary target $\mathcal{S}$.

The chapter is structured as follows: Section 3.1 is devoted to some notations, while Section 3.2 contains assumptions and statement of the main results. Detailed arguments begin in Section 3.3, which contains several lemmas whose geometrical meaning is illustrated, and ends with a result (Theorem 3.3.1) giving a representation of the normal cone to the hypograph of (T), under the wedgedness assumption. Section 3.4 is devoted to the conclusion of the proof of the main theorems, which is now only a simple use of the lemmas contained in Section 3.4. Section 3.5 is dedicated to an improvement of Theorems 3.2.1 and 3.2.2 for an *optimal point*, i.e. a point which is crossed by a time-optimal trajectory and Section 3.6 contains examples. Finally, our results will be extended for a class of differential inclusion in Section 3.7.

3.1 Nonlinear control system

We consider throughout the chapter a nonlinear control system of the form

$$\begin{cases} \dot{y}(t) \ = \ f(y(t), u(t)) & a.e. \\ u(t) \ \in \mathcal{U} & a.e. \\ y(0) \ = \ x, & \end{cases} \tag{3.1.1}$$

where the Lipschitz function $f : \mathbb{R}^N \times \mathcal{U} \longrightarrow \mathbb{R}^N$ and the control set $\mathcal{U}$, a compact nonempty subset of $\mathbb{R}^m$, are given. We recall that $\mathcal{U}_{\text{ad}}$ the set of admissible controls, i.e., the measurable functions $u : \mathbb{R} \to \mathbb{R}^m$, such that $u(t) \in \mathcal{U}$ a.e. For any $u(\cdot) \in \mathcal{U}_{\text{ad}}$, the unique Carathéodory solution of (3.1.1) is denoted by $y^{x,u}(\cdot)$.
The adjoint vectors associated with a trajectory $y^{x,u}(\cdot)$ can be represented using the fundamental solution matrix $M(\cdot, x, u)$ of the linear equation

$$\dot{p}(t) = D_x f(y^{x,u}(t), u(t)) \ p(t), \quad p(0) = \mathbb{I}^{N\times N}. \tag{3.1.2}$$

We also define $M^{-1}(\cdot, x, u)$ to be the fundamental solution matrix of the time reversed adjoint equation

$$\dot{q}(t) = -q(t) \ D_x f(y^{x,u}(t), u(t)), \quad q(0) = \mathbb{I}^{N\times N}. \tag{3.1.3}$$

Suppose we are now given a closed nonempty set $\mathcal{S} \subset \mathbb{R}^N$, which is called the target. For a fixed $x \in \mathbb{R}^N \setminus \mathcal{S}$, we define

$$\theta(x, u) := \min \ \{t \geq 0 \mid y^{x,u}(t) \in \mathcal{S}\}.$$

Of course, $\theta(x, u) \in (0, +\infty]$, and $\theta(x, u)$ is the time taken for the trajectory $y^{x,u}(\cdot)$ to reach $\mathcal{S}$, provided $\theta(x, u) < +\infty$. The *minimum time* $T(x)$ to reach $\mathcal{S}$ from x is defined by

$$T(x) := \inf \ \{\theta(x, u) \mid u(\cdot) \in \mathcal{U}_{\text{ad}}\}. \tag{3.1.4}$$

In general, an *optimal trajectory*, i.e., a trajectory which attains the infimum in (4.1.4) does not exist. Therefore, we need also to consider *minimizing* *sequences* and *limiting optimal trajectories* steering x to the target $\mathcal{S}$. In particular, we will consider the limits of end-points (thus belonging to $\mathcal{S}$) of minimizing sequences of trajectories. More precisely,

$$\begin{aligned}\mathcal{S}_x \quad = \quad & \{\bar{x} \in \mathcal{S} \mid \text{there exist sequences } \{x_n\} \subset \mathcal{S}^c \text{ and } \{\bar{u}_n(\cdot)\} \subset \mathcal{U}_{\mathrm{ad}} \\ & \quad \text{such that } x_n \to x,\ \theta(x_n, \bar{u}_n) \to T(x),\ y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)) \to \bar{x}\}.\end{aligned}$$

Observe that if $T(x) < +\infty$, then $\emptyset \neq \mathcal{S}_x \subseteq \mathrm{bdry}\mathcal{S}$.
For any $\bar{x} \in \mathcal{S}_x$ we define also

$$\begin{aligned}\mathcal{U}_{\bar{x}} \quad = \quad & \{\{\bar{u}_n(\cdot)\} \subset \mathcal{U}_{ad} \mid \text{there exists a sequence } \{x_n\} \text{ satisfying} \\ & \quad x_n \to x,\, \theta(x_n, \bar{u}_n) \to T(x),\ \text{and } y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)) \to \bar{x}\},\end{aligned}$$

i. e., the set of *minimizing sequences of controls* steering x to $\bar{x}$. Together with $\mathcal{U}_{\bar{x}}$ we define also

$$\begin{aligned}\mathcal{T}_{\bar{x}} \quad = \quad & \{\{y^{x_n,\bar{u}_n}(\cdot)\} \mid x_n \to x,\ \bar{u}_n \in \mathcal{U}_{\mathrm{ad}}, \\ & \quad \theta(x_n, \bar{u}_n) \to T(x),\ \text{and } y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)) \to \bar{x}\},\end{aligned}$$

i. e., the set of trajectories corresponding to minimizing sequences of controls steering x to $\bar{x}$.
Correspondingly, the *limiting adjoint trajectories* related to *minimizing sequences of controls* are defined by the following

$$\begin{aligned}\mathcal{M}_{\bar{x}} = \ & \{M : [0, T(x)] \to \mathbb{M}^{N\times N} \mid \exists\ \{y^{x_n,\bar{u}_n}(\cdot)\} \subset \mathcal{T}_{\bar{x}} \text{ such that} \\ & \quad M(\cdot) \text{ is uniform limit on } [0, T(x)] \text{ of } M(\cdot, x_n, \bar{u}_n)\}.\end{aligned} \tag{3.1.5}$$

Remark 3.1.1 *If $T(\cdot)$ is everywhere finite, both S_x, $\mathcal{T}_{\bar{x}}$ are nonempty. By compactness, $\mathcal{M}_{\bar{x}}$ is nonempty as well for all $\bar{x} \in \mathcal{S}_x$. Moreover, if $F(x) := \{f(x,u) | u \in \mathcal{U}\}$ is convex for all x, then the infimum is attained and the sets $\mathcal{S}_x$, $\mathcal{U}_{\bar{x}}$, and $\mathcal{T}_{\bar{x}}$ can be substituted by the simpler sets*

$$\begin{aligned}\mathcal{S}_x \quad &= \quad \{\bar{x} \in \mathcal{S} \mid \textit{there exists } \bar{u} \in \mathcal{U}_{ad} \textit{ such that} \\ & \qquad \theta(x, \bar{u}) = T(x), \bar{x} = y^{x,\bar{u}}(T(x))\} \\ \mathcal{U}_{\bar{x}} \quad &= \quad \{\bar{u} \in \mathcal{U}_{ad} \mid \theta(x, \bar{u}) = T(x), y^{x,\bar{u}}(T(x)) = \bar{x}\} \\ \mathcal{T}_{\bar{x}} \quad &= \quad \{y^{x,\bar{u}} \mid \bar{u} \in \mathcal{U}_{\bar{x}}\}.\end{aligned}$$

Finally, the *Maximized Hamiltonian*, namely the function

$$H : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}, \quad H(x, p) = \max_{u\in\mathcal{U}} \langle f(x, u), p\rangle\,,$$

will be important in our analysis.

3.2 Statement of the main results

We repeat first the setting we are concerned with and specify our assumptions.
We consider the nonlinear system (3.1.1) under the following assumptions:

(H1) $\mathcal{U} \subset \mathbb{R}^N$ is compact.

(H2) $f : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^N$ is continuous and satisfies:

$$\|f(x,u) - f(y,u)\| \leq L\,\|x-y\| \quad \forall\, x,y \in \mathbb{R}^N,\ u \in \mathcal{U},$$

for a positive constant L. Moreover, the differential of f with respect to the x variable, $D_x f$, exists everywhere, is continuous with respect to both x and u and satisfies the following Lipschitz condition:

$$\|D_x f(x,u) - D_x f(y,u)\| \leq L_1\,\|x-y\| \quad \forall\, x,y \in \mathbb{R}^N,\ u \in \mathcal{U},$$

for a positive constant L_1.

(H3) The minimum time function $T : \mathbb{R}^N \longrightarrow [0,+\infty)$ is everywhere finite and continuous, (i.e. controllability and small time controllability hold).

(H4) The target $\mathcal{S}$ is nonempty, closed, and satisfies the internal sphere condition of radius $\rho > 0$.

Remark 3.2.1 *Conditions ensuring small time controllability when the target is not necessarily a singleton can be found in the previous chapter.*

Our analysis will be based on the transportation of certain vectors, normal to the closure of the complement of the target $\mathcal{S}$, by means of the (limiting) adjoint flow. More precisely, two sets of transported normals will be considered, according with the Hamiltonian:

$$N_0(x) = \{M^T(r)v \mid M(\cdot) \in \mathcal{M}_{\bar{x}}, v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x}), \bar{x} \in \mathcal{S}_x \text{ and } H(M^T(r)v, x) = 0\}$$

$$N_1(x) = \{M^T(r)v \mid M(\cdot) \in \mathcal{M}_{\bar{x}}, v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x}), \bar{x} \in \mathcal{S}_x \text{ and } H(M^T(r)v, x) = 1\}$$

Our main results are the following three theorems, together with the corollary.

Theorem 3.2.1 *Let $x \in \mathcal{S}^c$ and $r = T(x)$. Under the conditions (H1), (H2), (H3), and (H4), together with the further assumption*

$$N^P_{hypo(T)}(x, T(x)) \text{ is wedged,} \tag{3.2.1}$$

the (proximal) horizontal supergradient of the minimum time function $T(\cdot)$ at the point x can be computed as follows:

$$\partial^\infty T(x) \;=\; -\mathrm{co}(N_0(x)). \tag{3.2.2}$$

Theorem 3.2.2 *Let $x \in \mathcal{S}^c$ and $r = T(x)$. Under the same assumptions of Theorem 3.2.1, the proximal supergradient of the minimum time function at the point x can be computed as follows:*

$$\partial^P T(x) \;=\; -[\mathrm{co}(N_1(x)) + \mathrm{co}(N_0(x))]. \tag{3.2.3}$$

Theorem 3.2.3 *Let the assumptions of Theorem 3.2.1 hold for all $x \in \mathcal{S}^c$. Then for every closed set $\mathcal{S}' \subset \mathcal{S}^c$, hypo$(T) \cap (\mathcal{S}' \times \mathbb{R})$ has positive reach.*

Corollary 3.2.1 *Let the assumptions of Theorem 3.2.1 hold. Then the minimum time function T satisfies all the properties listed in Theorem 2.2.2.*

The last result is concerned with the case where the wedgedness assumption (3.2.1) does not hold. We will present here, for the sake of brevity, only a partial result together with two examples, a thorough analysis being postponed to the next chapter.

Proposition 3.2.1 *Let the assumptions (H1), (H2) , (H3), and (H4) hold. Then the hypograph of the minimum time function T satisfies the external sphere condition with a locally uniform radius, namely for every $x \in \mathcal{S}^c$ there exists a unit proximal normal v to hypo(T) at $(x, T(x))$ which is realized by a sphere with a locally constant radius $\sigma > 0$.*

The proof of the previous Proposition is a straightforward consequence of Lemmas 3.3.2 and 3.3.3.

Remark 3.2.2 *The constant σ can be explicitly computed, and depend only on x, on f and $\mathcal{U}$, and on the constants L, L_1 and ρ appearing in the assumptions (H2) and (H4).*

3.3 Preparatory Lemmas

This section is devoted to several partial results which are needed to prove Theorem 3.2.2 and Theorem 3.2.2. In particular, the proof of "$\supseteq$" inclusions in (3.2.1) and (3.2.2) will be based on Lemma 3.3.2 and Lemma 3.3.3 below.

3.3.1 Transporting proximal normals

In this subsection we do not assume that $\mathcal{S}$ satisfies the internal sphere condition, nor that the normal cone to the hypograph of $T(\cdot)$ at $(x, T(x))$ is wedged.

The following notation for sublevels of the minimum time function will be used: for $r > 0$ we set

$$\mathcal{S}(r) := \{x \in \mathbb{R}^N \mid T(x) < r\}$$
$$\mathcal{S}^c(r) := \{x \in \mathbb{R}^N \mid T(x) \geq r\}$$

We state first a technical lemma, showing that the limiting adjoint flow transports proximal normals to the complement of the target to proximal normals to the complement of sublevels of T. Moreover, the radius of the ball which realizes the transported normal can be explicitly estimated.

Lemma 3.3.1 *Assume that $\mathcal{S}$ is closed and let the assumptions (H1), (H2), and (H3) hold. Let $x \in \mathcal{S}^c$ and set $r = T(x) > 0$. Fix $\bar{x} \in \mathcal{S}_x$, $v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$ and $M(\cdot) \in \mathcal{M}_{\bar{x}}$. Then*

$$M^T(r)v \in N^P_{\mathcal{S}^c(r)}(x).$$

More precisely, assume that v is realized by a ball of radius $\rho > 0$. Then there exists an explicitly computable continuous function K depending only on r, $\|x\|$, ρ such that for all $z \in \mathcal{S}^c(r)$ we have

$$\left\langle M^T(r)v, z - x\right\rangle \leq \; K(r, \|x\|, \rho) \; \left\|M^T(r)v\right\| \|z - x\|^2 . \tag{3.3.1}$$

Proof. Let $x_n \to x$, $\bar{x} \in \mathcal{S}_x$, and $\{\bar{u}_n\} \subset \mathcal{U}_{\mathrm{ad}}$ be such that $\{y^{x_n,\bar{u}_n}(\cdot)\} \in \mathcal{T}_{\bar{x}}$ and $M(\cdot, x_n, \bar{u}_n)$ converges to $M(\cdot)$ uniformly on $[0, T(x)]$. By definition of proximal normal realized by a ρ-ball,

$$\langle v \;,\; \bar{z} - \bar{x}\rangle \;\leq\; \frac{\|v\|}{2\rho} \;\|\bar{z} - \bar{x}\|^2 \quad \text{for all } \bar{z} \in \overline{\mathcal{S}^c}.$$

Fix $z \in \mathcal{S}^c(r)$. We define

$$\bar{x}_n \;=\; y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)),\; \bar{z}_n \;=\; y^{z,\bar{u}_n}(\theta(x_n, \bar{u}_n)),$$

and observe that $\bar{x}_n \in \mathcal{S}$, $\bar{x}_n \to \bar{x}$ and we can assume without loss of generality that $\bar{z}_n$ converges to a point $\bar{z}$ which belongs to $\overline{\mathcal{S}^c}$ since $\theta(x_n, \bar{u}_n) \to r \leq T(z)$.

We set for simplicity $\alpha_n(\cdot) = y^{x_n,\bar{u}_n}(\cdot)$, $\beta_n(\cdot) = y^{z,\bar{u}_n}(\cdot)$, $t_n = \theta(x_n, \bar{u}_n)$, so that

$$\bar{x}_n = x_n + \int_0^{t_n} f(\alpha_n(s), \bar{u}_n(s))ds \;,\; \bar{z}_n \;=\; z + \int_0^{t_n} f(\beta_n(s), \bar{u}_n(s))ds,$$

whence

$$\bar{z}_n - \bar{x}_n = z - x_n + \int_0^{t_n}\!\left(\int_0^1 D_x f(\alpha_n(s) + \tau(\beta_n(s) - \alpha_n(s)), \bar{u}_n(s))d\tau\right)(\beta_n(s) - \alpha_n(s))ds.$$

We define now

$$\begin{aligned} A_n^1(s) &= D_x f(\alpha_n(s), \bar{u}_n(s)), \\ A_n^2(s) &= \int_0^1 D_x f(\alpha_n(s) + \tau(\beta_n(s) - \alpha_n(s)), \bar{u}_n(s))d\tau, \end{aligned}$$

and observe that, thanks to (H2), for all $s \in [0, t_n]$ we have

$$\left\|A_n^2(s) - A_n^1(s)\right\| \leq \frac{L_1}{2} \left\|\beta_n(s) - \alpha_n(s)\right\|. \tag{3.3.2}$$

Using (iv) in Lemma 7.1.1 and the definition of L_2 in (7.1.1), we obtain

$$\left\|A_n^1(s)\right\| \leq L_2(s, \|x_n\|) \tag{3.3.3}$$

for all $s \in [0, t_n]$. Thus

$$\left\|A_n^2(s)\right\| \leq L_2(s, \|x_n\|) + \frac{L_1}{2} \left\|\beta_n(s) - \alpha_n(s)\right\| \tag{3.3.4}$$

for all $s \in [0, t_n]$. Now, Gronwall's Lemma yields

$$\|\beta_n(s) - \alpha_n(s)\| \leq e^{Ls} \|z - x_n\|, \tag{3.3.5}$$

so that combining (3.3.4) and (3.3.5) we obtain

$$\left\|A_n^2(s)\right\| \leq L_2(s, \|x_n\|) + \frac{L_1}{2} e^{Ls} \|z - x_n\|. \tag{3.3.6}$$

Define $M_n^2(\cdot)$ to be the solution of the problem

$$\dot{p}(s) = A_n^2(t)p(s), \quad p(0) = \mathbb{I}^{N \times N}.$$

Recalling that $M(\cdot, x, u)$ is the fundamental solution of (3.1.2) set $M_n^1(\cdot) = M(\cdot, x_n, \bar{u}_n)$, $z_n^1(s) = M_n^1(s)(z - x_n)$ and $z_n^2(s) = M_n^2(s)(z - x_n)$, for all $s \in [0, t_n]$. Using these notations, we can write

$$\begin{aligned}
\langle v, \bar{z}_n - \bar{x}_n \rangle &= \left\langle v, z_n^2(t_n) \right\rangle \\
&= \left\langle v, z_n^1(t_n) \right\rangle + \left\langle v, z_n^2(t_n) - z_n^1(t_n) \right\rangle \\
&= \left\langle v, M_n^1(t_n)(z - x_n) \right\rangle + \left\langle v, (M_n^2(t_n) - M_n^1(t_n))(z - x_n) \right\rangle \\
&\geq \left\langle v, M_n^1(t_n)(z - x_n) \right\rangle - \|v\| \left\|(M_n^2(t_n) - M_n^1(t_n))(z - x_n)\right\|.
\end{aligned} \tag{3.3.7}$$

To simplify our writing, we set, for all $s \geq 0$ and $y, z \in \mathbb{R}^N$, $L_3(s, y, z) = \frac{L_1}{2} e^{Ls} \|z - y\|$. By (3.3.3), (3.3.6), Lemma 7.1.3, and (3.3.2), we have

$$\begin{aligned}
&\left\|(M_n^2(t_n) - M_n^1(t_n))(z - x_n)\right\| \\
&\quad \leq e^{[2L_2(t_n, \|x_n\|) + L_3(t_n, x_n, z)]t_n} \int_0^{t_n} \left\|A_n^2(s) - A_n^1(s)\right\| ds \ \|z - x_n\| \\
&\quad \leq \frac{L_1}{2} e^{[2L_2(t_n, \|x_n\|) + L_3(t_n, x_n, z)]t_n} \int_0^{t_n} \|\beta_n(s) - \alpha_n(s)\| ds \ \|z - x_n\|.
\end{aligned}$$

Recalling (3.3.5), we obtain

$$\begin{aligned}&\left\|(M_n^2(t_n)-M_n^1(t_n))(z-x_n)\right\| \\ &\qquad\leq \frac{L_1}{2}e^{[2L_2(t_n,\|x_n\|)+L_3(t_n,x_n,z)+L]t_n}\left\|z-x_n\right\|^2 .\end{aligned} \tag{3.3.8}$$

Therefore, by passing to the limit in (3.3.7) and (3.3.8) (recall that $M_n^1(\cdot)\to M(\cdot)$ uniformly), we have

$$\begin{aligned}&\left\langle M^T(r)v, z-x\right\rangle \\ &\qquad\leq \langle v,\bar z-\bar x\rangle+\|v\|\,\frac{L_1}{2}e^{[2L_2(r,\|x\|)+L_3(r,x,z)+L]r}\left\|z-x\right\|^2 \\ &\qquad\leq \frac{\|v\|}{2\rho}\,\left\|\bar z-\bar x\right\|^2+\|v\|\,\frac{L_1}{2}e^{[2L_2(r,\|x\|)+L_3(r,x,z)+L]r}\,\left\|z-x\right\|^2 .\end{aligned}$$

Moreover, from (3.3.5) we have $\|\bar z-\bar x\|\leq e^{Lr}\ \|z-x\|$. Therefore,

$$\left\langle M^T(r)v, z-x\right\rangle \leq \left(\frac{L_1}{2}e^{[2L_2(r,\|x\|)+L_3(r,x,z)+L]r}+\frac{e^{2Lr}}{2\rho}\right)\|v\|\left\|z-x\right\|^2 \tag{3.3.9}$$

for all $z\in\mathcal{S}^c(r)$.
Observe that

$$\begin{aligned}\|v\| &= \left\|(M^T(r))^{-1}M^T(r)v\right\| \\ &\leq \left\|M(r)^{-1}\right\|\left\|M^T(r)v\right\| .\end{aligned}$$

By (ii) in Lemma 7.1.2 we obtain

$$\left\|M(r)^{-1}\right\| \;\leq\; e^{L_2(r,\|x\|)r}.$$

Combining the above inequalities with (3.3.9) we thus have

$$\begin{aligned}&\left\langle M^T(r)v, z-x\right\rangle \\ &\leq \left(\frac{L_1}{2}e^{[3L_2(r,\|x\|)+L_3(r,x,z)+L]r}+\frac{e^{2Lr+L_2(r,\|x\|)}}{2\rho}\right)\ \left\|M^T(r)v\right\|\left\|z-x\right\|^2 .\end{aligned} \tag{3.3.10}$$

In order to complete the proof, we consider two cases.

If $\|z-x\|<1$, then $L_3(r,x,z)\leq\frac{L_1}{2}e^{Lr}$. Thus, by (3.3.10) we have

$$\begin{aligned}&\left\langle M^T(r)v, z-x\right\rangle \\ &\quad\leq \left(\frac{L_1}{2}e^{[3L_2(r,\|x\|)+\frac{L_1}{2}e^{Lr}+L]r}+\frac{e^{2Lr+L_2(r,\|x\|)}}{2\rho}\right)\left\|M^T(r)v\right\|\left\|z-x\right\|^2 .\end{aligned} \tag{3.3.11}$$

If instead $\|z-x\| \geq 1$, then $\langle M^T(r)v, z-x \rangle \leq \|M^T(r)v\| \, \|z-x\|^2$. Therefore, in both cases we have that

$$\langle M^T(r)v, z-x \rangle \leq K(r, \|x\|, \rho) \ \|M^T(r)v\| \ \|z-x\|^2 \qquad \text{for all } z \in \mathcal{S}^c(r), \tag{3.3.12}$$

where the continuous function K, defined for $r, \delta \geq 0$ and $\rho > 0$ as

$$K(r,\delta,\rho) := \max\left\{1, \frac{L_1}{2} e^{[3L_2(r,\delta)+\frac{L_1}{2}e^{Lr}+L]r} + \frac{e^{2Lr+L_2(r,\delta)}}{2\rho}\right\}, \tag{3.3.13}$$

depends only on the variables r, δ, ρ and on the constants L, L_1, K_1, K_2. The proof is complete. □

Remark 3.3.1 *It follows from (3.3.13) that $K(r,\delta,\rho)$ is nondecreasing with respect to both r and δ.*

The next lemma establishes that normals transported along the *limiting adjoint flow* generate horizontal proximal normals to the hypograph of $T(\cdot)$, provided their Hamiltonian is zero. Moreover, the radius of the ball realizing them can be explicitly estimated.

Lemma 3.3.2 *Let $\mathcal{S}$ be closed and let the assumptions (H1), (H2), and (H3) hold. Let $x \in \mathcal{S}^c$, set $r := T(x) > 0$, and let $\xi \in N_0(x)$. Then $-\xi \in \partial^\infty T(x)$, or, equivalently, $(\xi, 0) \in N^P_{hypo(T(x))}(x, T(x))$.*

More precisely, let $\bar{x} \in \mathcal{S}_x$ and let $v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$, $M(\cdot) \in \mathcal{M}_{\bar{x}}$ be such that $H(M^T(r)v, x) = 0$. Assume that v is realized by a ball of radius ρ. Then there exists an explicitly computable continuous function $K_3(r,x,\rho)$, depending only on r, x, ρ, such that for all $z \in \overline{\mathcal{S}^c}$ and all $\beta \leq T(z)$ we have

$$\langle M^T(r)v, z-x \rangle \leq \ K_3(r,x,\rho) \ \|M^T(r)v\| \left(\|z-x\|^2 + |\beta - T(x)|^2\right). \tag{3.3.14}$$

Proof. Let $v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$ be such that

$$\langle v, \bar{z} - \bar{x} \rangle \leq \frac{\|v\|}{2\rho} \|\bar{z} - \bar{x}\|^2 \quad \forall \bar{z} \in \overline{\mathcal{S}^c}. \tag{3.3.15}$$

Recalling Lemma 3.3.1, for all $z \in \mathcal{S}^c(r)$ we have

$$\langle M^T(r)v, z-x \rangle \leq K(r, \|x\|, \rho) \, \|M^T(r)v\| \, \|z-x\|^2. \tag{3.3.16}$$

Let $z \in \overline{\mathcal{S}^c}$. Two cases may occur:
(i) $T(z) \geq T(x)$,
(ii) $T(z) < T(x)$.
In the first case, (3.3.14) follows immediately from (3.3.16).
In the second case, define $r_1 = T(z)$ and take sequences $\{x_n\}$, with $x_n \to x$,

$\{\bar{u}_n\} \subset \mathcal{U}_{\rm ad}$ and $\{\alpha_n(\cdot) := y^{x_n,\bar{u}_n}(\cdot)\}$ corresponding to $M(\cdot)$, according to the definition given in (3.1.5). For all n large enough there exists $r_n^1 < r$ for which

$$\bar{x}_n^1 := \alpha_n(r - r_n^1) = x_n + \int_0^{r-r_n^1} f(\alpha_n(s), \bar{u}_n(s))ds$$

is such that $T(\bar{x}_n^1) = r_1$. We can assume without loss of generality that $\alpha_n(\cdot)$ converges uniformly to some $\alpha(\cdot)$ and that $r_n^1 \to \bar{r}_1$. Observe that $\bar{r}_1 < r$. Setting $\bar{x}^1 = \alpha(r - \bar{r}_1)(= \lim \bar{x}_n^1)$, one can easily see that $T(\bar{x}^1) = r_1$ by the continuity of $T(x)$. Then, by Lemma 3.3.1 we obtain that

$$\left\langle M^T(r_1)v, z - \bar{x}^1 \right\rangle \leq K(r_1, \left\|\bar{x}^1\right\|, \rho) \left\|M^T(r_1)v\right\| \left\|z - \bar{x}^1\right\|^2. \tag{3.3.17}$$

We write

$$\left\langle M^T(r)v, z - x \right\rangle = \left\langle M^T(r)v, z - \bar{x}^1 \right\rangle + \left\langle M^T(r)v, \bar{x}^1 - x \right\rangle$$

and perform some estimates.
First, we consider

$$\left\langle M^T(r)v, z - \bar{x}^1 \right\rangle = \left\langle M^T(r_1)v, z - \bar{x}^1 \right\rangle + \left\langle (M^T(r) - M^T(r_1))v, z - \bar{x}^1 \right\rangle.$$

By (3.3.17) we have

$$\begin{aligned}\left\langle M^T(r)v, z - \bar{x}^1 \right\rangle \leq\ & K(r_1, \left\|\bar{x}^1\right\|, \rho) \left\|M^T(r_1)v\right\| \left\|z - \bar{x}^1\right\|^2 \\ & + \left\|(M^T(r) - M^T(r_1))v\right\| \left\|z - \bar{x}^1\right\|.\end{aligned}$$

Moreover from (ii) in Lemma 7.1.2 we have

$$\begin{aligned}\left\|M^T(r_1)v\right\| &\leq \left\|(M^T(r - r_1))^{-1}\right\| \left\|M^T(r)v\right\| \\ &\leq e^{L_2(r-r_1,\|x\|)(r-r_1)} \left\|M^T(r)v\right\| \\ &\leq e^{L_2(r,\|x\|)r} \left\|M^T(r)v)\right\|.\end{aligned}$$

Also, using (iv) in Lemma 7.1.1 we obtain

$$\begin{aligned}\left\|(M^T(r) - M^T(r_1))v\right\| &\leq \int_{r_1}^r \left\|\dot{M}^T(s)v\right\| ds \\ &\leq \int_{r_1}^r e^{L_2(r,\|x\|)r} \left\|M^T(r)v)\right\| ds \\ &= e^{L_2(r,\|x\|)r} \left\|M^T(r)v)\right\| |r - r_1|.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle M^T(r)v, z - \bar{x}^1 \right\rangle \leq\ & K(r_1, \left\|\bar{x}^1\right\|, \rho)\, e^{L_2(r,\|x\|)r} \left\|M^T(r)v\right\| \left\|z - \bar{x}^1\right\|^2 \\ & + e^{L_2(r,\|x\|)r} \left\|(M^T(r)v)\right\| |r - r_1| \left\|z - \bar{x}^1\right\|.\end{aligned}$$

Recalling (i) in Lemma 7.1.1 for $\alpha(\cdot) = y^{x_n, \bar{u}_n(\cdot)}$, $t = r - r_1$, and then taking $n \to \infty$, we obtain

$$\left\|\bar{x}^1 - x\right\| \leq \frac{(L\|x\| + K_1)(e^{L(r-r_1)} - 1)}{L} \leq \frac{(L\|x\| + K_1)(e^{Lr} - 1)}{L}, \quad (3.3.18)$$

from which it follows that $\left\|\bar{x}^1\right\| \leq e^{Lr}\|x\| + \frac{(e^{Lt}-1)K_1}{L}$. Hence,

$$\begin{aligned}\left\langle M^T(r)v, z - \bar{x}^1\right\rangle \; &\leq R_1(r, \|x\|, \rho)\; e^{L_2(r,\|x\|)r} \left\|M^T(r)v\right\| \left\|z - \bar{x}^1\right\|^2 \\ &\quad + e^{L_2(r,\|x\|)r} \left\|M^T(r)v)\right\| \; |r - r_1| \; \left\|z - \bar{x}^1\right\|,\end{aligned} \quad (3.3.19)$$

where

$$R_1(r, \delta, \rho) = K\left(r, e^{Lr}\delta + \frac{(e^{Lt} - 1)K_1}{L}, \rho\right), \qquad \text{for } r, \delta \geq 0, \rho > 0.$$

Observe also that we obtain from (iii) in Lemma 7.1.1 that

$$\begin{aligned}\left\|z - \bar{x}^1\right\| \;&\leq\; \lim_{n\to\infty}\left(\|z - x_n\| + \int_0^{r-r_n^1} \|f(\alpha_n(s), \bar{u}_n(s))\|\, ds\right) \\ &\leq\; \lim_{n\to\infty}\left(\|z - x_n\| + \int_0^{r-r_n^1} \left(Le^{Ls}\|x_n\| + e^{Ls}K_1\right) ds\right) \\ &\leq\; \|z - x\| + L_4(r, \|x\|)\; |r - r_1|,\end{aligned}$$

where $L_4(s, \delta) = Le^{Ls}\delta + e^{Ls}K_1$ for $s, \delta \geq 0$.
Combining the above inequality and (3.3.19), we obtain

$$\left\langle M^T(r)v, z - \bar{x}^1\right\rangle \leq R_2(r, \|x\|, \rho)\left\|M^T(r)v\right\|(\|z - x\|^2 + |r - r_1|^2), \quad (3.3.20)$$

where we have defined, for $r, \delta \geq 0$, $\rho > 0$,

$$R_2(r, \delta, \rho) = e^{L_2(r,\delta)r}\left(2R_1(r, \delta, \rho)\Big(\frac{3}{2} + L_4^2(r, \delta)\Big) + L_4(r, \delta)\right). \quad (3.3.21)$$

Second, we consider

$$\begin{aligned}&\left\langle M^T(r)v, \bar{x}_n^1 - x\right\rangle \\ &\quad = \left\langle M^T(r)v, x_n - x\right\rangle + \left\langle M^T(r)v, \int_0^{r-r_n^1} f(\alpha_n(s), \bar{u}_n(s))ds\right\rangle \\ &\quad = \left\langle M^T(r)v, x_n - x\right\rangle + \left\langle M^T(r)v, \int_0^{r-r_n^1} f(x, \bar{u}_n(s))ds\right\rangle \\ &\qquad + \left\langle M^T(r)v, \int_0^{r-r_n^1} \left(f(\alpha_n(s), \bar{u}_n(s)) - f(x, \bar{u}_n(s))\right) ds\right\rangle.\end{aligned}$$

Recalling that $H(M^T(r)v,x)=0$, we obtain from the above expression that

$$\begin{aligned}
&\langle M^T(r)v, \bar{x}_n^1 - x\rangle \leq \langle M^T(r)v, x_n - x\rangle + \\
&\qquad \left\langle M^T(r)v, \int_0^{r-r_n^1} \left(f(\alpha_n(s),\bar{u}_n(s)) - f(x,\bar{u}_n(s))\right) ds \right\rangle \\
&\leq \left\|M^T(r)v\right\| \left(\|x_n - x\| + \int_0^{r-r_n^1} \|f(\alpha_n(s),\bar{u}_n(s)) - f(x,\bar{u}_n(s))\| \, ds \right) \\
&\qquad \leq \left\|M^T(r)v\right\| \left(\|x_n - x\| + L\int_0^{r-r_n^1} \|\alpha_n(s) - x\| \, ds \right) \\
&\leq \left\|M^T(r)v\right\| \left((L+1)\|x_n - x\| + L\int_0^{r-r_n^1}\int_0^s \|f(\alpha_n(\tau),\bar{u}_n(\tau))\| \, d\tau ds \right).
\end{aligned}$$

By (iii) in Lemma 7.1.1, recalling that $\bar{r}_1 < r$ we now obtain that

$$\begin{aligned}
\left\langle M^T(r)v, \bar{x}_n^1 - x\right\rangle \leq \left\|M^T(r)v\right\| \Big(&(L+1)\|x_n - x\| \\
&+ L\int_0^{r-r_1}\int_0^s \left(Le^{Lr}\|x_n\| + e^{Lr}K_1\right) d\tau ds \Big),
\end{aligned}$$

whence, taking $n \to \infty$,

$$\langle M^T(r)v, \bar{x}^1 - x\rangle \leq \frac{L(Le^{Lr}\|x\| + e^{Lr}K_1)}{2} \left\|M^T(r)v\right\| |r - r_1|^2. \tag{3.3.22}$$

Set now, for $r, \delta \geq 0$, $\rho > 0$,

$$K_3(r,\delta,\rho) = R_2(r,\delta,\rho) + \frac{L(Le^{Lr}\delta + e^{Lr}K_1)}{2}. \tag{3.3.23}$$

Recalling (3.3.20) and (3.3.22), the proof is complete. □

Now we prove a similar result for normals such that the Hamiltonian along the *limiting adjoint flow* is 1. Actually, if ξ is such a vector, we show that $(\xi, 1)$ is a proximal normal to the hypograph of $T(\cdot)$, and again the radius of the sphere which realizes it can be explicitly estimated.

Lemma 3.3.3 *Let $\mathcal{S}$ be closed and let the assumptions (H1), (H2), and (H3) hold. Let $x \in \mathcal{S}^c$, set $r := T(x) > 0$, and let $\xi \in N_1(x)$. Then $-\xi \in \partial^P T(x)$, or, equivalently, $(\xi,1) \in N^P_{hypo(T(x))}(x,T(x))$.*
More precisely, let $\bar{x} \in \mathcal{S}_x$ and let $v \in N^P_{\mathcal{S}^c}(\bar{x})$, $M(\cdot) \in \mathcal{M}_{\bar{x}}$ be such that $H(M^T(r)v,x) = 1$ and assume that v is realized by a ball of radius $\rho > 0$. Then there exists an explicitly computable continuous function $K_6(r,\|x\|,\rho)$ depending only on r, $\|x\|$, ρ such that for all $z \in \overline{\mathcal{S}^c}$ and all $\beta \leq T(z)$ we have

$$\langle M^T(r)v, z - x\rangle + \beta - r \leq K_6(r,\|x\|,\rho) \left\|(M^T(r)v,1)\right\| (\|z-x\|^2 + |\beta - r|^2). \tag{3.3.24}$$

Proof. Let $v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$ be such that

$$\langle v, \bar{z} - \bar{x} \rangle \leq \frac{\|v\|}{2\rho} \|\bar{z} - \bar{x}\|^2 \quad \forall \bar{z} \in \overline{\mathcal{S}^c}.$$

Let $z \in \mathcal{S}^c$. Two cases may occur:
(i) $T(z) \geq T(x)$,
(ii) $T(z) < T(x)$.
First case. Recalling that $H(M^T(r)v, x) = 1$, one can find $\bar{u} \in \mathcal{U}$ such that

$$\left\langle M^T(r)v, f(x, \bar{u}) \right\rangle = 1.$$

Set $z_{\bar{u}}(\cdot) := y^{z,\bar{u}}(\cdot)$ to be the trajectory starting from z with the constant control $\bar{u}$, namely $z_{\bar{u}}(t) = z + \int_0^t f(z_{\bar{u}}(s), \bar{u})ds$.

Taking $T(x) \leq r_1 \leq T(z)$, we have that $z_{\bar{u}}(r_1 - r) \in \mathcal{S}^c(r)$. Recalling Lemma 3.3.1, we obtain that

$$\left\langle M^T(r)v, z_{\bar{u}}(r_1 - r) - x \right\rangle \leq K(r, \|x\|, \rho) \, \left\|M^T(r)v\right\| \|z_{\bar{u}}(r_1 - r) - x\|^2 . \tag{3.3.25}$$

We estimate

$$\begin{aligned} \left\langle M^T(r)v, z - z_{\bar{u}}(r_1 - r) \right\rangle &= \left\langle M^T(r)v, -\int_0^{r_1 - r} f(z_{\bar{u}}(t), \bar{u})dt \right\rangle \\ &= \left\langle M^T(r)v, -\int_0^{r_1 - r} f(x, \bar{u})dt \right\rangle \\ &+ \left\langle M^T(r)v, \int_0^{r_1 - r} \left(f(x, \bar{u}) - f(z_{\bar{u}}(t), \bar{u})\right) dt \right\rangle \\ &\leq r - r_1 + L \left\|M^T(r)v\right\| \int_0^{r_1 - r} \|z_{\bar{u}}(t) - x\| \, dt. \end{aligned}$$

Combining the above inequality with (3.3.25) we get

$$\begin{aligned} \left\langle M^T(r)v, z - x \right\rangle \leq r - r_1 + L \left\|M^T(r)v\right\| \int_0^{r_1 - r} \|z_{\bar{u}}(t) - x\| \, dt \\ + K(r, \|x\|, \rho) \left\|M^T(r)v\right\| \|z_{\bar{u}}(r_1 - r) - x\|^2 . \end{aligned} \tag{3.3.26}$$

Moreover,

$$\begin{aligned} \|z_{\bar{u}}(s) - x\| &\leq \|z - x\| + \int_0^s \|f(z_{\bar{u}}(\tau), \bar{u})\| \, dt \\ &\leq \|z - x\| + \tilde{K}(\|x\|)s + L \int_0^s \|z_{\bar{u}}(\tau) - x\| \, d\tau, \end{aligned}$$

where we set, for $\delta \geq 0$, $\tilde{K}(\delta) := L\delta + K_1$. Thus, Gronwall's inequality yields, for all $0 \leq s \leq r_1 - r$,

$$\|z_{\bar{u}}(s) - x\| \leq e^{Ls} \|z - x\| + \tilde{K}(\|x\|)\Big(s + \frac{e^{Ls} - Ls - 1}{L}\Big). \tag{3.3.27}$$

Since $e^{Ls} - Ls - 1 \le L(e^L - 1)s$ for all $s \in [0,1]$, we obtain from (3.3.27)

$$\|z_{\bar{u}}(s) - x\| \le e^L \|z - x\| + \tilde{K}(\|x\|)e^L s \quad \text{for all } s \in [0,1]. \tag{3.3.28}$$

Now we consider two subcases.

First subcase: $0 \le r_1 - r \le 1$. Combining (3.3.28) with (3.3.26) we obtain

$$\left\langle M^T(r)v, z - x\right\rangle + r_1 - r \le K_5(r, \|x\|, \rho) \left\|M^T(r)v\right\| (\|z - x\|^2 + |r_1 - r|^2), \tag{3.3.29}$$

where for $r, \delta \ge 0$, $\rho > 0$ we set

$$K_5(r, \delta, \rho) = e^L \left(\frac{L}{2} + 2e^L K(r, \delta, \rho)\big(1 + \tilde{K}(\delta)^2\big) + \frac{\tilde{K}(\delta)}{2} \right). \tag{3.3.30}$$

Second subcase: $r_1 - r > 1$. Recalling Lemma 3.3.1, we obtain

$$\begin{aligned} &\left\langle M^T(r)v, z - x\right\rangle + r_1 - r \\ &\qquad \le (K(r, \|x\|, \rho) + 1) \left\|(M^T(r)v, 1)\right\| (\|z - x\|^2 + |r_1 - r|^2). \end{aligned} \tag{3.3.31}$$

Observe now that, if $\beta \le T(x)$, recalling Lemma 3.3.1 we have

$$\left\langle M^T(r)v, z - x\right\rangle + \beta - T(x) \le K(r, \|x\|, \rho) \ \left\|M^T(r)v\right\| (\|z - x\|^2 + |\beta - T(x)|^2). \tag{3.3.32}$$

We are now ready to conclude the first case. Indeed, it suffices to combine (3.3.29), (3.3.32), and (3.3.31) and recall (3.3.30), obtaining, for all $z \in \mathcal{S}^c(r)$ and $\beta \le T(z)$,

$$\begin{aligned} &\left\langle M^T(r)v, z - x\right\rangle + \beta - T(x) \\ &\quad \le (K_5(r, \|x\|, \rho) + 1) \left\|(M^T(r)v, 1)\right\| (\|z - x\|^2 + |\beta - T(x)|^2). \end{aligned} \tag{3.3.33}$$

Second case. It is entirely similar to the proof of the second case of Lemma (3.3.2). Indeed, by using the condition $H(M^T(r)v, x) = 1$ we can replace (3.3.22) with

$$\left\langle M^T(r)v, \bar{x}^1 - x\right\rangle \le T(x) - T(z) + \frac{L(Le^{Lr}\|x\| + e^{Lr}K_1)}{2}|r - r_1|^2. \tag{3.3.34}$$

Then, combining (3.3.20) and (3.3.34) we obtain

$$\begin{aligned} &\left\langle M^T(r)v, z - x\right\rangle \ + \beta - T(x) \\ &\qquad \le K_3(r, \|x\|, \rho) \left\|M^T(r)v\right\| (\|z - x\|^2 + |\beta - T(x)|^2) \end{aligned} \tag{3.3.35}$$

for all $\beta \le T(z)$, $z \in \overline{\mathcal{S}^c}$ and $T(z) \le T(x)$.
To conclude the proof of the Lemma we recall (3.3.35), (3.3.33), (3.3.30) and set, for $r, \delta \ge 0$, $\rho > 0$,

$$K_6(r, \delta, \rho) = \max\{K_5(r, \delta, \rho) + 1, K_3(r, \delta, \rho)\}. \tag{3.3.36}$$

□

The next subsection is to show that singularities of T may be only of "*upwards type*". Assuming that the target satisfies the internal sphere condition of radius ρ, we show that if ξ belongs to the proximal subgradient of $T(\cdot)$ at x, then it belongs also to the proximal supergradient. Moreover $-\xi$ is the transported vector by the limiting adjoint flow of a normal to $\mathcal{S}^c$, which is realized by ρ, and the radius of the sphere realizing $(-\xi, 1)$ as a proximal normal to the hypograph of $T(\cdot)$ can be explicitly estimated. In this lemma, the internal sphere condition (H4) is used for the first time.

3.3.2 Type of singularities of the minimum time function

In order to simplify our writing, we will replace the functions K, K_3, and K_6 appearing respectively in Lemma 3.3.1, Lemma 3.3.2, and Lemma 3.3.3 by the explicit (continuous) function

$$k(r, \|x\|, \rho) = \max\{K_6(r, \|x\|, \rho), K(r, \|x\|, \rho)\}. \tag{3.3.37}$$

Lemma 3.3.4 *Let the assumptions (H1) – (H4) hold and let $x \in \mathcal{S}^c$ and let $\xi \in \partial_P T(x)$. Then*
(i) $\xi \in \partial^P T(x)$ *and therefore T is differentiable at x;*
(ii) $-\xi \in N_1(x)$.
Moreover, for all $z \in \overline{\mathcal{S}^c}$ and for all $\beta \leq T(z)$,

$$\langle -\xi, z - x\rangle + \beta - T(x) \leq k(T(x), \|x\|, \rho)\, \|(-\xi, 1)\| \,(\|z - x\|^2 + |\beta - T(x)|^2). \tag{3.3.38}$$

Proof. Set $r = T(x)$ and let $\xi \in \partial_P T(x)$. By Proposition IV.2.3 in [12], $H(x, -\xi) \geq 1$, so that $\xi \neq 0$. It follows from the definition of proximal subgradient that there exists $\sigma \geq 0$ such that

$$\langle \xi, z - x\rangle \leq \sigma \|z - x\|^2, \quad \forall z \in \mathcal{S}(r). \tag{3.3.39}$$

Let $\bar{x} \in \mathcal{S}_x$ and $M(\cdot) \in \mathcal{M}_{\bar{x}}$, and take a sequence $\{y^{x_n, \bar{u}_n}(\cdot)\} \subset \mathcal{T}_{\bar{x}}$ such that $M(\cdot)$ is the uniform limit of $M(\cdot, x_n, \bar{u}_n)$. We claim that $(M^T(r))^{-1}\xi \in N^P_{\mathcal{S}}(\bar{x})$.
Indeed, take $\bar{z} \in \mathcal{S}$ and set $\bar{z}_n^-(\cdot) = y^-(\cdot, \bar{z}, \bar{u}_n)$ where $y^-(\cdot, \bar{z}, \bar{u}_n)$ is the solution of

$$\begin{cases} \dot{y}(t) &= -f(y(t), \bar{u}_n(r - t)) \text{ a.e.} \\ y(0) &= \bar{z}. \end{cases}$$

We set $z_n = z_n^-(\theta(x_n, \bar{u}_n))$ and consider $\bar{z}_n = y^{z_n, \bar{u}_n}(\theta(x_n, \bar{u}_n))$. We can assume without loss of generality that $\{z_n\}$ converges to some z, which is easily seen belonging to $\mathcal{S}(r)$.
To simplify our writing, we set $t_n = \theta(x_n, \bar{u}_n)$, $\alpha_n(\cdot) = y^{x_n, \bar{u}_n}(\cdot)$, $\bar{x}_n = \alpha_n(t_n)$, and $M_n^1(\cdot) = M(\cdot, x_n, \bar{u}_n)$. Let also $\beta_n(\cdot) = y^{z_n, \bar{u}_n}(\cdot)$, $A_n(t) =$

$\int_0^1 D_x f(\alpha_n(t)+\tau(\beta_n(t)-\alpha_n(t)), \bar{u}_n(t))\, d\tau$ and let $M_n^2(\cdot)$ be the fundamental solution of $\dot{p}(t) = A_n(t)p(t)$, $p(0) = \mathbb{I}^{N\times N}$. Finally, we set $w_n^i(t) = M_n^i(z_n - x_n)$ for $i \in \{1,2\}$.
Using Lemma 7.1.2 and the same argument leading to (3.3.8) we can perform the following estimate:

$$
\begin{aligned}
\left\langle M^T(r)^{-1}\xi, \bar{z}_n - \bar{x}_n \right\rangle &= \left\langle M^T(r)^{-1}\xi, w_n^2(t_n) \right\rangle \\
&= \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \left\langle M^T(r)^{-1}\xi, w_n^2(t_n) - w_n^1(t_n) \right\rangle \\
&\le \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \left\| M^T(r)^{-1} \right\| \ \|\xi\| \left\| w_n^2(t_n) - w_n^1(t_n) \right\| \\
&\le \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \tilde{K}_0 \left\| z_n - x_n \right\|^2 \\
&\le \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \tilde{K}_1 \left\| \bar{z} - \bar{x}_n \right\|^2 ,
\end{aligned}
$$

where $\tilde{K}_0$ and $\tilde{K}_1$ are suitable constants. Taking $n \to \infty$ in the above inequalities, we obtain

$$
\begin{aligned}
\left\langle M^T(r)^{-1}\xi, \bar{z} - \bar{x} \right\rangle &\le \left\langle M^T(r)^{-1}\xi, M^T(r)(z-x) \right\rangle + \tilde{K}_1 \left\| \bar{z} - \bar{x} \right\|^2 \\
&= \langle \xi, z - x \rangle + \tilde{K}_1 \left\| \bar{z} - \bar{x} \right\|^2 .
\end{aligned}
$$

Recalling (3.3.39) and Lemma 7.1.2, we thus obtain

$$
\begin{aligned}
\left\langle M^T(r)^{-1}\xi, \bar{z} - \bar{x} \right\rangle &\le \sigma \|\xi\| \left\| z - x \right\|^2 + \tilde{K}_1 \left\| \bar{z} - \bar{x} \right\|^2 \\
&\le \tilde{K}_2 \left\| \bar{z} - \bar{x} \right\|^2 ,
\end{aligned}
$$

for a suitable constant $\tilde{K}_2$. The above inequality in turn implies that

$$
(M^T(r))^{-1}\xi \in N_{\mathcal{S}}^P(\bar{x}). \tag{3.3.40}
$$

Thanks to (H4), there exists $0 \neq \zeta \in N_{\overline{\mathcal{S}^c}}^P(\bar{x})$. Therefore, both $\mathcal{S}$ and $\overline{\mathcal{S}^c}$ admit at $\bar{x}$ an external nonzero proximal normal. This means that $\mathcal{S}$ is smooth at $\bar{x}$, and so, by (H4), the unique external normal to $\overline{\mathcal{S}^c}$ at $\bar{x}$, namely $-M^T(r)^{-1}\xi$, must be realized by a ball of radius ρ.
Using Proposition IV.2.3 in [12] we see that $H(x, -\xi) \ge 1$, and so we can choose $\lambda \in (0,1)$ such that $H(-\lambda\xi, x) = 1$. Applying Lemma 3.3.3 for $v = \lambda M^T(r)^{-1}\xi$, we obtain that $\lambda\xi \in \partial^P T(x)$. Therefore, T is differentiable at x and so $\lambda\xi = \xi$. Thus both (i) and (ii) are proved.
In order to complete the proof, we apply the last statement of Lemma 3.3.3.
□

The next lemma classifies limiting normals, and shows that limiting subgradients generate proximal normals to the hypograph which are *horizontal/ non-horizontal* according to the *unboundedness/boundedness* of the corresponding sequence of proximal subgradients. Also, the radius of the sphere realizing the limiting vector can be explicitly estimated.

Lemma 3.3.5 *Let the assumptions (H1) – (H4) hold, and let* $\{x_n\}$ *be a sequence converging to* $x \in \mathcal{S}^c$. *Assume that there exists a sequence* $\{\xi_n\}$ *satisfying* $\xi_n \in \partial_P T(x_n)$.
Then the following alternatives hold true:
(i) If $\limsup_{n\to\infty} \|\xi_n\| < +\infty$ *then there exists a subsequence* $\{\xi_{n_k}\}$ *converging to a vector* ξ *such that* $-\xi \in N_1(x)$. *Moreover,* $(-\xi, 1) \in N^P_{hypo(T)}(x, T(x))$ *and, for all* $z \in \mathcal{S}^c$ *and all* $\beta \leq T(z)$ *the inequality*

$$\langle -\xi, z - x\rangle + \beta - T(x) \ \leq \ k(T(x), \|x\|, \rho)\, \|(-\xi, 1)\|\, (\|z - x\|^2 + |\beta - T(x)|^2) \tag{3.3.41}$$

holds.
(ii) If $\limsup_{n\to\infty} \|\xi_n\| = +\infty$ *then there exists a subsequence of* $\{\frac{\xi_n}{\|\xi_n\|}\}$ *converging to a vector* ξ *such that* $-\xi \in N_0(x)$. *Moreover,* $(-\xi, 0)$ *is in* $N^P_{hypo(T)}(x, T(x))$ *and for all* $z \in \mathcal{S}^c$ *and all* $\beta \leq T(z)$, *the inequality*

$$\langle -\xi, z - x\rangle \ \leq \ k(T(x), \|x\|, \rho)(\|z - x\|^2 + |\beta - T(x)|^2) \tag{3.3.42}$$

holds.

Proof. Set $r = T(x)$. Recalling Lemma (3.3.4), the function $T(\cdot)$ is differentiable at x_n. Taking $\bar{x}_n \in \mathcal{S}_{x_n}$ and $M_n(\cdot) \in \mathcal{M}_{\bar{x}_n}$, it follows from Lemma 3.3.4 that for all $n \in \mathbb{N}$
a) $-M_n^T(T(x_n))^{-1}\xi_n \in N^P_{\overline{\mathcal{S}^c}}(\bar{x}_n)$ and each $-M_n^T(T(x_n))^{-1}\xi_n$ is realized by a ball of radius ρ, namely

$$\left\langle -M_n^T(T(x_n))^{-1}\xi_n, \bar{z} - \bar{x}_n\right\rangle \leq \frac{\left\|M_n^T(T(x_n))^{-1}\xi_n\right\|}{2\rho}\, \|\bar{z} - \bar{x}_n\|^2, \quad \forall \bar{z} \in \overline{\mathcal{S}^c}. \tag{3.3.43}$$

b) $H(-\xi_n, x_n) = 1$.

If $\limsup_{n\to\infty} \|\xi_n\| < +\infty$, we choose subsequences $\{\bar{x}_{n_k}\}$ and $\{\xi_{n_k}\}$ converging respectively to $\bar{x} \in \mathcal{S}$ and $\bar{\xi}$. By compactness, without loss of generality we can assume that $\{M_{n_k}(\cdot)\}$ converges uniformly to $M(\cdot)$. We now take $n_k \to \infty$ in (3.3.43) and obtain

$$\left\langle -M^T(r)^{-1}\bar{\xi}, \bar{z} - \bar{x}\right\rangle \leq \frac{\left\|M^T(r)^{-1}\bar{\xi}\right\|}{2\rho}\, \|\bar{z} - \bar{x}\|^2. \tag{3.3.44}$$

Thus $-M^T(r)^{-1}\bar{\xi} \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$ and $-M^T(r)^{-1}\bar{\xi}$ is realized by a ball of radius ρ .
On the other hand, we also take $n_k \to \infty$ in b) and obtain $H(-\bar{\xi}, x) = 1$.
One can also easily show that $M^T(\cdot) \in \mathcal{M}_{\bar{x}}$, so that $-\bar{\xi} \in N_1(x)$. Recalling Lemma 3.3.3 and setting $\xi := \bar{\xi}$ the proof of (i) is concluded.

Analogously, if $\limsup_{n\to\infty} \|\xi_n\| = +\infty$, we can assume that $-\bar{\xi} = -\lim_{n_k\to\infty} \frac{\xi_{n_k}}{\|\xi_{n_k}\|}$, together with $-M^T(r)^{-1}\bar{\xi} \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$ and $H(-\bar{\xi}, x) = 0$. Thus $-\bar{\xi} \in N_0(x)$. Finally, recalling Lemma 3.3.2 and setting $\xi := \bar{\xi}$ we conclude the proof of (ii). □

3.3.3 Wedgedness and exposed rays

The final results of this section use for the first time the wedgedness assumption for the normal cone $N^P_{hypo(T)}(x, T(x))$. They show essentially that $N^P_{hypo(T)}(x, T(x))$ is a closed cone, and that horizontal (resp. non-horizontal) exposed rays of $N^P_{hypo(T)}(x, T(x))$ belong to $N_0(x)$ (resp. $N_1(x)$). As a byproduct of our argument we obtain a representation of $N^P_{hypo(T)}(x, T(x))$ through $N_0(x)$ and $N_1(x)$ (see Theorem 3.3.1).

Lemma 3.3.6 *Let $s \in \mathcal{S}^c$ and let the assumptions (H1) - H(4) hold. Assume that $N^P_{hypo(T)}(x, T(x))$ is wedged and set*

$$\begin{aligned} \tilde{N}_0(x) &= \{(\xi, 0) \mid \xi \in N_0(x)\}, \\ \tilde{N}_1(x) &= \{\lambda(\xi, 1) \mid \xi \in N_1(x),\ \lambda \geq 0\}, \\ N(x) &= \text{co}\tilde{N}_0(x) + \text{co}\tilde{N}_1(x). \end{aligned}$$

Then $N(x) \subseteq N^P_{hypo(T)}(x, T(x))$ is a closed, convex, and wedged cone.

Proof. Thanks to Lemmas 3.3.2 and 3.3.3 and the definition of k in (3.3.37), every $\zeta \in \tilde{N}_0(x) \cup \tilde{N}_1(x)$ satisfies the following property: for every $y \in \mathcal{S}^c$ and every $\beta \leq T(y)$, the inequality

$$\langle \zeta, (y - x, \beta - T(x)) \rangle \leq k(T(x), \|x\|, \rho) \|\zeta\| \left(\|y - x\|^2 + |\beta - T(x)|^2 \right) \tag{3.3.45}$$

holds. It follows immediately from the above property that both $\tilde{N}_0(x)$ and $\tilde{N}_1(x)$ are cones contained in $N^P_{hypo(T)}(x, T(x))$. Thus $\text{co}\tilde{N}_0(x)$ and $\text{co}\tilde{N}_1(x)$ are contained in $N^P_{hypo(T)}(x, T(x))$, and so they are wedged. Set $N_0^1 = \{\xi \in \mathbb{R}^N \mid \xi \in N_0(x), \|\xi\| = 1\}$, and observe that on one hand $\tilde{N}_0(x) = \{\lambda(\xi, 0) \mid \xi \in N_0^1, \lambda \geq 0\}$, on the other N_0^1 (by the continuity of the Hamiltonian) is compact and $0 \notin N_0^1$. Analogously, observe that $N_1(x)$ is compact and does not contain zero. Therefore, using Lemma 7.2.1, we obtain that both $\text{co}\tilde{N}_0(x)$ and $\text{co}\tilde{N}_1(x)$ are closed, and the proof is concluded. □

Lemma 3.3.7 *Let $x \in \mathcal{S}^c$ and let the assumptions of Theorem 3.2.1 hold. Let $\tilde{N}$ be a closed convex cone in $\mathbb{R}^{N+1}$ with the property*

$$N(x) \subseteq \tilde{N} \subseteq N^P_{hypo(T)}(x, T(x)). \tag{3.3.46}$$

Let ζ belong to an exposed ray of $\tilde{N}$. The following statements hold true:

(i) if $\zeta = (\xi, 0)$, with $\xi \in \mathbb{R}^N$, then $\xi \in N_0(x)$;

(ii) if $\zeta = (\xi, \lambda)$, with $\xi \in \mathbb{R}^N$ and $\lambda > 0$, then $\xi/\lambda \in N_1(x)$.

Moreover, ζ satisfies (3.3.45) for all $y \in \mathcal{S}^c$ and all $\beta \leq T(y)$.

Proof. By our assumption on ζ, there exists $\bar{v} = (v_0, \lambda_0)$ satisfying $v_0 \in \mathbb{R}^N$, $\|v_0\| = 1$, and $\lambda_0 \in \mathbb{R}$ such that

$$\begin{cases} \langle (v_0, \lambda_0), \zeta \rangle &= 0 \\ \langle (v_0, \lambda_0), w \rangle &\leq 0 \quad \forall w \in \tilde{N} \\ \langle (v_0, \lambda_0), w \rangle &= 0, \text{ and } 0 \neq w \in \tilde{N} \Rightarrow \frac{w}{\|w\|} = \frac{\zeta}{\|\zeta\|}. \end{cases} \tag{3.3.47}$$

We now begin proving (i). Since $\zeta = (\xi, 0) \in N^P_{hypo(T)}(x, T(x))$, there exists a constant $\sigma \geq 0$ such that, for all $z \in \overline{\mathcal{S}^c}$ and all $\beta \leq T(z)$, the inequality

$$\langle \xi, z - x \rangle \ \leq \ \sigma(\|z - x\|^2 + |\beta - T(x)|^2) \tag{3.3.48}$$

holds. Set now $x_n = x + \frac{v_0}{n} + \frac{\xi}{n\sqrt{n}}$. Then, by the Density Theorem (see [28, Theorem 1.3.1]), for each n there exists z_n such that

$$\partial_P T(z_n) \ \neq \ \emptyset, \tag{3.3.49}$$

$$\|z_n - x_n\| \ \leq \ \frac{1}{n^2}. \tag{3.3.50}$$

First, we show that

$$T(z_n) \leq T(x) \quad \text{ for all } n \text{ large enough.} \tag{3.3.51}$$

Indeed, assume by contradiction that $T(z_n) > T(x)$. Taking $z = z_n$ and $\beta = T(x)$ in (3.3.48), we obtain

$$\langle \xi, z_n - x \rangle \ \leq \ \sigma \ \|z_n - x\|^2 .$$

It follows from the above inequality, (3.3.47), and (3.3.50) that there exists a suitable constant σ_1 for which

$$\frac{\|\xi\|^2}{n\sqrt{n}} \ \leq \ \frac{\sigma_1}{n^2}$$

for all n large enough, a contradiction.
Second, we claim that there exists σ_2 such that

$$|T(z_n) - T(x)| > \sigma_2 n^{-\frac{3}{4}} \quad \text{for all } n \text{ large enough.} \tag{3.3.52}$$

Indeed, taking $z = z_n$ and $\beta = T(z_n)$ in (3.3.48) we obtain

$$\langle \xi, z_n - x \rangle \leq \sigma \ (\|z_n - x\|^2 + |T(z_n) - T(x)|^2).$$

From the above inequality, (3.3.47), and (3.3.50), one can easily see that (3.3.52) holds.

On the other hand, by (3.3.49) and Lemma 3.3.4 we know that T is differentiable at z_n and we write $\xi_n = DT(z_n)$. Recalling (3.3.38), for all $z \in \mathcal{S}^c$ and all $\beta \leq T(z)$ the inequality

$$\begin{aligned}&\langle -\xi_n, z - z_n\rangle + \beta - T(z_n)\\&\qquad \leq k(T(z_n), \|z_n\|, \rho)\,\|(-\xi_n, 1)\|\,(\|z - z_n\|^2 + |\beta - T(z_n)|^2) \qquad (3.3.53)\end{aligned}$$

holds.
We claim that $\|\xi_n\| \to +\infty$.
Assume by contradiction that there exists a constant Q such that $\|\xi_n\| \leq Q$ for all n. Taking $z = x$, $\beta = T(x)$ in (3.3.53) and recalling (3.3.51), we obtain that

$$\begin{aligned}&(T(x) - T(z_n))\Big(1 - k(T(z_n), \|z_n\|, \rho)\sqrt{Q^2+1}|T(x) - T(z_n)|\Big)\\&\qquad \leq \|x - z_n\| \left(Q + k(T(z_n), \|z_n\|, \rho)\sqrt{Q^2+1}\,\|x - z_n\|\right).\end{aligned}$$

By the continuity of $T(\cdot)$ and $k(\cdot)$ and by (3.3.51), (3.3.50), and (3.3.52), there exists a constant $Q_1 > 0$ such that

$$\frac{Q_1}{n^{\frac{3}{4}}} \leq \frac{1}{n} \quad \text{for all } n \text{ large enough,}$$

a contradiction.
Now, recalling (ii) in Lemma 3.3.5 and assuming without loss of generality that $\lim_{n\to\infty} -\frac{\xi_n}{\|\xi_n\|} = -\bar{\xi}$, we see that $(-\bar{\xi}, 0) \in \tilde{N}_0(x) \subseteq \tilde{N}$. By (3.3.51) we can take $z = x$ and $\beta = T(z_n)$ in (3.3.53), obtaining

$$\left\langle -\frac{\xi_n}{\|(-\xi_n, 1)\|}, \frac{x - z_n}{\|x - z_n\|}\right\rangle \;\leq\; k(T(z_n), \|z_n\|, \rho)\,\|x - z_n\|\,.$$

Taking $n \to \infty$ in the above inequality and recalling (3.3.50) we obtain

$$\langle -\bar{\xi}, -v_0\rangle \leq 0,$$

or, equivalently, $\big\langle(-\bar{\xi}, 0), (v_0, \lambda_0)\big\rangle \geq 0$. Therefore, we obtain from (3.3.47) that $(-\bar{\xi}, 0) = \frac{(\xi,0)}{\|\xi\|}$. Thus $\xi = -\bar{\xi}$ and the proof of claim (i) is concluded .

Ad (ii). We now take $\zeta = (\xi, 1)$ and take $\bar{v} = (v_0, \lambda_0)$ satisfying (3.3.47). Set $x_n = x + \frac{v_0}{n}$. Then by the Density Theorem (see Theorem 1.3.1 in [28]) for each n there exists z_n such that

$$\partial_P T(z_n) \;\neq\; \emptyset, \qquad (3.3.54)$$

$$\|z_n - x_n\| \;\leq\; \frac{1}{n^2}. \qquad (3.3.55)$$

Recalling Lemma 3.3.4, (3.3.54) implies that $T(\cdot)$ is differentiable at z_n. Moreover, if we set $\xi_n = DT(z_n)$, then $-\xi_n \in N_1(z_n)$ and for all $z \in \mathcal{S}^c$ and $\beta \le T(z)$ we have

$$\begin{aligned} &\langle -\xi_n, z - z_n \rangle + \beta - T(z_n) \\ &\qquad \le\ k(T(z_n), \|z_n\|, \rho)\, \|(-\xi_n, 1)\|\, (\|z - z_n\|^2 + |\beta - T(z_n)|^2). \end{aligned} \tag{3.3.56}$$

We claim that the sequence $\{\xi_n\}$ is bounded.
Suppose by contradiction that $\limsup_{n\to\infty} \|\xi_n\| = +\infty$. Then assuming without loss of generality that $-\frac{\xi_n}{\|\xi_n\|} \to -\bar{\xi}$, (ii) of Lemma 3.3.5 yields that $-\bar{\xi} \in N_0(x)$ and $(-\bar{\xi}, 0) \in \tilde{N}_0(x)$.
In order to obtain a contradiction, we consider two cases:
a) $T(x) \ge T(z_n)$ for infinitely many n;
b) $T(x) < T(z_n)$ for infinitely many n.

In the first case, we can choose $z = x$, $\beta = T(z_n)$ in (3.3.56), obtaining

$$\left\langle -\frac{\xi_n}{\|(-\xi_n, 1)\|}\ ,\ \frac{x - z_n}{\|x - z_n\|} \right\rangle\ \le\ k(T(z_n), z_n, \rho)\, \|x - z_n\|\,.$$

Taking $n \to \infty$ and recalling (3.3.55) we get

$$\left\langle -\bar{\xi}\ ,\ -v_0 \right\rangle\ \le\ 0, \tag{3.3.57}$$

which implies $\left\langle (-\bar{\xi}, 0), (v_0, \lambda_0) \right\rangle \ge 0$. Thus, combining $(-\bar{\xi}, 0) \in \tilde{N}_0(x)$ with (3.3.47) we obtain $\frac{(-\bar{\xi},0)}{\|-\bar{\xi}\|} = \frac{(\xi,1)}{\|(\xi,1)\|}$, a contradiction.

In the second case, since $(\xi, 1) \in N^P_{hypo(T)}(x, T(x))$ there exists $\sigma \ge 0$ such that

$$\langle \xi, z_n - x \rangle + T(z_n) - T(x)\ \le\ \sigma\ (\|z_n - x\|^2 + |T(z_n) - T(x)|^2) \quad \text{for all } n. \tag{3.3.58}$$

The above inequality implies that there exists σ_1 such that, for all n large enough,

$$T(z_n) - T(x)\ =\ |T(z_n) - T(x)|\ \le\ \sigma_1\, \|z_n - x\|\,. \tag{3.3.59}$$

Recalling (3.3.56) and taking $z = x$, $\beta = T(x)$, we have, for all n large enough,

$$\begin{aligned} &\left\langle \frac{-\xi_n}{\|(-\xi_n, 1)\|}, \frac{x - z_n}{\|z_n - x\|} \right\rangle + \frac{T(x) - T(z_n)}{\|(-\xi_n, 1)\|\, \|z_n - x\|} \le \\ &\qquad\qquad \le k(T(z_n), \|z_n\|, \rho) \left(\|x - z_n\| + \frac{|T(x) - T(z_n)|^2}{\|x - z_n\|} \right). \end{aligned} \tag{3.3.60}$$

Taking $n \to \infty$ in both (3.3.59) and (3.3.60) we obtain

$$\left\langle -\bar{\xi}\ ,\ v_0 \right\rangle\ \ge\ 0, \tag{3.3.61}$$

which implies in turn that $\langle(-\bar{\xi},0),(v_0,\lambda_0)\rangle \geq 0$. Thus, combining the condition $(-\bar{\xi},0) \in \tilde{N}_0(x)$ with (3.3.47), we obtain $\frac{(-\bar{\xi},0)}{\|\bar{\xi}\|} = \frac{(\xi,1)}{\|(\xi,1)\|}$, a contradiction.

We can now assume that

$$\|\xi_n\| \leq Q \quad \text{for all } n, \tag{3.3.62}$$

for a suitable constant Q, and without loss of generality that

$$\lim_{n\to\infty} \xi_n \;=\; \bar{\xi}. \tag{3.3.63}$$

From (i) of Lemma 3.3.5 we have that $-\bar{\xi} \in N_1(x)$, $(-\bar{\xi},1) \in \tilde{N}_1(x)$, and (3.3.41) with $\bar{\xi}$ in place of ξ holds.
We claim that there exists a constant σ_2 such that

$$|T(z_n) - T(x)| \;\leq\; \sigma_2 \|z_n - x\| \quad \forall n. \tag{3.3.64}$$

In the case $T(x) < T(z_n)$, this was already proved (see (3.3.59)).
Assume now $T(x) \geq T(z_n)$. Then, using (3.3.56) with $z = x$ and $\beta = T(x)$, we obtain, for all n large enough,

$$\begin{aligned}&\langle -\xi_n, x - z_n\rangle + T(x) - T(z_n)\\ &\qquad \leq k(T(z_n), \|z_n\|, \rho)\,\|(-\xi_n,1)\|\,(\|x - z_n\|^2 + |T(x) - T(z_n)|^2).\end{aligned} \tag{3.3.65}$$

The above inequality and (3.3.62) imply, for all n large enough,

$$\begin{aligned}T(x) - T(z_n) \leq k(T(z_n), \|z_n\|, \rho)\sqrt{Q^2+1}(\|x - z_n\|^2&\\ + |T(x) - T(z_n)|^2) + Q\,\|z_n - x\|&,\end{aligned}$$

from which, by the local boundedness of k, the inequality (3.3.64) follows.
Summing (3.3.58) and (3.3.65) we obtain, for a suitable constant $\sigma_3 \geq 0$ that for all n large enough

$$\left\langle \xi_n + \xi, \frac{z_n - x}{\|z_n - x\|}\right\rangle \leq \sigma_3 \left(\|z_n - x\| + \frac{|T(z_n) - T(x)|^2}{\|z_n - x\|}\right).$$

Taking $n \to \infty$ in the above inequality and using (3.3.64), (3.3.55) we obtain

$$\langle \bar{\xi} + \xi \;,\; v_0\rangle \;\leq\; 0,$$

or, equivalently,

$$\langle(\xi,1)\;,\;(v_0,\lambda_0)\rangle \;\leq\; \langle(-\bar{\xi},1)\;,\;(v_0,\lambda_0)\rangle\,.$$

Recalling (3.3.47), we have $\langle(\xi,1)\ ,\ (v_0,\lambda_0)\rangle = 0$, whence $\big\langle(-\bar\xi,1),(v_0,\lambda_0)\big\rangle \geq 0$. Note that $(-\bar\xi,1) \in \tilde N_1(x)$, so that $\big\langle(-\bar\xi,1)\ ,\ (v_0,\lambda_0)\big\rangle = 0$ by (3.3.47). Moreover, using again (3.3.47), we finally arrive to

$$\frac{(-\bar\xi,1)}{\big\|(-\bar\xi,1)\big\|} = \frac{(\xi,1)}{\|(-\xi,1)\|}.$$

Therefore we see that $\xi = -\bar\xi \in N_1(x)$ and the proof is concluded. □

The lemmas contained in this section yield immediately the following result.

Theorem 3.3.1 *Let $x \in \mathcal{S}^c$ and let the assumptions of Theorem 3.2.1 hold. Then*

$$N^P_{hypo(T)}(x,T(x)) = N(x),$$

where $N(x)$ was defined in the statement of Lemma 3.3.6, so that $N^P_{hypo(T)}(x,T(x))$ is a closed (convex) cone.

Proof. Assume by contradiction that there exists $\zeta \in N^P_{hypo(T)}(x,T(x)) \setminus N(x)$. Set

$$\tilde N = \mathrm{co}\,(N(x) \cup \{\lambda\zeta \mid \lambda \geq 0\})$$

and observe that $\tilde N$ is a closed convex cone which satisfies (3.3.46). Clearly, ζ belongs to an exposed ray of $\tilde N$, so that, by Lemma 3.3.7, $\zeta \in \tilde N_0(x)\cup\tilde N_1(x)$, a contradiction. □

3.4 Proof of the main results of Chapter 3

3.4.1 Proof of Theorem 3.2.1

It is clear that the "$\supseteq$" inclusion in (3.2.2) follows from Lemma 3.3.2 and the convexity of $\partial^\infty T(x)$.

In order to prove the "$\subseteq$" inclusion, take $\xi \in \partial^\infty T(x)$, i.e, $(-\xi,0) \in N^P_{hypo(T)}(x,T(x))$. Since $N^P_{hypo(T)}(x,T(x))$ is wedged and closed (see Theorem 3.3.1), recalling (2.2.3) we can find numbers $\alpha_i,\ \beta_i \geq 0$ and vectors $\xi_i,\ \zeta_i \in \mathbb{R}^N$, $i \in \{1,\dots,N+2\}$, such that

$$\begin{cases} (-\xi_i,1) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x,T(x)) \\ (-\zeta_i,0) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x,T(x)) \\ (-\xi,0) = & \sum_{i=1}^{N+2}\alpha_i(-\xi_i,1) + \sum_{i=1}^{N+2}\beta_i(-\zeta_i,0). \end{cases} \quad (3.4.1)$$

From the above equality we deduce that $\alpha_i = 0$ for all $i \in \{1,\dots,N+2\}$. Thus, we have

$$(-\xi,0) = \sum_{i=1}^{N+2}\beta_i(-\zeta_i,0). \quad (3.4.2)$$

Recalling (i) in Lemma 3.3.7 we obtain $-\zeta_i \in N_0(x)$. Setting $\bar{\zeta}_i = (\sum_{j=1}^{N+2} \beta_j)\zeta_i$ and $\bar{\beta}_i = \frac{\beta_i}{\sum_{i=1}^{N+2} \beta_i}$, one can easily see $-\bar{\zeta}_i \in N_0(x)$ and $\sum_{i=1}^{N+2} \bar{\beta}_i = 1$.
From (3.4.2), we obtain

$$\xi \ = \ -\sum_{i=1}^{N+2} \bar{\beta}_i(-\bar{\zeta}_i).$$

The proof is concluded. □

Proof of Theorem 3.2.2. Observe that from the very definition it follows that if $\xi \in \partial^P T(x)$ and $\zeta \in \partial^\infty T(x)$ then $\xi + \zeta \in \partial^P T(x)$. Thus the "$\supseteq$" inclusion in (3.2.3) follows from Lemma 3.3.3, Lemma 3.3.2 and the above observation.

In order to prove the "$\subseteq$" inclusion, take $\xi \in \partial^P T(x)$, i.e, $(-\xi, 1) \in N^P_{hypo(T)}(x, T(x))$. Since $N^P_{hypo(T)}(x, T(x))$ is wedged and closed (see Theorem 3.3.1), recalling (2.2.3) we can find numbers $\alpha_i,\ \beta_i \geq 0$ and vectors $\xi_i,\ \zeta_i \in \mathbb{R}^N$, $i \in \{1, ..., N+2\}$, such that

$$\begin{cases} (-\xi_i, 1) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x)) \\ (-\zeta_i, 0) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x)) \\ (-\xi, 1) \ = & \sum_{i=1}^{N+2} \alpha_i(-\xi_i, 1) + \sum_{i=1}^{N+2} \beta_i(-\zeta_i, 0). \end{cases} \tag{3.4.3}$$

From the above equality we deduce that $\sum_{i=1}^{N+2} \alpha_i = 1$. Thus, recalling (ii) in Lemma 3.3.7 we obtain that $\sum_{i=1}^{N+2} \alpha_i(-\xi_i) \in \mathrm{co}(N_1(x))$.

On the other hand, arguing similarly to the above proof we see that $\sum_{i=1}^{N+2} \beta_i(-\zeta_i) \in \mathrm{co}(N_0(x))$. Therefore,

$$\xi = -\left(\sum_{i=1}^{N+2} \alpha_i(-\xi_i) + \sum_{i=1}^{N+2} \bar{\beta}_i(-\bar{\zeta}_i)\right) \in -[\mathrm{co}(N_1(x)) + \mathrm{co}(N_0(x))].$$

The proof is concluded. □

3.4.2 Proof of Theorem 3.2.3

We need the following technical lemma.

Lemma 3.4.1 *Assume that $N^P_{hypo(T)}(x, T(x))$ is wedged for all $x \in \mathcal{S}^c$. Then for each continuous function $\theta : \mathcal{S}^c \to [0, \infty)$, there exists a continuous function $\psi_\theta : \mathcal{S}^c \to (0, 1]$ such that*

$$\langle \zeta_1, \zeta_2 \rangle \geq \psi_\theta(x) - 1 \tag{3.4.4}$$

for all $x \in \mathcal{S}^c$ and for all $\zeta_1,\ \zeta_2 \in N^P_{hypo(T)}(x, T(x))$ satisfying both $\|\zeta_1\| = \|\zeta_2\| = 1$ and

$$\langle \zeta_j, (z - x, \beta - T(x)) \rangle \leq \theta(x)(\|z - x\|^2 + |\beta - T(x)|^2) \tag{3.4.5}$$

for all $z \in \mathcal{S}^c$, $\beta \leq T(x)$, and $j = 1, 2$.

Proof. We only need to show that for every $n \in \mathbb{N}$ there exists a continuous function $\psi_n : \overline{B(0,n)} \cap \mathcal{S}^c \to (0,1]$ satisfying (3.4.4) with $\psi_\theta(x)$ replaced by $\psi_n(x)$. It is easy to see that the following statement is sufficient to this aim.

Let, for all $m, n \in \mathbb{N}$, $\mathcal{K}_n^m = \overline{B(0,n)} \cap \mathcal{S}^c(\frac{1}{m})$, and observe that, by the continuity of $T(\cdot)$, $\mathcal{K}_n^m$ is compact. Fix n. We claim that for each $m \in \mathbb{N}$ there exists a constant $k_m \in (0,1]$ such that

$$\langle \zeta_1 \ , \ \zeta_2 \rangle \geq k_m - 1, \tag{3.4.6}$$

for all $x \in \mathcal{K}_n^m$, $\zeta_1, \ \zeta_2 \in N^P_{hypo(T)}(x, T(x))$ satisfying $\|\zeta_1\| = \|\zeta_2\| = 1$ and (3.4.5).

Indeed, assume by contradiction that there exists a sequence $\{x_i\} \subset \mathcal{K}_n^m$ together with vectors $\zeta_1^i, \zeta_2^i \in N^P_{hypo(T)}(x_i, T(x_i))$ satisfying $\|\zeta_1^i\| = \|\zeta_2^i\| = 1$ and

$$\left\langle \zeta_j^i, (z - x_i, \beta - T(x_i)) \right\rangle \leq \theta(x_i)(\|z - x_i\|^2 + |\beta - T(x_i)|^2), \tag{3.4.7}$$

for all $z \in \mathcal{S}^c$, $\beta \leq T(x_i)$ and $j \in \{1, 2\}$, but such that

$$\lim_{i \to \infty} \left\langle \zeta_1^i \ , \ \zeta_2^i \right\rangle = -1. \tag{3.4.8}$$

We can assume without loss of generality that $\{x_i\}$, $\{\zeta_1^i\}$ and $\{\zeta_2^i\}$ converge respectively to $\bar{x} \in \mathcal{K}_n^m$, $\bar{\zeta}_1$ and $\bar{\zeta}_2$. By the continuity of $T(\cdot)$, $\theta(\cdot)$ and (3.4.7) we obtain

$$\bar{\zeta}_i \in N^P_{hypo(T)}(\bar{x}, T(\bar{x})) \quad \text{for } i \in \{1, 2\}.$$

On the other hand, from $\|\zeta_1^i\| = \|\zeta_2^i\| = 1$ and (3.4.8) we get

$$\bar{\zeta}_1 = -\bar{\zeta}_2.$$

But then the normal cone $N^P_{hypo(T)}(\bar{x}, T(\bar{x}))$ contains a line, and this is a contradiction. □

End of the proof of Theorem 3.2.3.

We need to find a continuous function $\varphi : \mathcal{S}^c \to [0, \infty)$ such that for all $x \in \mathcal{S}^c$, $\zeta \in N^P_{hypo(T)}(x, T(x))$ and for all $z \in \mathcal{S}^c$, $\beta \leq T(z)$ we have

$$\langle \zeta, (z - x, \beta - T(x)) \rangle \leq \varphi(x) \, \|\zeta\| \, (\|z - x\|^2 + |\beta - T(x)|^2). \tag{3.4.9}$$

Observe that for every $\zeta \in N^P_{hypo(T)}(x, T(x))$, by the wedgedness assumption and recalling Theorem 3.3.1, we have

$$\zeta = \sum_{i=1}^{N+2} \zeta_i, \tag{3.4.10}$$

where each ζ_i belongs to an exposed ray of $N^P_{hypo(T)}(x, T(x))$. For $k \in \{1, 2, .., N+2\}$, we set

$$N^P_k(x) = \Big\{\zeta \mid \zeta = \sum_{i=1}^{k} \zeta_i, \\ \text{where } \zeta_i \text{ belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x))\Big\}. \tag{3.4.11}$$

Of course $N^P_k(x) \subseteq N^P_{hypo(T)}(x, T(x))$ and $N^P_{N+2}(x) = N^P_{hypo(T)}(x, T(x))$.

Now, we are going to construct by induction a continuous function $\varphi_k(\cdot)$ such that

$$\Big\langle \zeta^k, (z - x, \beta - T(x)) \Big\rangle \leq \ \varphi_k(x) \left\|\zeta^k\right\| (\|z - x\|^2 + |\beta - T(x)|^2), \tag{3.4.12}$$

for all $x \in \mathcal{S}^c$, $\zeta^k \in N^P_k(x)$ and for all $z \in \mathcal{S}^c$, $\beta \leq T(z)$.
For $k = 1$ we choose $\varphi_1(x) := k(T(x), \|x\|, \rho)$. Recalling Lemma 3.3.7 and Lemma 3.3.3, 3.3.2, we obtain that for all $\zeta^1 \in N^P_1(x)$ and for all $z \in \mathcal{S}^c$, $\beta \leq T(z)$

$$\big\langle \zeta^1, (z - x, \beta - T(x)) \big\rangle \leq \varphi_1(x) \left\|\zeta^1\right\| (\|z - x\|^2 + |\beta - T(x)|^2). \tag{3.4.13}$$

Thus (3.4.12) holds.
Assume now that (3.4.12) is satisfied for $k = h \geq 1$. We want to show that (3.4.12) holds for $k = h + 1$, with

$$\varphi_{h+1}(x) = \sqrt{\frac{\varphi_h(x)^2 + \varphi_1(x)^2}{\psi_{\max\{\varphi_1, \varphi_h\}}(x)}}, \tag{3.4.14}$$

where the function $\psi_{\max\{\varphi_1, \varphi_h\}}(\cdot)$ is given by Lemma 3.4.1 for $\theta(\cdot) = \max\{\varphi_1(\cdot), \varphi_h(\cdot)\}$. Indeed, given $\zeta^{h+1} \in N^P_{h+1}(x)$, one can write

$$\zeta^{h+1} \ = \ \zeta^h + \zeta^1,$$

where $\zeta^h \in N^P_h(x)$ and $\zeta^1 \in N^P_1(x)$. From (3.4.13) and the inductive assumption, one can easily see that

$$\Big\langle \zeta^{h+1}, (z - x, \beta - T(x)) \Big\rangle \\ \leq \Big(\varphi_1(x) \left\|\zeta^1\right\| + \varphi_h(x) \left\|\zeta^h\right\| \Big) (\|z - x\|^2 + |\beta - T(x)|^2), \tag{3.4.15}$$

for all $z \in \mathcal{S}^c$, $\beta \leq T(z)$.
On the other hand, by inductive assumption, (3.4.13) and Lemma 3.4.1 applied for $\theta(\cdot) = \max\{\varphi_1(\cdot), \varphi_h(\cdot)\}$, we obtain

$$\left\langle \frac{\zeta^h}{\|\zeta^h\|}, \frac{\zeta^1}{\|\zeta^1\|} \right\rangle \geq \psi_{\max\{\varphi_1, \varphi_h\}}(x) - 1.$$

Thus, since $\psi(x) \in (0,1]$, we see that

$$\left\|\zeta^h + \zeta^1\right\|^2 \geq \psi_{\max\{\varphi_1,\varphi_h\}}(x)\Big(\left\|\zeta^h\right\|^2 + \left\|\zeta^1\right\|^2\Big).$$

Therefore,

$$\left\|\zeta^h + \zeta^1\right\|^2 \geq \frac{\psi_{\max\{\varphi_1,\varphi_h\}}(x)}{\varphi_h(x)^2 + \varphi_1(x)^2}\Big(\varphi_h(x)\left\|\zeta^h\right\| + \varphi_1(x)\left\|\zeta^1\right\|\Big)^2.$$

Combining the above inequality, (3.4.14) and (3.4.15) we obtain that

$$\left\langle \zeta^{h+1}, (z-x, \beta - T(x))\right\rangle \leq \varphi_{h+1}(x)\left\|\zeta^{h+1}\right\|\Big(\|z-x\|^2 + |\beta - T(x)|^2\Big),$$

for all $z \in \mathcal{S}^c$, $\beta \leq T(z)$.
To conclude the proof, we choose $\varphi(\cdot) = \varphi_{N+2}(\cdot)$. □

3.5 The case of optimal points

This section is devoted to the representation of *supergradient* and *horizontal gradient* at optimal points. The corresponding formulas are easier than in the general case and the structure of the Hamiltonian exhibits special properties.

The definition of optimal points here is based on the classical definition (see, e.g., Definition 2.24, p. 119 in [12]), but is adapted to *limiting optimal trajectories*, since optimal trajectories may not exist.

Definition 3.5.1 *Let $x \in \mathcal{S}^c$ and set $r = T(x)$. The point x is called an optimal point if there exist $\tau > 0$ and $x_\tau \in \mathcal{S}^c$ such that*

(i) $T(x_\tau) = r + \tau$;

(ii) there exist $\bar{x}_\tau \in \mathcal{S}_{x_\tau}$ and $\{\bar{u}_n\} \subset \mathcal{U}_{\bar{x}_\tau}$, together with the corresponding sequence $x_n \to x_\tau$, such that $y^{x_n,\bar{u}_n}(\tau) \to x$.

At optimal points, the Hamiltonian has a special behavior. More precisely, let x be an optimal point with $T(x) = r > 0$. Then the Hamiltonian $H(x,\cdot)$ is additive on the proximal normal cone to $\mathcal{S}^c(r)$. It follows from this property that the *supergradient* and *horizontal supergradient* of T are contained, respectively, in the 1-*level set* and the 0-*level set* of the Hamiltonian.

Theorem 3.5.1 *Let $x \in \mathcal{S}^c$ be an optimal point. Under the same assumptions of Theorem 3.2.1, the (proximal) horizontal gradient and the supergradient of the minimum time function $T(\cdot)$ at the point x can be computed as follows:*

(a) $\partial^\infty T(x) \;=\; [-\mathrm{co}(N(x))] \cap \{-\xi \mid H(\xi, x) = 0\}$,

(b) $\partial^P T(x) \;=\; [-\mathrm{co}(N(x))] \cap \{-\xi \mid H(\xi, x) = 1\}$,

where

$$N(x) = \{M^T(r)v \mid M(\cdot) \in \mathcal{M}_{\bar{x}}, v \in N^P_{\mathcal{S}^c}(\bar{x}), \bar{x} \in \mathcal{S}_x\}. \tag{3.5.1}$$

The proof of Theorem 3.5.1 requires some preliminary lemmas. The first one gives an information on a lower bound of the Hamiltonian computed at a proximal normal of the sublevel of T at an optimal point.

Lemma 3.5.1 *Let $x \in \mathcal{S}^c$ be an optimal point, and let $\xi \in N^P_{\mathcal{S}^c(T(x))}(x)$. Then $H(x, \xi) \geq 0$.*

Proof. Set $r = T(x)$. Let τ, x_τ, $\bar{x}_\tau$, $\bar{u}_n$ and x_n be with the properties stated in Definition 3.5.1. To simplify our writing, we set $\gamma_n(\cdot) = y^{x_n, \bar{u}_n}(\cdot)$. Assuming without loss of generality that $\gamma_n(\cdot)$ converges uniformly to $\gamma(\cdot)$, one can easily check that $\gamma(t) \in \mathcal{S}^c(r)$ for all $t \in [0, \tau]$. For, should $\bar{t} \in (0, \tau]$ exist such that $T(\gamma(\bar{t})) < r$, then one would have $T(x_\tau) < r + \tau$, a contradiction. Now, since $\xi \in N^P_{\mathcal{S}^c(r)}(x)$ there exists $\sigma > 0$ such that for all $t \in [0, \tau]$ we have

$$\langle \xi \;,\; \gamma(t) - x \rangle \;\leq\; \sigma \;\|\gamma(t) - x\|^2 , \tag{3.5.2}$$

namely, for all $t \in [0, \tau]$,

$$\lim_{n\to\infty} \langle \xi, \gamma_n(t) - x \rangle \leq \sigma \lim_{n\to\infty} \|\gamma_n(t) - x\|^2 .$$

Equivalently, for all $t \in [0, \tau]$

$$\lim_{n\to\infty} \left\langle \xi, \gamma_n(\tau) - \int_{\tau-t}^{\tau} f(\gamma_n(s), \bar{u}_n(s))ds - x \right\rangle \leq$$
$$\leq \;\; \sigma \lim_{n\to\infty} \left\| \gamma_n(\tau) - \int_{\tau-t}^{\tau} f(\gamma_n(s), \bar{u}_n(s))ds - x \right\|^2 .$$

Recalling (ii) in Definition 3.5.1, we obtain that for all $t \in [0, \tau]$

$$\lim_{n\to\infty} \left\langle \xi \;,\; -\int_{\tau-t}^{\tau} f(\gamma_n(s), \bar{u}_n(s))ds \right\rangle \;\leq\; \sigma \;\lim_{n\to\infty} \left\| \int_{\tau-t}^{\tau} f(\gamma_n(s), \bar{u}_n(s))ds \right\|^2 .$$

From (iii) of Lemma 7.1.1 and (i), (ii) in Definition 3.5.1, one can see that

$$\lim_{n\to\infty} \left\langle \xi \;,\; -\int_{\tau-t}^{\tau} f(\gamma_n(s), \bar{u}_n(s))ds \right\rangle \;\leq\; O(t^2) \text{ for } t \to 0^+.$$

Thus, for $t \to 0^+$,

$$\limsup_{n\to\infty} \left\langle \xi, -\int_{\tau-t}^{\tau} f(x, \bar{u}_n(s))ds \right\rangle$$
$$\leq O(t^2) + \limsup_{n\to\infty} \left\langle \xi, \int_{\tau-t}^{\tau} \big(f(\gamma_n(s), \bar{u}_n(s)) - f(x, \bar{u}_n(s))\big)ds \right\rangle .$$

Applying the Lipschitz condition of the function $f(\cdot,\cdot)$ and (iii) of Lemma 7.1.1 we easily obtain that

$$\limsup_{n\to\infty}\left\langle \xi, -\int_{\tau-t}^{\tau} f(x,\bar{u}_n(s))ds\right\rangle \;\leq\; O(t^2) \qquad \text{for } t\to 0^+.$$

Therefore, there exists a constant $Q>0$ such that for each $t\in[0,\tau]$ one can find $n_t\in\mathbb{N}$ with the property

$$\left\langle \xi, -\frac{\int_{\tau-t}^{\tau} f(x,\bar{u}_{n_t}(s))ds}{t}\right\rangle \;\leq\; Qt.$$

Set $\bar{f}_t = \frac{\int_{\tau-t}^{\tau} f(x,\bar{u}_{n_t}(s))ds}{t}$. Since $\bar{f}_t\in \mathrm{co}(f(x,\mathcal{U}))$, by the compactness of $\mathcal{U}$, there exits a sequence $\{t_n\}\subseteq[0,\tau]$ converging to 0 and $\bar{f}\in \mathrm{co}f(x,\mathcal{U})$ such that both

$$\bar{f} = \lim_{n\to\infty}\bar{f}_{t_n}$$

and

$$\langle \xi\ ,\ \bar{f}\rangle \;\geq\; 0$$

hold. Since

$$H(x,\xi) \;=\; \max\{\langle \xi\ ,\ f\rangle \mid f\in \mathrm{co}f(x,\mathcal{U})\},$$

the proof is concluded. □

The next Lemma is the key point in order to obtain the additivity of the Hamiltonian.

Lemma 3.5.2 *Let $x\in\mathcal{S}^c$ be an optimal point, and set $T(x)=r$. Then there exists $\bar{f}\in \mathrm{co}f(x,\mathcal{U})$ such that, for all $\xi\in N^P_{\mathcal{S}^c(r)}(x)$,*

$$H(x,\xi) \;=\; \langle \xi\ ,\ \bar{f}\rangle .$$

Proof. Let τ, x_τ, $\bar{x}_\tau$, $\bar{u}_n$ and x_n be with the properties stated in Definition 3.5.1.
To simplify our writing, we set $\gamma_n(\cdot) = y^{x_n,\bar{u}_n}(\cdot)$. Assuming without loss of generality that $\gamma_n(\cdot)$ converges uniformly to $\gamma(\cdot)$, we see that $\gamma(\tau)=x$ and $T(\gamma(\tau-t)) = r+t$ for all $t\in[0,\tau]$. Pick $v\in\mathcal{U}$, and define, for each $t\in[0,\tau]$, $\beta_{v,t}(\cdot) = y^{\gamma(\tau-t),v}(\cdot)$, where $v(\cdot)$ is the constant control $v(t)\equiv v$. Observe that $\beta_{v,t}(t)\in\mathcal{S}^c(r)$ for all $t\in[0,\tau]$.

Let now $\xi\in N^P_{\mathcal{S}^c(r)}$, together with a constant $\sigma\geq 0$ such that for all $t\in[0,\tau]$

$$\langle \xi\ ,\ \beta_{v,t}(t)-x\rangle \;\leq\; \sigma\ \|\beta_{v,t}(t)-x\|^2 .$$

Recalling (ii) in Definition 3.5.1, the latter is equivalent to

$$\begin{aligned}\lim_{n\to\infty}&\left\langle \xi, \int_0^t \left(f(\beta_{v,t}(s),v) - f(\gamma_n(\tau-t+s),\bar{u}_n(\tau-t+s))\right)ds\right\rangle\\ &\leq\; \sigma\;\lim_{n\to\infty}\left\|\int_0^t \left(f(\beta_{v,t}(s),v) - f(\gamma_n(\tau-t+s),\bar{u}_n(\tau-t+s))\right)ds\right\|^2\end{aligned} \tag{3.5.3}$$

for all $t \in [0,\tau]$. Moreover, by (iii) of Lemma 7.1.1, there exists a constant M such that, for all $n \in \mathbb{N}$, $t \in [0,\tau]$ and $s \in [0,t]$,

$$\|\gamma_n(\tau - t + s) - \gamma_n(\tau)\| \ \leq \ Mt,$$

so that for all $t \in [0,\tau]$ and $s \in [0,t]$

$$\lim_{n\to\infty} \|\gamma_n(\tau - t + s) - x\| \ \leq \ Mt.$$

Combining the above inequality with (3.5.3) and recalling the Lipschitz condition on f, we obtain that, for $t \to 0^+$,

$$\limsup_{n\to\infty} \left\langle \xi, \int_0^t \big(f(x,v) - f(x,\bar{u}_n(r-t+s))\big)ds \right\rangle \leq O(t^2),$$

or, equivalently,

$$\limsup_{n\to\infty} \left\langle \xi, f(x,v) - \frac{\int_0^t f(x,\bar{u}_n(r-t+s))ds}{t} \right\rangle \ \leq \ O(t).$$

By arguing as in the proof of Lemma 3.5.1, we can find $\bar{f} \in \mathrm{co}(f(x,U))$ independent of ξ and v such that

$$\langle \xi, f(x,v)\rangle \leq \langle \xi, \bar{f}\rangle .$$

The proof is therefore complete. □

The desired additivity property follows immediately from the above Lemma.

Corollary 3.5.1 *Let $x \in \mathcal{S}^c$ be an optimal point, and set $T(x) = r$. Then for all $\xi_1,\ \xi_2 \in N^P_{\mathcal{S}^c(r)}(x)$, the property*

$$H(x, \xi_1 + \xi_2) = H(x,\xi_1) + H(x,\xi_2)$$

holds.

We are now ready to prove Theorem 3.5.1.
Proof of Theorem 3.5.1.
Proof of part a). It is clear that the "$\subseteq$" inclusion of the equality in (a) follows from Theorem 3.2.1 and Corollary 3.5.1.

To prove the "$\supseteq$" inclusion, take $\xi \in\ [-\mathrm{co}(N(x))] \cap \{-\xi \mid H(x,\xi) = 0\}$, namely,

$$\xi = -\sum_{i=1}^m M_i^T(r)v_i, \quad \text{where} \quad M_i^T(r)v_i \in N(x) \tag{3.5.4}$$

and

$$H(x, \sum_{i=1}^m M_i^T(r)v_i,) = 0. \tag{3.5.5}$$

Applying Lemma 3.3.1 we get that $M_i^T(r)v_i \in N^P_{\mathcal{S}^c(r)}(x)$ for all $i \in \{1,2,...,m\}$. Thus it follows from Lemma 3.5.1 that

$$H(x, M_i^T(r)v_i) \geq 0 \quad \text{for all } i \in \{1,2,...,m\}. \tag{3.5.6}$$

Combining (3.5.5) and (3.5.6), we obtain from Corollary 3.5.1 that for all $i \in \{1,2,...,m\}$, $H(x, M_i^T(r)v_i) = 0$. Therefore $M_i^T(r)v_i \in N_0(x)$ for all $i \in \{1,2,...,m\}$. We conclude the proof using (3.5.4) and Theorem 3.2.1.

Proof of part b). Similarly to part (a), that the "$\subseteq$" inclusion of the equality in (b) follows from Theorem 3.2.2 and Corollary 3.5.1.

To show the "$\supseteq$" inclusion, let $\xi \in \ [-\text{co}(N(x))] \cap \{-\xi \mid H(x,\xi) = 1\}$. Recalling Lemma 3.5.1, ξ can be represented as

$$\xi = -\sum_{i=1}^{m} \alpha_i M_{0i}^T(r)v_i - \sum_{j=1}^{m} \beta_j M_{1j}^T(r)w_j, \tag{3.5.7}$$

where $\alpha_i \geq 0$, $\beta_j \geq 0$ and $M_{0i}^T(r)v_i \in N_0(x)$, $M_{1j}^T(r)w_j \in N_1(x)$.
From $M_{0i}^T(r)v_i \in N^P_{\mathcal{S}^c(r)(x)}$, $M_{1j}^T(r)w_j \in N^P_{\mathcal{S}^c(r)(x)}$, and Corollary 3.5.1, we have

$$H(x,\xi) = \sum_{i=1}^{m} \alpha_i H(x, M_{0i}^T(r)v_i) + \sum_{j=1}^{m} \beta_j H(x, M_{1j}^T(r)w_j) = \sum_{j=1}^{m} \beta_j, \tag{3.5.8}$$

so that $\sum_{j=1}^{m} \beta_j = 1$. The proof is concluded by using (3.5.8), (3.5.7), and Theorem 3.2.2. □

3.6 Examples

In this section we present some examples which illustrate our results. In particular, we provide an example showing that Theorem 3.2.3 is no longer valid if the wedgedness assumption (3.2.1) is dropped.

Example 1. Consider the dynamics $x''(\cdot) \in [-1,1] =: \mathcal{U}$, i.e.

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = A \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix}, \ u \in \mathcal{U}, \ \text{ where } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.6.1}$$

with the initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$. The target is the set (see Figure 1)

$$\begin{aligned} \mathcal{S} \ &= \ \{ (x_1,x_2) \in \mathbb{R}^2 \mid x_1 \leq 0 \} \ \cup \ \{ (x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, \ x_2 \leq \ -x_1 \} \\ &\cup \ \{ (x_1,x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, \ x_2 \geq x_1 \} \\ &\cup \ \{ (x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geq 1, \ x_2 \geq 1 \}. \end{aligned}$$

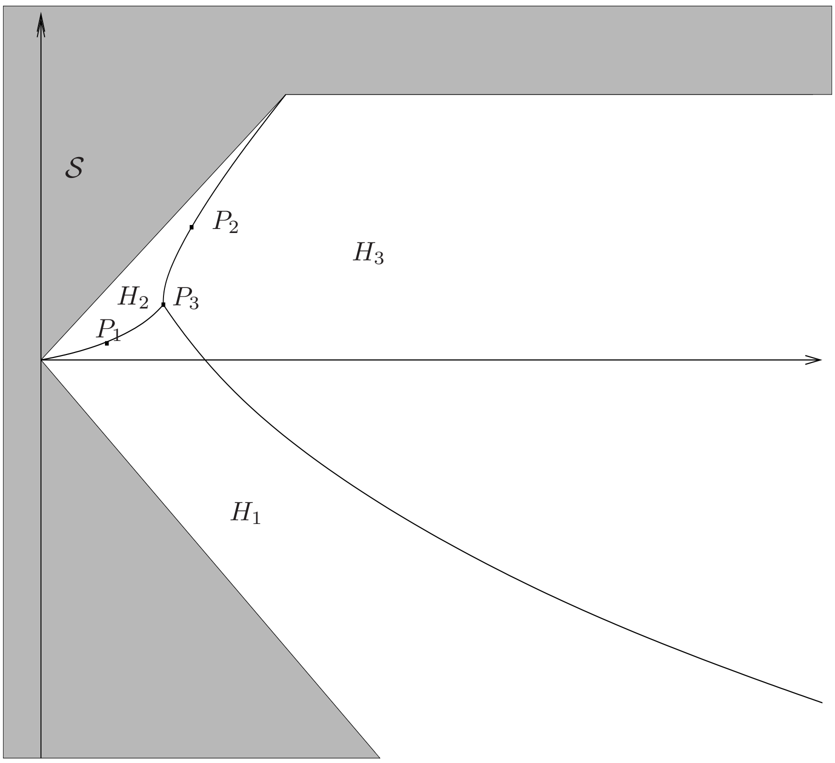

Figure 1

Optimal trajectories are arcs of parabolas

$$x_1 = \frac{1}{2}(x_2)^2 - \frac{1}{2}(x_2^0)^2 + x_1^0 \quad \text{(corresponding to the control } u \equiv 1\text{)},$$

and

$$x_1 = -\frac{1}{2}(x_2)^2 + \frac{1}{2}(x_2^0)^2 + x_1^0 \quad \text{(corresponding to the control } u \equiv -1\text{)}.$$

By direct computation, the minimum time function T is everywhere finite, continuous on the whole of $\mathbb{R}^2$, and the open set S^c can be divided into three regions, say H_1, H_2 and H_3, where T has a different explicit expression. More precisely, consider the curves

$$\begin{aligned}
\gamma_1(t) &= \left(\sqrt{2t}(1-t), t\right), & 0 < t \leq 2-\sqrt{3}, \\
\gamma_2(t) &= \left(\frac{1+t^2}{2}, t\right), & 2-\sqrt{3} < t < 1, \\
\gamma_3(t) &= \left(\frac{3-8t+3t^2}{2}, t\right), & t \geq 2-\sqrt{3}.
\end{aligned}$$

Observe that $\gamma_1(2-\sqrt{3}) = \gamma_2(2-\sqrt{3}) = \gamma_3(2-\sqrt{3}) = 4-2\sqrt{3}$ and moreover all points $\gamma_2(t)$, with $2-\sqrt{3} < t < 1$, are optimal (according to Definition

3.5.1), while all points $\gamma_1(t)$, $\gamma_2(t)$ are not optimal. Set

$$\begin{aligned} H_1 &= \{(x_1,x_2)\in\mathcal{S}^c \mid 0\le x_2\le 2-\sqrt{3},\ \gamma_1(x_2)\le x_1\le\gamma_3(x_2)\} \\ &\cup\ \{(x_1,x_2)\in\mathcal{S}^c \mid x_2\le 0,\ -x_2\le\gamma_3(x_2)\}, \end{aligned}$$

$$\begin{aligned} H_2 &= \{(x_1,x_2)\in\mathcal{S}^c \mid 0\le x_2\le 2-\sqrt{3},\ x_2\le x_1\le\gamma_1(x_2)\} \\ &\cup\ \{(x_1,x_2)\in\mathcal{S}^c \mid 2-\sqrt{3}\le x_2\le 1,\ x_2\le x_1\le\gamma_2(x_2)\}, \end{aligned}$$

$$\begin{aligned} H_3 &= \{(x_1,x_2)\in\mathcal{S}^c \mid 2-\sqrt{3}\le x_2\le 1,\ x_1\ge\gamma_2(x_2)\} \\ &\cup\ \{(x_1,x_2)\in\mathcal{S}^c \mid x_2\le 2-\sqrt{3},\ x_1\ge\gamma_3(x_2)\}. \end{aligned}$$

The minimum time function $T:\mathcal{S}^c\to\mathbb{R}$ can be explicitly computed as

$$T(x_1,x_2)=\begin{cases} x_2-1+\sqrt{1+2x_1+(x_2)^2} & :=\theta_1(x_1,x_2), \quad (x_1,x_2)\in H_1 \\ 1-x_2-\sqrt{1-2x_1+(x_2)^2} & :=\theta_2(x_1,x_2), \quad (x_1,x_2)\in H_2 \\ 1-x_2 & :=\theta_3(x_1,x_2), \quad (x_1,x_2)\in H_3. \end{cases}$$

In the interior of each region H_i, $i=1,2,3$, T is differentiable. Singularities appear of each point of the curves γ_i, $i=1,2,3$. Moreover T is Hölder continuous with exponent $\frac{1}{2}$.
In order to appreciate the role of nonsmoothness of the target, as well as optimality/non optimality of a point and failure of Petrov's condition, we compute the generalized differential of T at the three points

$$P_1=\left(\frac{7}{16},\frac{1}{8}\right),\quad P_2=\left(\frac{5}{8},\frac{1}{2}\right),\quad P_3=\left(4-2\sqrt{3},2-\sqrt{3}\right).$$

Observe that $T(P_1)=\frac{1}{2}$, $T(P_2)=\frac{1}{2}$, $T(P_3)=\sqrt{3}-1$.
To this aim we compute the adjoint flow:

$$e^{A^T t}=\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

and the Hamiltonian

$$H\left((x_1,x_2),(\xi_1,\xi_2)\right) = x_2\xi_1+|\xi_2|.$$

The point P_1 belongs to the curve γ_1, and is steered optimally in time $\frac{1}{2}$ to both $(\frac{5}{8},\frac{5}{8})$ and $(\frac{3}{8},-\frac{3}{8})$, where the normal cones to $\mathcal{S}^c$ are respectively $\mathbb{R}^+(-1,1)$ and $\mathbb{R}^+(-1,-1)$, while P_2 belongs to the curve γ_2, and is steered optimally to $(1,1)$ in time $\frac{1}{2}$, where the normal cone to $\overline{\mathcal{S}^c}$ is $\mathbb{R}^+\mathrm{co}\{(-1,1),\ (0,1)\}$. P_2 is an optimal point. Finally, P_3 is steered optimally to both $(2\sqrt{3}-3,3-2\sqrt{3})$ and $(1,1)$ in time $\sqrt{3}-1$. Observe that $H\left((1,1),(-1,1)\right)=0$, i.e., Petrov's condition fails, while at all other (nonzero) points P of the boundary of $\mathcal{S}$ we have $H(P,\zeta)>0$ for all

$\zeta \in N^P_{\mathcal{S}^c}(P)$, $\zeta \neq 0$.

According to Theorem 3.2.2, and, of course, also to explicit computations from the expression of T, we have

$$\begin{aligned}
\partial^c T(P_1) &= \partial^P T(P_1) \\
&= -\mathrm{co}\left\{ \begin{array}{c} e^{A^T \frac{1}{2}} v \mid v = \begin{pmatrix} -\lambda \\ \lambda \end{pmatrix} \\ \text{or } v = \begin{pmatrix} -\lambda \\ -\lambda \end{pmatrix},\ H(P_1, e^{A^T \frac{1}{2}} v) = 1 \end{array} \right\} \\
&= -\mathrm{co}\left\{ \begin{pmatrix} \frac{8}{3} \\ -\frac{4}{3} \end{pmatrix}, \begin{pmatrix} \frac{8}{11} \\ -\frac{12}{11} \end{pmatrix} \right\}, \\
\partial^\infty T(P_1) &= \{0\};
\end{aligned}$$

$$\begin{aligned}
\partial^c T(P_2) &= \partial^P T(P_2) \\
&= -\mathrm{co}\left\{ e^{A^T \frac{1}{2}} v \mid v \in N^P_{\mathcal{S}^C}(1,1), H(P_2, e^{A^T \frac{1}{2}} v) = 1 \right\} \\
&\quad -\mathrm{co}\left\{ e^{A^T \frac{1}{2}} v \mid v \in N^P_{\mathcal{S}^C}(1,1), H(P_2, e^{A^T \frac{1}{2}} v) = 0 \right\} \\
&= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \left\{ \lambda \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mid \lambda \geq 0 \right\} \\
&= \left\{ \begin{pmatrix} \lambda \\ -1 - \frac{\lambda}{2} \end{pmatrix} \mid \lambda \geq 0 \right\} \\
&= [-N(P_2)] \cap \{ \zeta \mid H(P_2, -\zeta) = 1 \} \\
&\quad \text{(where } N(P_2) \text{ was defined in (3.5.1)),} \\
\partial^\infty T(P_2) &= \left\{ \lambda \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mid \lambda \geq 0 \right\};
\end{aligned}$$

$$\begin{aligned}
\partial^c T(P_3) &= \partial^P T(P_3) \\
&= -\mathrm{co}\Big\{ e^{A^T(\sqrt{3}-1)v} \mid v \in N^P_{\mathcal{S}^c}(1,1) \\
&\qquad \text{or } v \in N^P_{\mathcal{S}^c}(2\sqrt{3}-3, 3-2\sqrt{3}), \text{and } H(P_3, v) = 1 \Big\} \\
&= -\mathrm{co}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{-1}{2(\sqrt{3}-1)} \\ \frac{-\sqrt{3}}{2(\sqrt{3}-1)} \end{pmatrix} \right\} \\
&\qquad - \left\{ \begin{pmatrix} \lambda \\ (2-\sqrt{3})\lambda \end{pmatrix} \mid \lambda \geq 0 \right\}, \\
\partial^\infty T(P_3) &= \left\{ \begin{pmatrix} \lambda \\ (2-\sqrt{3})\lambda \end{pmatrix} \mid \lambda \geq 0 \right\}.
\end{aligned}$$

Observe that the vector $\bar{f} \in \mathrm{co}(f(P_2, \mathcal{U}))$ appearing in the statement of Lemma 3.5.2 is given by $\bar{f} = (1/2, -1)$.

If the target is modified to become $\mathcal{S}' := \mathcal{S} \setminus \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq$

$1, x_1 \geq (x_2)^2/2 + 1/2 + (x_2 - 1)^4\}$ (note that the boundary of $\mathcal{S}'$ is $\mathcal{C}^2$ at $(1,1)$, see Figure 2), then the graph of the new minimum time function T' is smooth at all points of γ_2, but the unique normal is horizontal, so that T' is not differentiable at those points.

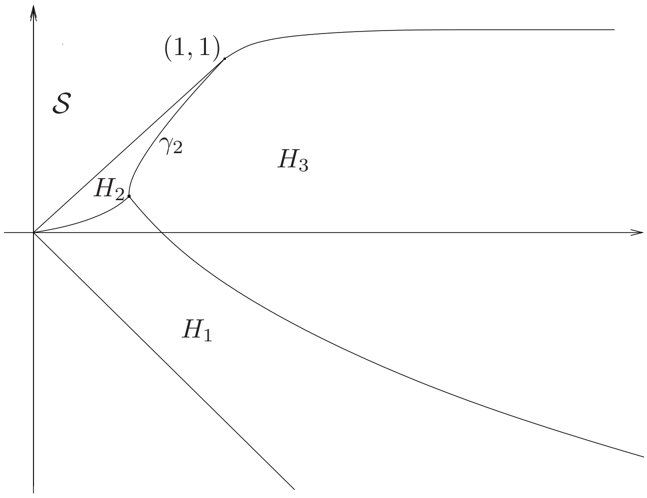

Figure 2

The next two examples deal with the case where the normal cone to the hypograph of T is not wedged. We show first that Theorem 3.2.3 does not hold in general. Next we provide an example where – although the normal cone is not wedged – the situation is entirely analogous to the case where the cone is wedged.

Example 2. Set

$$\gamma_1(y) = \begin{cases} (1 - \sqrt{-y^2 - 2y}, y) & -2 \leq y \leq -1 \\ (-1 + \sqrt{-y^2 - 2y}, y) & -1 \leq t \leq 0 \\ (-1 - \sqrt{-y^2 + 4y}, y) & 0 \leq y \leq 3, \end{cases}$$

and

$$\gamma_2(y) = \begin{cases} (1 + \sqrt{-y^2 - 2y}, y) & -2 \leq y \leq 0 \\ (1 - \sqrt{-y^2 + 2y}, y) & 0 \leq y \leq 1 \\ (0, y) & 1 \leq y \leq 2 \\ (-1 + \sqrt{-y^2 + 4y}, y) & 2 \leq y \leq 3. \end{cases}$$

Observe now that the concatenation of γ_1 with γ_2 defines a $\mathcal{C}^{1,1}$-curve γ. We set the target $\mathcal{S}$ to be the unbounded component of $\mathbb{R}^2 \setminus \{\gamma\}$ (see Figure 3) and the dynamics to be

$$\begin{cases} \dot{x}(t) &= u \\ \dot{y}(t) &= 0 \\ u &\in \mathcal{U} = [-1, 1]. \end{cases}$$

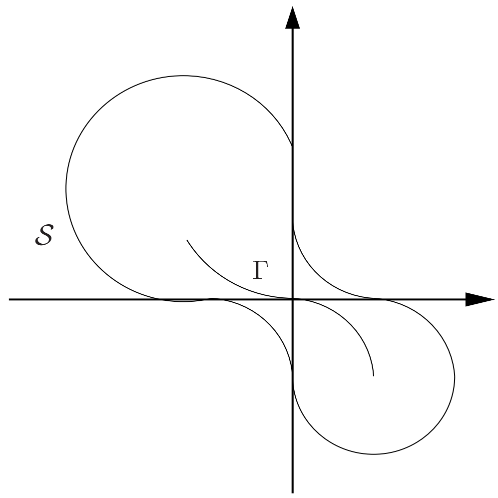

Figure 3

It is readily verified that the minimum time function is everywhere defined and continuous. Observe furthermore that Petrov's condition holds at no points of the segment $[-1,1]\times\{0\}$.
Consider now the curve

$$\Gamma(t)=\frac{\gamma_1(t)+\gamma_2(t)}{2},\qquad t\in[-1,1],$$

together with the function

$$T(t)=\gamma_2(t)-\Gamma(t)(=\Gamma(t)-\gamma_1(t)),\qquad t\in[-1,1],$$

which is the minimum time to reach $\mathcal{S}$ from the point $\Gamma(t)$.
Observe that $T(t)$ is constantly equal to 1 for $-1\le t\le 0$ and in this interval all points of Γ are maximum points for T. Therefore $(0,0,1)$ is a unit limiting normal vector to the hypograph of T at $(0,0,1)$.
On the other hand, it can be easily computed that a unit tangent vector to the graph of T at $(0,0,1)$ is

$$\left(-\frac{2+\sqrt{2}}{2\sqrt{3}},0,\frac{2-\sqrt{2}}{2\sqrt{3}}\right).$$

Since the latter has positive scalar product with the limiting normal $(0,0,1)$, it is clear that the hypograph of T is not regular at $(0,0,1)$. In particular, the normal vector $(0,0,1)$ is not proximal, thus showing that $hypo(T)$ doesn't have positive reach (see (4) in Theorem 2.2.1).
Observe that both $(1,0,0)$ and $(-1,0,0)$ are unit proximal normals to hypo(T) at $(0,0,1)$, so that $N^C_{hypo(T)}(0,0,1)$ contains a line. Therefore, the assumption (3.2.1) in Theorem 3.2.3 cannot be dropped in general.

Observe finally that the hypograph of T satisfies the external sphere condition with radius ρ for a suitable $\rho > 0$. Therefore this is a simple example showing that this condition is weaker than positive reach. □

Example 3. We consider again the dynamics (3.6.1) appearing in Example 1 and modify the target in order to allow lines in the normal cone to the hypograph of T.
The target is the set (see Figure 4)

$$\begin{aligned}\mathcal{S} \;=\; & \{\ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le 0\ \} \;\cup\; \{\ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 1\} \\ & \cup \{\ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1,\ x_2 \le\ x_1 - 1\} \\ & \cup \{\ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1,\ x_2 \ge x_1\}.\end{aligned}$$

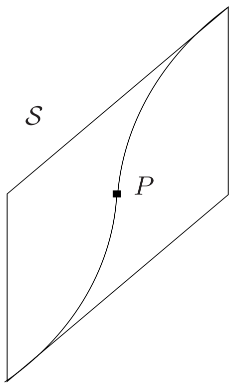

Figure 4

The minimum time function is everywhere finite and continuous, but Petrov's condition does not hold. Computations of the same type of Example 1 show that the normal cone to the hypograph of T at $(1/2, 0, 1)$ is not wedged, however $N^C_{hypo(T)}(1/2, 0, 1)$ can be represented exactly as in (3.2.3) and the hypograph of T has positive reach. More precisely,

$$\begin{aligned}N^P_{hypo(T)}(1/2, 0, 1) &= N^C_{hypo(T)}(1/2, 0, 1) \\ &= \mathbb{R}\begin{pmatrix}1\\0\\0\end{pmatrix} + \mathbb{R}^+ \mathrm{co}\left\{\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}\right\}\end{aligned}$$

and

$$\partial^P T(1/2, 0) = \partial^C T(1/2, 0) = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix} + \mathrm{co}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}.$$

Observe that $H((1/2, 0), (1, 0)) = 0$, while

$$H((1/2, 0), (1, 1)) = H((1/2, 0), (-1, -1)) = 1,$$

so that the conclusion of Theorem 3.2.2 holds. An explicit computation of the minimum time function shows also that the conclusion of Theorem 3.2.3 holds as well. □

3.7 The case of differential inclusions

This section is concerned with the time optimal control problem for the differential inclusion

$$\begin{cases} \dot{x}(t) \in F(x(t)) & a.e.\ t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{3.7.1}$$

with a closed target $\mathcal{S} \subset \mathbb{R}^n$. Here, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex-valued Lipschitz continuous multifunction describing the dynamics, and will be subject to further conditions in Hamiltonian form (see subsection 3.7.1 below). For each trajectory $y^{x_0}(\cdot)$ of (3.7.1), we denote by $\theta(y^{x_0}(\cdot)) := \inf \{t \geq 0 \mid y^{x_0}(t) \in \mathcal{S}\}$ the transition time from x_0 to $\mathcal{S}$ along $y^{x_0}(\cdot)$. Clearly, $\theta(y^{x_0}(\cdot)) \in [0, \infty]$. The *minimum time* $T(x_0)$ to reach $\mathcal{S}$ from x_0 is defined by

$$T(x_0) := \inf \{\theta(y^{x_0}(\cdot)) \mid y^{x_0}(\cdot) \text{ is a trajectory of (3.7.1)}\}. \tag{3.7.2}$$

Observe that, in general, $T(\cdot) : \mathbb{R}^n \to [0, \infty]$. The controllable set $\mathcal{C}$ consists of all points $x \in \mathbb{R}^n$ such that $T(x)$ is finite.

The regularity of the minimum time function, which is related to the controllability of (3.7.1), has been the subject of an extensive literature. Most papers study the case where F is given with a parameterization, which means that F has the form

$$F(x) = \big\{ f(x, u) \mid u \in U \big\} \qquad \forall x \in \mathbb{R}^n \tag{3.7.3}$$

with $U \subseteq \mathbb{R}^m$ compact and $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfying

- f is continuous in (x, u), and
- there exists $k > 0$ so that

  $$|f(x_0, u) - f(x_1, u)| \leq k|x_0 - x_1|$$

  for all $u \in U$ and $x_0, x_1 \in \mathbb{R}^n$.

On the contrary, fewer results are available for systems modeled by differential inclusions such as (3.7.1)—an exception to that being [71], where a representation formula for the proximal subgradient of $T(\cdot)$ is recovered. Although Lipschitz multifunctions with convex values always admit parameterizations by Lipschitz functions (see [1] and [56]), it is a challenging open problem to determine which multifunctions F admit parameterizations with *smooth* functions. Observe that a certain smoothness of the parameterization—essentially that $f(\cdot, u)$ be differentiable with $D_x f(\cdot, u)$ Lipschitz—is crucial in order to derive further regularity properties of the value function, such as *semiconcavity*, by known methods (see [18], and also [22]). This fact explains why parameterization theorems have so far proved of little use for regularity purposes.

Instead of searching for smooth parameterizations, in [23] Cannarsa and Wolenski proposed an alternative strategy to obtain semiconcavity results for the value function of the *Mayer problem* for system (3.7.1). Unlike previous approaches, the proof of [23] exploits the semiconvexity in x of the Hamiltonian

$$H(x,p) = \sup_{v \in F(x)} \langle v, p \rangle \qquad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{3.7.4}$$

as well as the nonsmooth maximum principle.

As for time optimal control problems, when F is parameterized as in (3.7.3) and $D_x f(\cdot, u)$ is Lipschitz, the minimum time function is known to be semiconcave on $\mathcal{C} \setminus \mathrm{int}(\mathcal{S})$ provided $\mathcal{S}$ has the inner ball property, and $T(\cdot)$ is dominated by the distance from $\mathcal{S}$ (see [21], and also [22]). The latter assumption is equivalent to *Petrov's controllability condition:* for some constant $\mu > 0$ and all points $x \in \partial\mathcal{S}$,

$$\min_{u \in U} \langle f(x,u), \nu \rangle \leq -\mu |\nu| \tag{3.7.5}$$

for all proximal normal vectors ν to $\mathcal{S}$ at x. This is a strong hypothesis on the control system since, when $f(x,u) = u$, it is equivalent to the fact that U contains an open neighborhood of the origin. Nevertheless, it is also necessary for the semiconcavity of $T(\cdot)$ up to the boundary of the target, because it is equivalent to the Lipschitz continuity of $T(\cdot)$ in a neighborhood of $\mathcal{S}$—a direct consequence of semiconcavity. On the other hand, the assumption that $\mathcal{S}$ satisfies a uniform interior sphere condition[1] can be removed by the method introduced in [18], where the local semiconcavity of $T(\cdot)$ in $\mathcal{C} \setminus \mathcal{S}$ is derived from an analogous geometric property of $f(x, U)$. Moreover, the approach of [23] to obtain the semiconcavity of the value function, can be adapted to the minimum time function for (3.7.1), keeping Petrov's condition and the inner ball property of $\mathcal{S}$ as standing assumptions (see [24]).

The main purpose of this section is to study the regularity of $T(\cdot)$ for the general system (3.7.1), assuming neither the inner ball property of $\mathcal{S}$ nor Petrov's condition (3.7.5). In view of the above discussion, the expected regularity of $T(\cdot)$ will be weaker than semiconcavity, since the Lipschitz continuity of $T(\cdot)$ near $\mathcal{S}$ is no longer guaranteed. Similarly, the lack of the inner ball property of $\mathcal{S}$ will result in the fact that the regularity of $T(\cdot)$ will be just local in $\mathcal{C} \setminus \mathcal{S}$.

[1] In this section, the expressions "$\mathcal{S}$ has the inner ball property" and "$\mathcal{S}$ satisfies a uniform interior sphere condition" have the same meaning.

3.7.1 Hypotheses and some consequences

We list below our hypotheses on F and H—the problem data introduced in (3.7.1) and (3.7.4), repectively—together with some of their consequences.
Hypotheses (F):

(F1) $F(x)$ is nonempty, convex, and compact for each $x \in \mathbb{R}^n$.

(F2) F is Lipschitz continuous with respect to the Hausdorff distance. Thus, if K is the Lipschitz constant of F, then $K|p|$ is the Lipschitz constant of $H(\cdot, p)$, i.e.,

$$|H(y,p) - H(x,p)| \ \leq \ K|p||y-x| \qquad \forall x, y \in \mathbb{R}^n\,,\ \forall p \in \mathbb{R}^n\,. \quad (3.7.6)$$

Hypotheses (H):

(H1) There exists a constant $c_0 \geq 0$ such that $x \mapsto H(x,p)$ is semiconvex with semiconvexity constant $c_0|p|$.

(H2) For all $p \neq 0$, the gradient $\nabla_p H(\cdot, p)$ exists and is globally Lipschitz, i.e.,

$$|\nabla_p H(x,p) - \nabla_p H(y,p)| \ \leq \ K_1|y-x| \qquad \forall x, y \in \mathbb{R}^n\,,\ \forall p \in \mathbb{R}^n \setminus \{0\}\,, \quad (3.7.7)$$

for some constant $K_1 \geq 0$.

Remark 3.7.1 In particular, (F2) implies that

(F3) $\exists\, K_2 > 0$ such that $\max\{|v| \mid v \in F(x)\} \leq K_2(1+|x|)\,,$

which in turn guarantees that solutions to (3.7.1) are defined on $[0, \infty)$.

Global Lipschitz continuity in both (F2) and (H2) was assumed just to simplify computations. Indeed, our results still hold if F is locally Lipschitz with respect to the Hausdorff distance, and $\nabla_p H(\cdot, p)$ is locally Lipschitz in x, uniformly so over p in $\mathbb{R}^n \backslash \{0\}$. In that case, however, (F3) has to be assumed as an extra condition.

We immediately obtain the following.

Corollary 3.7.1 *Suppose H satisfies assumption (H1). Then*

$$\partial H(x,p) \subseteq \partial_x H(x,p) \times \partial_p H(x,p) \qquad \forall p \neq 0\,.$$

The following proposition is a consequence of [23, Proposition 1].

Proposition 3.7.1 *Suppose F satisfies (F) and (H1). Then*
(1) for each $x, z \in \mathbb{R}^n$, we have

$$H(x+z,p) + H(x-z,p) - 2H(x,p) \geq -c_0|p||z|^2; \text{ and}$$

(2) for each $x, y \in \mathbb{R}^n$, and $\xi \in \partial_x H(x,p)$, we have

$$H(y,p) - H(x,p) - \langle \xi, y-x \rangle \geq -c_0|p||y-x|^2.$$

The differentiability statement in assumption (H2) is equivalent to the argmax set of $v \mapsto \langle v, p\rangle$ being a singleton, which equals $\nabla_p H(x,p)$ and will also be denoted by $F_p(x)$. In view of (H2), for all $p \neq 0$ the function $F_p(\cdot)$ is globally Lipschitz with constant K_1. The main use of (H2) is given by the following result whose proof is straightforward.

Proposition 3.7.2 *Assume (F) and (H), and let $p(\cdot)$ be an absolutely continuous arc on $[0,T]$, with $p(t) \neq 0$ for all $t \in [0,T]$. Then, for each $x \in \mathbb{R}^n$, the problem*

$$\begin{cases} \dot{x}(t) \;=\; F_{p(t)}(x(t)) & a.e.\ t \in [0,T] \\ x(0) \;=\; x \end{cases} \tag{3.7.8}$$

has a unique solution $y(\cdot, x)$. Moreover, $x \mapsto y(t,x)$ is Lipschitz on $\mathbb{R}^n$ and

$$|y(t,z) - y(t,x)| \;\le\; e^{K_1 t}|z-x| \qquad \forall x, z \in \mathbb{R}^n\,,\ \forall t \in [0,T]\,. \tag{3.7.9}$$

We conclude this section with some simple consequences of Gronwall's lemma.

Lemma 3.7.1 *Let $G : [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous multifunction. Assume $G(t,\cdot)$ satisfies hypotheses (F1), (F2) uniformly in $t \in [0,T]$, and is such that, for some $K_0 > 0$,*

$$|v| \le K_0|p| \qquad \forall v \in G(t,p)\,,\ \forall (t,p) \in [0,T] \times \mathbb{R}^n\,.$$

Let $\bar{p}(\cdot)$ be a solution of the differential inclusion

$$\begin{cases} \dot{p}(t) \;\in\; G(t,p(t)) & a.e.\ t \in [0,T] \\ p(0) \;=\; p_0. \end{cases} \tag{3.7.10}$$

Then

$$e^{-K_0 t}|p(0)| \;\le\; |p(t)| \;\le\; e^{K_0 t}|p(0)| \quad \forall t \in [0,T].$$

Moreover, for all $0 \le t_1 \le t_2 \le T$,

$$e^{-K_0(t_2-t_1)}|p(t_2)| \;\le\; |p(t_1)| \;\le\; e^{K_0(t_2-t_1)}|p(t_2)|$$

and

$$|p(t_2) - p(t_1)| \;\le\; K_0 e^{K_0(t_2-t_1)}(t_2-t_1)|p(t_2)|.$$

Proof. Since $p(t) = p(0) + \int_0^t \dot{p}(s)ds$, we have

$$|p(t)| \le |p(0)| + \int_0^t |\dot{p}(s)|ds \le |p(0)| + K_0 \int_0^t |p(s)|ds.$$

So, using Gronwall's inequality, we get: $|p(t)| \le e^{K_0 t}|p(0)|$.

We are now going to prove that $e^{-K_0 t}|p(0)| \le |p(t)|$ for all $t > 0$. Fixing

$t > 0$, we define $g(s) := p(t-s)$ for all $s \in [0,t]$. Since $\dot{g}(s) = -\dot{p}(t-s)$ for almost $s \in [0,t]$, we have $g(s) = g(0) + \int_0^s \dot{g}(\tau)d\tau$ for all $s \in [0,t]$. Thus,

$$\begin{aligned} |g(s)| &\leq |g(0)| + \int_0^s |\dot{g}(\tau)|d\tau = |g(0)| + \int_0^s |\dot{p}(t-\tau)|d\tau \\ &\leq |g(0)| + K_0 \int_0^s |p(t-\tau)|d\tau = |g(0)| + K_0 \int_0^s |g(\tau)|d\tau \end{aligned}$$

Again by Gronwall's inequality, we obtain $|g(s)| \leq e^{K_0 s}|g(0)|$ for all $s \in [0,t]$. In particular, $|g(t)| \leq e^{K_0 t}|g(0)|$. The proof is completed noting that $g(t) = p(0)$ and $g(0) = p(t)$. □

Corollary 3.7.2 *Let $p(\cdot)$ be a solution of (3.7.10). Then either $\bar{p}(t) = 0$ for all $t \in [0,T]$ or $p(t) \neq 0$ for all $t \in [0,T]$.*

Lemma 3.7.2 *Let $y(\cdot, x_0)$ be a solution of (3.7.1). Then, for all $t > 0$, the following holds:*
i) $|y(t,x_0)| \leq (|x_0|+1)e^{K_2 t} - 1,$
ii) $|y(t,x_0) - x_0| \leq (|x_0|+1)(e^{K_2 t} - 1) \leq K_2(|x_0|+1)e^{K_2 t}t.$

Proof. Since

$$y(t,x_0) = x_0 + \int_0^t \dot{y}(s,x_0)ds,$$

recalling (F3) we have

$$|y(t,x_0)| \leq |x_0| + K_2 t + K_2 \int_0^t |y(s,x_0)|ds.$$

Hence, Gronwall's inequality yields (i). Then, observing that

$$|y(t,x_0) - x_0| \leq K_2 \int_0^t (1 + |y(s,x_0)|)ds,$$

(ii) follows using (i) in the above estimate. □

3.7.2 Main results

Part I

In this part, we will assume that $\mathcal{S}$ is nonempty, closed and has the inner ball property with balls of radius $\rho_0 > 0$. Moreover, assumptions (F) and (H) are also assumed throughout. Recall that c_0, K, K_1, K_2 are the constants in (H1), (F2), (H2), (F3) . Let us define, for any $r > 0$,

$$\mathcal{S}'(r) = \{x \mid T(x) \geq r\}, \ \mathcal{S}' = (\mathbb{R}^n \backslash \mathcal{S}) \cup \partial\mathcal{S}, \ \mathcal{C} = \{x \in \mathbb{R}^n \backslash \mathcal{S} \mid T(x) < +\infty\}$$

and $T_{|\mathcal{O}} : \mathcal{O} \to \mathbb{R}$ the restriction of T to $\mathcal{O}$, i.e., $T_{|\mathcal{O}}(x) = T(x)$ for $x \in \mathcal{O}$.

Our main results are the following theorem, together with the corollary.

Theorem 3.7.1 *Assume (F) and (H). Suppose further that $\mathcal{S}$ is nonempty, closed and has the inner ball property with balls of radius $\rho_0 > 0$ and $T(\cdot)$ is continuous in a open subset $\mathcal{O}$ of $\mathcal{C}$. Then, the hypograph of $T_{|\mathcal{O}}(\cdot)$ satisfies a $\rho_T(\cdot)$-exterior sphere condition for some continuous function $\rho_T(\cdot) : \mathcal{O} \to (0,\infty)$.*

Remark 3.7.2 The function $\rho_T(\cdot)$ can be explicitly computed and depends only on $x, T(x)$, and on c_0, K, K_1, K_2, ρ_0.

Consequently, under the assumptions of Theorem 3.7.1, $T_{|\mathcal{O}}(\cdot)$ enjoys the regularity properties described in Theorem 2.2.2. Moreover, the following corollary follows from Theorem 3.7.1 and [54, Theorem 21].

Corollary 3.7.3 *Under the assumptions of Theorem 3.7.1, if $T(\cdot)$ is locally Lipschitz in $\mathcal{O}$, then $T(\cdot)$ is locally semiconcave in $\mathcal{O}$.*

The main part of the proof of Theorem 3.7.1 is divided into three lemmas.

Lemma 3.7.3 *Suppose $\bar{x} \in \mathcal{O}$ is not a local maximum of $T(\cdot)$. Let $r = T(\bar{x})$ and let $\bar{x}^+(\cdot)$ be an optimal trajectory steering $\bar{x}$ to $\mathcal{S}$ in time r, and set $\bar{x}^-(s) = \bar{x}^+(r-s)$. Then, there exists an arc $\bar{p}(\cdot)$ defined on $[0,r]$, with $\bar{p}(s) \neq 0$ for all $s \in [0,r]$, such that*

$$\begin{cases} -\dot{\bar{p}}(s) \in \partial_x H(\bar{x}^-(s), -\bar{p}(s)) & a.e.\ s \in [0,r], \\ \dot{\bar{x}}^-(s) = -F_{-\bar{p}(s)}(\bar{x}^-(s)) & a.e.\ s \in [0,r]. \end{cases} \tag{3.7.11}$$

Moreover, $-\bar{p}(r-t) \in N^P_{\mathcal{S}'(r-t)}(\bar{x}^+(t))$ is realized by a ball of radius $\rho(r-t)$ for all $t \in [0,r]$, i.e.,

$$\left\langle \frac{-\bar{p}(r-t)}{|\bar{p}(r-t)|}, \bar{y} - \bar{x}^+(t) \right\rangle \leq \frac{1}{2\rho(r-t)} |\bar{y} - \bar{x}^+(t)|^2, \quad \forall\, \bar{y} \in \mathcal{S}'(r-t), \tag{3.7.12}$$

where

$$\rho(s) = \frac{\rho_0}{1 + 2c_0\rho_0 s} e^{-(K+2K_1)s}. \tag{3.7.13}$$

Proof. Set $\bar{x}_1 = \bar{x}^+(r)$. Of course, $\bar{x}_1 \in \partial\mathcal{S}$. Since $\mathcal{S}$ satisfies the ρ_0-internal sphere condition, there exists a proximal normal vector $v \neq 0$ to $\mathcal{S}'$ at $\bar{x}_1$ such that $\overline{B}\big(\bar{x}_1 + \rho_0 \frac{v}{|v|}, \rho_0\big) \subseteq \mathcal{S}$, i.e.,

$$\left\langle \frac{v}{|v|}, z - \bar{x}_1 \right\rangle \leq \frac{1}{2\rho_0}\ |z - \bar{x}_1|^2 \quad \forall z \in \mathcal{S}'. \tag{3.7.14}$$

Now, consider the reversed differential inclusion with initial data

$$\begin{cases} \dot{y}(s) \in -F(y(s)) & a.e.\ s \in [0,r], \\ y(0) \in \overline{B}\big(\bar{x}_1 + \rho_0 \frac{v}{|v|}, \rho_0\big) \subseteq \mathcal{S}. \end{cases} \tag{3.7.15}$$

The Hamiltonian associated with $-F$ is defined by

$$H^-(x,p) := \sup_{v \in -F(x)} \langle v, p \rangle = \sup_{w \in F(x)} \langle w, -p \rangle = H(x,-p). \tag{3.7.16}$$

Let us recall that the attainable set from $\overline{B}(\bar{x}_1 + \rho \frac{v}{|v|}, \rho)$, denoted $\mathcal{A}^-(r)$, is defined to be the set of all points $y(r)$ where $y(\cdot)$ is a trajectory satisfying (3.7.15). Since $\bar{x}^-(\cdot)$ is a solution of (3.7.15) with initial point $y(0) = \bar{x}_1$, and $T(\cdot)$ has not a local maximum at the point $\bar{x}$, one has that $\bar{x}^-(r) = \bar{x}$ is on the boundary of $\mathcal{A}^-(r)$. Indeed, suppose $\bar{x}$ is not on the boundary of $\mathcal{A}^-(r)$, then there exists $\epsilon > 0$ such that $B(\bar{x},\epsilon) \subset \mathcal{A}^-(r)$. Thus, $T(y) \leq r = T(\bar{x})$ for all $y \in B(\bar{x},\epsilon)$, and we get a contradiction since $T(\cdot)$ has not a local maximum at $\bar{x}$. Now, since $\bar{x}$ is on the boundary of $\mathcal{A}^-(r)$, by [27, Theorem 3.5.4], there is an arc $\bar{p}(\cdot)$, such that $\|\bar{p}(\cdot)\|_\infty > 0$, satisfying

$$(-\dot{\bar{p}}(s), \dot{\bar{x}}^-(s)) \in \partial H^-(\bar{x}^-(s), \bar{p}(s)) \quad a.e.\ s \in [0,r], \tag{3.7.17}$$

and

$$\bar{p}(0) \in N_{\overline{B}\left(\bar{x}_1 + \rho_0 \frac{v}{|v|}, \rho_0\right)}(\bar{x}_1). \tag{3.7.18}$$

From (3.7.17), (3.7.16) and Corollary 3.7.1, we have

$$-\dot{\bar{p}}(s) \in \partial_x H(\bar{x}^-(s), -\bar{p}(s)) \quad a.e.\ s \in [0,r]. \tag{3.7.19}$$

Moreover, owing to (3.7.6), for all $v \in \partial_x H(x,p)$ we have $|v| \leq K|p|$. Therefore, applying Lemma 3.7.1 to $G(s, -\bar{p}(s)) = \partial_x H(\bar{x}^-(s), -\bar{p}(s))$, we get

$$e^{-Ks}|\bar{p}(0)| \leq |\bar{p}(s)| \leq e^{Ks}|\bar{p}(0)| \quad \forall s \in [0,r]. \tag{3.7.20}$$

Since $\|\bar{p}(\cdot)\|_\infty > 0$, we have $\bar{p}(s) \neq 0$ for all $s \in [0,r]$. Therefore, from (3.7.17) and Corollary 3.7.1 we get

$$\dot{\bar{x}}^-(s) = -\partial_p H(\bar{x}^-(s), -\bar{p}(s)) = -F_{-\bar{p}(s)}(\bar{x}^-(s)) \quad a.e.\ s \in [0,r]. \tag{3.7.21}$$

We are now going to prove (3.7.12). Fix $t \in [0,r]$ and let $\bar{y} \in \mathcal{S}'(r-t)$, i.e., $T(\bar{y}) \geq r - t$. Let $\bar{y}^+(\cdot)$ be the solution of the Cauchy problem

$$\begin{cases} \dot{\bar{y}}^+(s) = F_{-\bar{p}(r-t-s)}(\bar{y}^+(s)) & a.e.\ s \in [0, r-t], \\ \bar{y}^+(0) = \bar{y}. \end{cases} \tag{3.7.22}$$

Note that $\bar{y}_1 := \bar{y}^+(r-t) \in \mathcal{S}'$. Then, $\bar{y}^-(s) := \bar{y}^+(r-t-s)$ satisfies $\bar{y}^-(r-t) = \bar{y}$ and

$$\begin{cases} \dot{\bar{y}}^-(s) = -F_{-\bar{p}(s)}(\bar{y}^-(s)) & a.e.\ s \in [0, r-t], \\ \bar{y}^-(0) = \bar{y}_1. \end{cases} \tag{3.7.23}$$

From (3.7.21), (3.7.23) and (3.7.9), we have

$$|\bar{y}^-(s) - \bar{x}^-(s)| \le e^{K_1(r-t)}|\bar{y}_1 - \bar{x}_1| \quad \forall s \in [0, r-t]. \tag{3.7.24}$$

In order to prove (3.7.12), observe that

$$\begin{aligned} &\langle -\bar{p}(r-t), \bar{y}^-(r-t) - \bar{x}^-(r-t)\rangle \\ &\quad = \langle -\bar{p}(0), \bar{y}^-(0) - \bar{x}^-(0)\rangle + \int_0^{r-t} \frac{d}{ds}\langle -\bar{p}(s), \bar{y}^-(s) - \bar{x}^-(s)\rangle ds. \end{aligned} \tag{3.7.25}$$

Moreover,

$$\begin{aligned} &\frac{d}{ds}\langle -\bar{p}(s), \bar{y}^-(s) - \bar{x}^-(s)\rangle \\ &\qquad = \langle -\dot{\bar{p}}(s), \bar{y}^-(s) - \bar{x}^-(s)\rangle + \langle -\bar{p}(s), \dot{\bar{y}}^-(s) - \dot{\bar{x}}^-(s)\rangle \\ &\quad = \langle -\dot{\bar{p}}(s), \bar{y}^-(s) - \bar{x}^-(s)\rangle + \langle -\bar{p}(s), -F_{-\bar{p}(s)}(\bar{y}^-(s)) + F_{-\bar{p}(s)}(\bar{x}^-(s))\rangle \\ &\qquad = \langle -\dot{\bar{p}}(s), \bar{y}^-(s) - \bar{x}^-(s)\rangle - H(\bar{y}^-(s), -\bar{p}(s)) + H(\bar{x}^-(s), -\bar{p}(s)). \end{aligned}$$

Recalling (3.7.19) and Proposition 3.7.1 it follows that

$$\begin{aligned} &\langle -\dot{\bar{p}}(s), \bar{y}^-(s) - \bar{x}^-(s)\rangle - H(\bar{y}^-(s), -\bar{p}(s)) + H(\bar{x}^-(s), -\bar{p}(s)) \\ &\qquad\qquad \le c_0\ |\bar{p}(s)|\ |\bar{y}^-(s) - \bar{x}^-(s)|^2. \end{aligned}$$

Therefore,

$$\frac{d}{ds}\langle -\bar{p}(s), \bar{y}^-(s) - \bar{x}^-(s)\rangle \le c_0\ |\bar{p}(s)|\ |\bar{y}^-(s) - \bar{x}^-(s)|^2. \tag{3.7.26}$$

Owing to (3.7.18) and the fact that $\bar{p}(0) \neq 0$, we have $-\frac{\bar{p}(0)}{|\bar{p}(0)|} = \frac{v}{|v|}$. Thus, by (3.7.14) and the fact that $\bar{y}_1 \in \mathcal{S}'$ we obtain

$$\left\langle -\frac{\bar{p}(0)}{|\bar{p}(0)|}, \bar{y}_1 - \bar{x}_1 \right\rangle \le \frac{1}{2\rho_0}\ |\bar{y}_1 - \bar{x}_1|^2. \tag{3.7.27}$$

Combining (3.7.25), (3.7.26), (3.7.27) and noting that $\bar{x}^-(0) = \bar{x}_1$, $\bar{y}^-(0) = \bar{y}_1$, $x^-(r-t) = x^+(t)$, $\bar{y}^-(r-t) = \bar{y}$, we conclude that

$$\langle -\bar{p}(r-t), \bar{y} - \bar{x}^+(t)\rangle \le \frac{|\bar{p}(0)|}{2\rho_0}\ |\bar{y}_1 - \bar{x}_1|^2 + c_0 \int_0^{r-t} |\bar{p}(s)||\bar{y}^-(s) - \bar{x}^-(s)|^2 ds.$$

Thus, by (3.7.20) and (3.7.24),

$$\left\langle \frac{-\bar{p}(r-t)}{|-\bar{p}(r-t)|}, \bar{y} - \bar{x}^+(t) \right\rangle \le \left(\frac{1}{2\rho_0} + c_0(r-t)\right) e^{(K+2K_1)(r-t)}|\bar{y} - x^+(t)|^2.$$

So, (3.7.12) follows (3.7.13), and the proof is complete. □

Lemma 3.7.4 *Suppose $\bar{x} \in \mathcal{O}$ is not a local maximum of $T(\cdot)$. Let $r = T(\bar{x})$ and let $\bar{x}^+(\cdot)$ be an optimal trajectory steering $\bar{x}$ to $\mathcal{S}$ in time r. If $\bar{p}(\cdot)$ is the arc in Lemma 3.7.3, then $H(\bar{x}, -\bar{p}(r)) \geq 0$.*

Proof. Fixing $t \in (0, r]$, by (3.7.12) and the fact that $\bar{x} \in \mathcal{S}'(r-t)$, we have

$$\langle -\bar{p}(r-t), \bar{x} - \bar{x}^+(t) \rangle \leq \frac{1}{2\rho(r-t)} \; |\bar{p}(r-t)| \; |\bar{x} - \bar{x}^+(t)|^2.$$

Equivalently,

$$\Big\langle - \bar{p}(r-t), \int_0^t -F_{-\bar{p}(r-s)}(\bar{x}^+(s))ds \Big\rangle \\ \leq \frac{1}{2\rho(r-t)} \; |\bar{p}(r-t)| \; \Big| \int_0^t -F_{-\bar{p}(r-s)}(\bar{x}^+(s))ds \Big|^2.$$

Dividing by t both sides of the above inequality, we get

$$\Big\langle - \bar{p}(r-t), \frac{\int_0^t -F_{-\bar{p}(r-s)}(\bar{x}^+(s))ds}{t} \Big\rangle \\ \leq \frac{1}{2\rho(r-t)} \; |\bar{p}(r-t)| \; \frac{|\int_0^t -F_{-\bar{p}(r-s)}(\bar{x}^+(s))ds|^2}{t}.$$

As $t \to 0$, we obtain

$$\langle -\bar{p}(r), -F_{-\bar{p}(r)}(\bar{x}) \rangle \; \leq 0.$$

This implies that $H(\bar{x}, -\bar{p}(r)) \geq 0$. □

Lemma 3.7.5 *Suppose $\bar{x} \in \mathcal{O}$ is not a local maximum of $T(\cdot)$. Let $r = T(\bar{x})$ and let $\bar{x}^+(\cdot)$ be an optimal trajectory steering $\bar{x}$ to $\mathcal{S}$ in time r. Let $\bar{p}(\cdot)$ be the arc given by Lemma 3.7.3, and set $\lambda = H(\bar{x}, -\bar{p}(r))$. Then there exists a positive constant ρ_T such that $(-\bar{p}(r), \lambda) \in N^P_{hypo(T_{|\mathcal{O}})}(\bar{x}, T_{|\mathcal{O}}(\bar{x}))$ is realized by a ball of radius ρ_T, i.e., for all $\bar{y} \in \mathcal{O}$ and $\beta \leq T(\bar{y})$*

$$\Big\langle \frac{(-\bar{p}(r), \lambda)}{|(-\bar{p}(r), \lambda)|} \;,\; (\bar{y} - \bar{x}, \beta - r) \Big\rangle \leq \frac{1}{2\rho_T}(|\bar{y} - \bar{x}|^2 + |\beta - r|^2)\,. \tag{3.7.28}$$

Moreover, $\rho_T = \rho_T(\bar{x})$ where $\rho_T(\cdot) : \mathcal{O} \to (0, \infty)$ is a continuous function that can be computed explicitly.

Proof. Let $\bar{y} \in \mathcal{O}$. Two cases may occur:
(i) $T(\bar{y}) < T(\bar{x})$,
(ii) $T(\bar{y}) \geq T(\bar{x})$.

First case: $T(\bar{y}) =: r_1 < r = T(\bar{x})$. Let $\bar{x}_1 = \bar{x}^+(r - r_1)$ and write

$$\langle -\bar{p}(r), \bar{y} - \bar{x} \rangle = \langle -\bar{p}(r), \bar{y} - \bar{x}_1 \rangle + \langle -\bar{p}(r), \bar{x}_1 - \bar{x} \rangle. \tag{3.7.29}$$

Recalling Lemma 3.7.3 and noting that $\bar{y} \in \mathcal{S}'(r_1)$, we can estimate the first term in the right-hand side of the above identity as follows

$$\begin{aligned}\langle -\bar{p}(r), \bar{y} - \bar{x}_1 \rangle =& \langle -\bar{p}(r_1), \bar{y} - \bar{x}_1 \rangle + \langle -\bar{p}(r) + \bar{p}(r_1), \bar{y} - \bar{x}_1 \rangle \\ &\leq \frac{1}{2\rho(r_1)} \, |\bar{p}(r_1)| \, |\bar{y} - \bar{x}_1|^2 + |\bar{p}(r) - \bar{p}(r_1)| \, |\bar{y} - \bar{x}_1|.\end{aligned}$$

From Lemmas 3.7.1 and 3.7.2, we have that

$$|\bar{p}(r_1)| \leq e^{K(r-r_1)} |\bar{p}(r)| \;, \; |\bar{p}(r) - \bar{p}(r_1)| \leq K e^{K(r-r_1)} (r - r_1) |\bar{p}(r)|$$

and

$$|\bar{y} - \bar{x}_1| \leq |\bar{y} - \bar{x}| + |\bar{x}_1 - \bar{x}| \leq |\bar{y} - \bar{x}| + K_2(|\bar{x}| + 1) e^{K_2(r-r_1)} (r - r_1).$$

Thus, observing that $\rho(r_1) \geq \rho(r)$, one can get the estimate

$$\langle -\bar{p}(r), \bar{y} - \bar{x}_1 \rangle \;\leq\; L_1(|x|, r) |\bar{p}(r)| (|\bar{y} - \bar{x}|^2 + |r - r_1|^2) \tag{3.7.30}$$

where

$$L_1(|x|, r) = \frac{1 + K_2^2(|x| + 1)^2 e^{2K_2 r}}{2\rho(r)} e^{Kr} + K K_2(|x| + 1) e^{(K + K_2) r} + 2K e^{Kr}.$$

We rewrite the right-most term of (3.7.29) as follows

$$\begin{aligned}\langle -\bar{p}(r), \bar{x}_1 - \bar{x} \rangle \;&=\; \Big\langle -\bar{p}(r), \int_0^{r-r_1} F_{-\bar{p}(r-s)}(\bar{x}^+(s)) ds \Big\rangle \\ &=\; \int_0^{r-r_1} \langle -\bar{p}(r), F_{-\bar{p}(r-s)}(\bar{x}^+(s)) \rangle ds\end{aligned}$$

and observe that

$$\begin{aligned}\langle -\bar{p}(r), F_{-\bar{p}(r-s)}(\bar{x}^+(s)) \rangle &= \langle -\bar{p}(r), F_{-\bar{p}(r-s)}(\bar{x}^+(s)) - F_{-\bar{p}(r)}(\bar{x}^+(s)) \rangle \\ &+ \langle -\bar{p}(r), F_{-\bar{p}(r)}(\bar{x}^+(s)) - F_{-\bar{p}(r)}(\bar{x}) \rangle + \langle -\bar{p}(r), F_{-\bar{p}(r)}(\bar{x}) \rangle.\end{aligned}$$

Moreover, recalling that $\lambda = H(\bar{x}, -\bar{p}(r))$, we have

$$\langle -\bar{p}(r), F_{-\bar{p}(r)}(\bar{x}) \rangle = H(\bar{x}, -\bar{p}(r)) = \lambda,$$

$$\begin{aligned}\langle -\bar{p}(r), F_{-\bar{p}(r)}(\bar{x}^+(s)) - F_{-\bar{p}(r)}(\bar{x}) \rangle \;&\leq\; K \, |\bar{p}(r)| \, |\bar{x}^+(s) - \bar{x}| \\ &\leq\; K K_2(|\bar{x}| + 1) e^{K_2 r} \, |\bar{p}(r)| \, s\end{aligned}$$

and

$$\begin{aligned}&\langle -\bar{p}(r), F_{-\bar{p}(r-s)}(\bar{x}^+(s)) - F_{-\bar{p}(r)}(\bar{x}^+(s)) \rangle \\ &\qquad = \langle -\bar{p}(r), F_{-\bar{p}(r-s)}(\bar{x}^+(s)) \rangle - H(\bar{x}^+(s), -\bar{p}(r)) \\ &= \langle -\bar{p}(r) + \bar{p}(r-s), F_{-\bar{p}(r-s)}(\bar{x}^+(s)) \rangle + H(\bar{x}^+(s), -\bar{p}(r-s)) - H(\bar{x}^+(s), -\bar{p}(r)) \\ &\quad \leq 2K_2(|\bar{x}^+(s)| + 1) \; |\bar{p}(r) - \bar{p}(r-s)| \leq 2K K_2(|\bar{x}| + 1) e^{(K_2 + K) r} |\bar{p}(r)| s.\end{aligned}$$

Therefore,

$$\langle -\bar{p}(r), F_{-\bar{p}(r-s)}(\bar{x}^+(s))\rangle \le \lambda + L_2(|\bar{x}|, r)\ |\bar{p}(r)|\ s$$

where $L_2(|\bar{x}|, r) = KK_2(|\bar{x}|+1)(2e^{Kr}+1)e^{K_2 r}$. Thus, in view of the above estimates,

$$\langle -\bar{p}(r), \bar{x}_1 - \bar{x}\rangle \le \lambda(r - r_1) + \frac{L_2(|\bar{x}|, r)}{2}\ |\bar{p}(r)|\ |r - r_1|^2. \tag{3.7.31}$$

Combining (3.7.29), (3.7.30) and (3.7.31), we get

$$\langle -\bar{p}(r), \bar{y} - \bar{x}\rangle + \lambda(r_1 - r) \le \frac{2L_1(|\bar{x}|, r) + L_2(|\bar{x}|, r)}{2}\ |\bar{p}(r)|\ (|\bar{y} - \bar{x}|^2 + |r_1 - r|^2).$$

From Lemma 3.7.4, we have that $\lambda \ge 0$. Therefore, since $r_1 < r$, we conclude that

$$\left\langle \frac{(-\bar{p}(r), \lambda)}{|(-\bar{p}(r), \lambda)|}\ ,\ (\bar{y} - \bar{x}, \beta - r)\right\rangle \le \frac{2L_1(|\bar{x}|, r) + L_2(|\bar{x}|, r)}{2}\ (|\bar{y} - \bar{x}|^2 + |\beta - r|^2) \tag{3.7.32}$$

for all $\beta \le r_1$. So, if $T(\bar{y}) < T(\bar{x})$, then (3.7.28) holds true provided ρ_T is such that

$$\rho_T \le \frac{1}{2L_1(|\bar{x}|, r) + L_2(|\bar{x}|, r)}. \tag{3.7.33}$$

Second case: $T(\bar{y}) = r_1 \ge r = T(\bar{x})$.
In view of Lemmas 3.7.3 and 3.7.4, we already know that (3.7.28) holds for all $\beta \le r$ provided $\rho_T \le \rho(r)$. So, we just need to prove (3.7.28) for $r < \beta \le r_1$. Let $\bar{y}^+(\cdot)$ be the solution of

$$\begin{cases} \dot{\bar{y}}(s) \in F_{-\bar{p}(r)}(\bar{y}(s)) & a.e.\ s \in [0, r_1 - \beta] \\ \bar{y}(0) = \bar{y}. \end{cases} \tag{3.7.34}$$

Set $\bar{y}_1 = \bar{y}^+(\beta - r)$ and compute

$$\langle -\bar{p}(r), \bar{y} - \bar{x}\rangle = \langle -\bar{p}(r), \bar{y} - \bar{y}_1\rangle + \langle -\bar{p}(r), \bar{y}_1 - \bar{x}\rangle. \tag{3.7.35}$$

Since $r < \beta \le r_1$, one can see that $T(\bar{y}_1) \ge r$. Thus, $\bar{y}_1 \in \mathcal{S}'(r)$. Then, recalling Lemma 3.7.3 we get

$$\langle -\bar{p}(r), \bar{y}_1 - \bar{x}\rangle \le \frac{1}{2\rho(r)}\ |\bar{p}(r)|\ |\bar{y}_1 - \bar{x}|^2.$$

Using Lemma 3.7.2, we also have

$$\begin{aligned} |\bar{y}_1 - \bar{x}| &\le |\bar{y}_1 - \bar{y}| + |\bar{y} - \bar{x}| \\ &\le K_2(|\bar{y}| + 1)e^{K_2(\beta - r)}|\beta - r| + |\bar{y} - \bar{x}|. \end{aligned}$$

So,

$$\langle -\bar{p}(r), \bar{y}_1 - \bar{x}\rangle \leq \frac{K_2^2(|\bar{y}|+1)^2 e^{2K_2(\beta-r)}+1}{2\rho(r)}\ |\bar{p}(r)|\ (|\bar{y}-\bar{x}|^2\ +\ |\beta-r|^2). \tag{3.7.36}$$

On the other hand, recalling (3.7.7) we have

$$\begin{aligned}
&\langle -\bar{p}(r), \bar{y}-\bar{y}_1\rangle \\
&= \Big\langle -\bar{p}(r), \int_0^{\beta-r} -F_{-\bar{p}(r)}(\bar{y}^+(s))ds\Big\rangle = \int_0^{\beta-r}\Big\langle -\bar{p}(r), -F_{-\bar{p}(r)}(\bar{y}^+(s))\Big\rangle ds \\
&= \int_0^{\beta-r}\Big\langle -\bar{p}(r), -F_{-\bar{p}(r)}(\bar{y}^+(s)) + F_{-\bar{p}(r)}(\bar{x})\Big\rangle ds + \int_0^{\beta-r}\Big\langle -\bar{p}(r), -F_{-\bar{p}(r)}(\bar{x})\Big\rangle ds \\
&\qquad \leq K_1\ |\bar{p}(r)| \int_0^{\beta-r} |\bar{y}^+(s)-\bar{x}|ds + \int_0^{\beta-r} -H(x,-\bar{p}(r))ds \\
&\qquad\qquad = K_1\ |\bar{p}(r)| \int_0^{\beta-r} |\bar{y}^+(s)-\bar{x})|ds + \lambda(r-\beta).
\end{aligned}$$

Owing to Lemma 3.7.2, for all $s \in [0, \beta - r]$

$$\begin{aligned}
|\bar{y}^+(s)-\bar{x}| &\leq |\bar{y}^+(s)-\bar{y}|\ +\ |\bar{y}-\bar{x}| \\
&\leq K_2(|\bar{y}|+1)e^{K_2(\beta-r)}|\beta-r| + |\bar{y}-\bar{x}|.
\end{aligned}$$

Therefore,

$$\langle -\bar{p}(r), \bar{y}-\bar{y}_1\rangle \leq \lambda(r-\beta) + K_1[1+K_2(|\bar{y}|+1)e^{K_2(\beta-r)}]\ |\bar{p}(r)|\ (|\bar{y}-\bar{x}|^2+|\beta-r|^2). \tag{3.7.37}$$

Combining (3.7.35), (3.7.36) and (3.7.37), we get

$$\Big\langle \frac{(-\bar{p}(r),\lambda)}{|(-\bar{p}(r),\lambda)|}\ ,\ (\bar{y}-\bar{x}, \beta-r)\Big\rangle \leq L_3\ (|\bar{y}-\bar{x}|^2+\ |\beta-r|^2)$$

where $L_3 = \frac{K_2^2(|\bar{y}|+1)^2 e^{2K_2(\beta-r)}+1}{2\rho(r)} + K_1[1+K_2(|\bar{y}|+1)e^{K_2(\beta-r)}]$. The dependence of L_3 on $|\bar{y}|$ can be easily disposed of taking

$$L_4(|\bar{x}|, r) = \frac{K_2^2(|\bar{x}|+2)^2 e^{2K_2}+1}{2\rho(r)} + K_1[1+K_2(|\bar{x}|+2)e^{K_2}] + 1.$$

Then, the above inequality yields

$$\langle \frac{(-\bar{p}(r),\lambda)}{|(-\bar{p}(r),\lambda)|}\ ,\ (\bar{y}-\bar{x}, \beta-r)\rangle \leq L_4(|\bar{x}|, r)\ (|\bar{y}-\bar{x}|^2+\ |\beta-r|^2). \tag{3.7.38}$$

Recalling (3.7.33) and (3.7.38), and taking

$$\rho_T := \Big(\max\big\{2L_1(|\bar{x}|, T(\bar{x})) + L_2(|\bar{x}|, T(\bar{x})), 2L_4(|\bar{x}|, T(\bar{x}))\big\}\Big)^{-1}, \tag{3.7.39}$$

we obtain (3.7.28). Finally, since $T(\cdot)$ is continuous on $\mathcal{O}$, one can easily see that if we set $\rho_T(\bar{x}) = \rho_T$ then $\rho_T(\cdot)$ is also continuous on $\mathcal{O}$. The proof is complete. □

Proof of Theorem 3.7.1. Let $\bar{x} \in \mathcal{O}$. Let $r = T(\bar{x})$ and let $\bar{x}^+(\cdot)$ be an optimal trajectory steering $\bar{x}$ to $\mathcal{S}$ in time r. By the dynamic programming principle, $T(\bar{x}^+(t)) = r - t$ for all $t \in (0, r)$. This implies that $\bar{x}^+(t)$ is not a local maximum of $T(\cdot)$ for all $t \in (0, r)$. Therefore, by applying Lemma 3.7.5, we obtain that for all $t > 0$ sufficiently small, there exists a unit vector $\bar{q}(t) \in N^P_{\mathrm{hypo}(T_{|\mathcal{O}})}(\bar{x}^+(t), T_{|\mathcal{O}}(\bar{x}^+(t)))$ realized by a ball of radius $\rho_T(\bar{x}^+(t))$ where $\rho_T(\cdot)$ is given by (3.7.39), i.e, for all $\bar{y} \in \mathcal{O}$ and $\beta \le T(\bar{y})$

$$\Big\langle \bar{q}(t)\ ,\ \big(\bar{y} - \bar{x}^+(t), \beta - T(\bar{x}^+(t))\big) \Big\rangle \le \frac{1}{2\rho_T(x^+(t))}(|\bar{y} - \bar{x}^+(t)|^2 + |\beta - T(x^+(t))|^2). \quad (3.7.40)$$

Since $\bar{q}(t)$ is a unit vector in $\mathbb{R}^{n+1}$ for all $t > 0$ sufficiently small, there exists a sequence $\{t_k\}$ which converges to 0^+ such that the sequence $\{\bar{q}(t_k)\}$ converges to a unit vector $\bar{q}$ in $\mathbb{R}^{n+1}$. Taking $t = t_k$ and then letting $k \to \infty$ in (3.7.40), by the continuity of $T(\cdot)$ and $\rho_T(\cdot)$ in $\mathcal{O}$, we obtain that for all $\overline{y} \in \mathcal{O}$ and $\beta \le T(\overline{y})$,

$$\Big\langle \bar{q}\ ,\ \big(\bar{y} - \bar{x}, \beta - T(\bar{x})\big) \Big\rangle \le \frac{1}{2\rho_T(\bar{x})}(|\bar{y} - \bar{x}|^2 + |\beta - T(\overline{y})|^2)$$

where $\rho_T(\cdot)$ is given by (3.7.39). Therefore, $\bar{q} \in N^P_{\mathrm{hypo}(T_{|\mathcal{O}})}(\bar{x}, T_{|\mathcal{O}}(\bar{x}))$ is realized by a ball of radius $\rho_T(T(\bar{x}))$. The proof is complete. □

We conclude this part with an example where Petrov's controllability condition does not hold, and the minimum time function T is just continuous. Moreover, multifunction F admits no C^1 parameterization even though F and H satisfy assumptions (F) and (H). Therefore, the results in [22, 20, 51] do not apply to this example while our results do.
Example 1. Set

$$\gamma(t) = \begin{cases} (1, t) & t \le 0 \\ \big(1 - \sqrt{-t^2 + 2t}, t\big) & 0 \le t \le 1 \\ \big(0, t\big) & t \ge 1. \end{cases}$$

We set the target $\mathcal{S}$ to be the right part of $\mathbb{R}^2 \backslash \{\gamma\}$ (see Figure 1) and the differential inclusion to be

$$\big(\dot{x}_1(t), \dot{x}_2(t)\big) \in F\big(x_1(t), x_2(t)\big) = \Big\{\big(u_1, h(x_2(t))u_2\big) \mid u_1, u_2 \in [0, 1]\Big\},$$

where
$$h(x_2) = \begin{cases} 0 & \text{if } x_2 \le 1 \\ x_2 - 1 & \text{if } x_2 \ge 1. \end{cases}$$

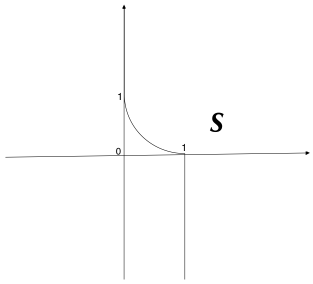

Figure 1

Observe first that $\mathcal{S}$ has the inner ball property. Observe furthermore that for $0 < t \le 1$, the point $z_t = \big(1 - \sqrt{-t^2 + 2t}, t\big)$ is on the boundary of $\mathcal{S}$, and
$$\min_{v \in F(z_t)} \langle v, \nu \rangle = -\sqrt{-t^2 + 2t}\ |\nu|,$$
where ν is the proximal vector to $\mathcal{S}$. Therefore, since $\lim_{t \to 0^+} \sqrt{-t^2 + 2t} = 0$, one can see that Petrov's controllability condition (3.7.5) does not hold in a neighborhood of $(1, 0)$. Moreover, the minimum time function T equals
$$T(x_1, x_2) = \begin{cases} 1 - x_1 & \text{if } x_1 \le 1, x_2 \le 0 \\ 1 - \sqrt{-x_2^2 + 2x_2} - x_1 & \text{if } x_1 \le 1 - \sqrt{-x_2^2 + 2x_2}, 0 < x_2 \le 1 \\ -x_1 & \text{if } x_1 \le 0, x_2 > 1\,. \end{cases}$$
So, T is continuous, but is not Lipschitz at points $(x_1, 0)$ for $x_1 \le 1$.

We next show that F admits no C^1 parameterization. We first recall a criterion from [23, p3]: if F admits a C^1 parameterization, then the Hamiltonian H (see (3.7.4)) necessarily has the property
$$H(x, p) = -H(x, -p) \quad \Longrightarrow \quad \partial_x H(x, p) = -\partial_x H(x, -p), \tag{3.7.41}$$
where ∂_x denotes the Clarke partial subgradient in x. In this example, the Hamiltonian H is computed as
$$H\big((x_1, x_2), (p_1, p_2)\big) = \begin{cases} 0 & p_1 < 0, p_2 < 0, \\ p_1 & p_1 \ge 0, p_2 < 0, \\ h(x_2)p_2 & p_1 < 0, p_2 \ge 0, \\ p_1 + h(x_2)p_2 & p_1 \ge 0, p_2 \ge 0. \end{cases}$$
At the point $(x_1, x_2) = (1, 1)$, one has $H\big((1,1), (0, -1)\big) = H\big((1,1), (0,1)\big) = 0$. However,
$$\partial_x H\big((1,1), (0, -1)\big) = (0, 0) \quad \text{and} \quad \partial_x H\big((1,1), (0,1)\big) = (0, [0,1])\,.$$

Since (3.7.41) is violated at the point $(1,1)$, there is no C^1 parameterization of F.

Finally, since h is a convex function, one can also prove that F and H satisfies the assumptions (F) and (H). Therefore, by applying Theorem 3.7.1, the hypograph of T satisfies a $\rho_T(\cdot)$-exterior sphere condition.

Part II

In this second part, we will study the attainable set $\mathcal{A}(T)$ from 0 for the reversed differential inclusion

$$\begin{cases} \dot{x}(t) \in -F(x(t)) \quad a.e. \\ x(0) = 0. \end{cases} \tag{3.7.42}$$

For any $T > 0$, such set is defined by

$$\mathcal{A}(T) := \{y(t) \mid t \in [0,T] \text{ and } y(\cdot) \text{ is a solution of (3.7.42)}\}.$$

Let us recall that c_0, K and K_1 are the constants appearing in (H_1), (3.7.6) and (3.7.7), respectively.

Theorem 3.7.2 *Assume F satisfies (F) and (H). In addition, suppose that, for some $R > 0$ and all $x \in \mathbb{R}^n$, $F(x)$ has the inner ball property of radius R. If $T > 0$ and $e^{-3KT} > 2c_0RT^2$, then the attainable set $\mathcal{A}(T)$ has the inner ball property of radius*

$$R(T) = R\frac{\left(e^{-3KT} - 2c_0RT^2\right)}{(1 + KT + K_1T)^2}. \tag{3.7.43}$$

Proof. Let $\bar{x} \in \partial\mathcal{A}(T)$ and let $\bar{x}^-(\cdot)$ be a trajectory of (3.7.42) steering 0 to $\bar{x}$ in time T. By the Pontryagin maximum principle, there exists an arc $\bar{p}(\cdot)$ defined in $[0,T]$, with $\bar{p}(s) \neq 0$ for all $s \in [0,T]$, such that

$$\begin{cases} -\dot{\bar{p}}(s) \in \partial_x H(\bar{x}^-(s), -\bar{p}(s)) \quad a.e.\ s \in [0,T] \\ \dot{\bar{x}}^-(s) = -F_{-\bar{p}(s)}(\bar{x}^-(s)) \quad a.e.\ s \in [0,T]. \end{cases} \tag{3.7.44}$$

We want to prove that, for $r_0 := R(T)$ (where $R(T)$ is defined in (3.7.43)),

$$B\Big(\bar{x} + r_0T\frac{-\bar{p}(T)}{|\bar{p}(T)|}, r_0T\Big) \subseteq \mathcal{A}(T). \tag{3.7.45}$$

Equivalently,

$$\bar{x} - r_0T\Big(\frac{\bar{p}(T)}{|\bar{p}(T)|} - \theta\Big) \in \mathcal{A}(T) \text{ for all } \theta \in B(0,1).$$

Let $\theta \in B(0,1)$. Considering the adjoint equation associated with $\bar{p}(\cdot)$, that is,

$$\begin{cases} \dot{\bar{z}}(s) = -\frac{\langle \dot{\bar{p}}(s), \bar{z}(s)\rangle}{|\bar{p}(s)|^2}\bar{p}(s) \ a.e. \\ \\ \bar{z}(T) = \frac{\bar{p}(T)}{|\bar{p}(T)|} - \theta, \end{cases} \tag{3.7.46}$$

one can see that

$$\langle \dot{\bar{z}}(s), \bar{p}(s) \rangle = -\langle \dot{\bar{p}}(s), \bar{z}(s) \rangle \text{ for a.e. } s \in [0, T]. \tag{3.7.47}$$

This implies that $\frac{d}{ds}\langle \bar{z}(s), \bar{p}(s) \rangle = 0$ for a.e. $s \in [0, T]$. Therefore, $\langle \bar{z}(s), \bar{p}(s) \rangle$ is constant for all $s \in [0, T]$. In particular,

$$\langle \bar{z}(s), \bar{p}(s) \rangle = \langle \bar{z}(T), \bar{p}(T) \rangle\,. \tag{3.7.48}$$

On the other hand, from (3.7.46) we have $|\dot{\bar{z}}(s)| \le K|\bar{z}(s)|$. Thus, recalling Lemma 3.7.1 we obtain

$$e^{-K(t_2-t_1)}|\bar{z}(t_2)| \le |\bar{z}(t_1)| \le e^{K(t_2-t_1)}|\bar{z}(t_2)| \quad \text{for all } 0 \le t_1 \le t_2 \le T. \tag{3.7.49}$$

Set

$$\bar{y}_\theta(s) = \bar{x}(s) - r_0 s \bar{z}(s), \tag{3.7.50}$$

we have $\bar{y}_\theta(T) = \bar{x} - r_0 T(\frac{\bar{p}(T)}{|\bar{p}(T)|} - \theta)$. Thus, our aim is now to prove that $\bar{y}_\theta(T) \in \mathcal{A}(T)$. Since $\bar{y}_\theta(0) = \bar{x}(0) = 0$, we only need to show

$$\dot{\bar{y}}_\theta(s) \in -F(\bar{y}_\theta(s)) \quad \text{for a.e. } s \in [0, T].$$

Observe that $F_{-\bar{p}(s)}(\bar{y}_\theta(s)) \in \partial F(\bar{y}_\theta(s))$. Since $F(\bar{y}_\theta(s))$ is convex and has the inner ball property of radius R, we have that $\bar{p}(s)$ is an inner normal vector to $\partial F(\bar{y}_\theta(s))$ at the point $F_{-\bar{p}(s)}(\bar{y}_\theta(s))$. Thus, $-\dot{\bar{y}}_\theta(s) \in F(\bar{y}_\theta(s))$ (equivalently, $\dot{\bar{y}}_\theta(s) \in -F(\bar{y}_\theta(s))$) if $-\dot{\bar{y}}_\theta(s) \in B(F_{-\bar{p}(s)}(\bar{y}_\theta(s)) + R\frac{\bar{p}(s)}{|\bar{p}(s)|}, R)$. Therefore,[2] our conclusion will follow from

$$\Big\langle \frac{\bar{p}(s)}{|\bar{p}(s)|}, -\dot{\bar{y}}_\theta(s) - F_{-\bar{p}(s)}(\bar{y}_\theta(s)) \Big\rangle \ge \frac{1}{2R}\, |\dot{\bar{y}}_\theta(s) + F_{-\bar{p}(s)}(\bar{y}_\theta)(s)|^2. \tag{3.7.51}$$

Equivalently,

$$\Big\langle -\frac{\bar{p}(s)}{|\bar{p}(s)|}, \dot{\bar{y}}_\theta(s) + F_{-\bar{p}(s)}(\bar{y}_\theta(s)) \Big\rangle \ge \frac{1}{2R}\, |\dot{\bar{y}}_\theta(s) + F_{-\bar{p}(s)}(\bar{y}_\theta)(s)|^2. \tag{3.7.52}$$

We are now going to prove (3.7.52). On account of (3.7.50), we have

$$\dot{\bar{y}}_\theta(s) = -F_{-\bar{p}(s)}(\bar{x}(s)) - r_0 \bar{z}(s) - r_0 s \dot{\bar{z}}(s).$$

Thus,

$$\begin{aligned}
&\Big\langle -\frac{\bar{p}(s)}{|\bar{p}(s)|}, \dot{\bar{y}}_\theta(s) + F_{-\bar{p}(s)}(\bar{y}_\theta(s)) \Big\rangle \\
&\quad = \Big\langle -\frac{\bar{p}(s)}{|\bar{p}(s)|}, F_{-\bar{p}(s)}(\bar{y}_\theta(s)) - F_{-\bar{p}(s)}(\bar{x}(s)) - r_0 \bar{z}(s) - r_0 s \dot{\bar{z}}(s) \Big\rangle \\
&\qquad = \frac{1}{|\bar{p}(s)|}\, \big[H(\bar{y}_\theta(s), -\bar{p}(s)) - H(\bar{x}(s), -\bar{p}(s))\big] \\
&\qquad\qquad + \frac{r_0}{|\bar{p}(s)|}\langle \bar{p}(s), \bar{z}(s) \rangle + r_0 s \frac{1}{|\bar{p}(s)|}\langle \bar{p}(s), \dot{\bar{z}}(s) \rangle.
\end{aligned}$$

[2] Observe that for all $R > 0$ and $x \in \mathbb{R}^N$, $y \in \overline{B}(x + Rv, R) \Leftrightarrow \langle v, y - x \rangle \ge \frac{1}{2R}|y - x|^2$ where $v \in \mathbb{R}^N$ is any unit vector.

Recalling (3.7.47), (3.7.50) and (3.7.48), we conclude that

$$\begin{aligned}
&\Big\langle -\frac{\bar p(s)}{|\bar p(s)|}, \dot{\bar y}_\theta(s) + F_{-\bar p(s)}(\bar y_\theta(s))\Big\rangle \\
&\quad = \frac{1}{|\bar p(s)|}\big[H(\bar y_\theta(s), -\bar p(s)) - H(\bar x(s), -\bar p(s)) - \langle -\dot{\bar p}(s), \bar y_\theta(s) - \bar x(s)\rangle\big] \\
&\qquad\qquad + \frac{r_0}{|\bar p(s)|}\langle \bar p(T), \bar z(T)\rangle \geq -c_0\ |\bar y_\theta(s) - \bar x(s)|^2 \\
&+ r_0\frac{|\bar p(T)|}{|\bar p(s)|}\Big\langle \frac{\bar p(T)}{|\bar p(T)|}, \frac{\bar p(T)}{|\bar p(T)|} - \theta\Big\rangle \geq -c_0 r_0^2 s^2 |\bar z(s)|^2 + \frac{r_0}{2}\frac{|\bar p(T)|}{|\bar p(s)|}\Big|\frac{\bar p(T)}{|\bar p(T)|} - \theta\Big|^2 \\
&\qquad\qquad = -c_0 r_0^2 s^2|\bar z(s)|^2 + \frac{r_0}{2}\frac{|\bar p(T)|}{|\bar p(s)|}|\bar z(T)|^2.
\end{aligned}$$

Recalling Lemma 3.7.1 and (3.7.49), we obtain

$$\Big\langle -\frac{\bar p(s)}{|\bar p(s)|}, \dot{\bar y}_\theta(s) + F_{-\bar p(s)}(\bar y_\theta(s))\Big\rangle \geq \frac{r_0}{2}(-2c_0 r_0 T^2 + e^{-3KT})|\bar z(s)|^2. \quad (3.7.53)$$

Observe that $0 < r_0 = R(T) = R\frac{(e^{-3KT} - 2c_0 RT^2)}{(1+KT+K_1T)^2} \leq R$. Then,

$$\Big\langle -\frac{\bar p(s)}{|\bar p(s)|}, \dot{\bar y}_\theta(s) + F_{-\bar p(s)}(\bar y_\theta(s))\Big\rangle \geq \frac{r_0}{2}(-2c_0 RT^2 + e^{-3KT})\ |\bar z(s)|^2. \quad (3.7.54)$$

On the other hand,

$$\begin{aligned}
&|\dot{\bar y}_\theta(s) + F_{-\bar p(s)}(\bar y_\theta(s))| \\
&\qquad \leq |F_{-\bar p(s)}(\bar y_\theta(s)) - F_{-\bar p(s)}(\bar x(s))| + r_0|\bar z(s)| + r_0 s|\dot{\bar z}(s)| \\
&\qquad\qquad \leq K_1 r_0 s|\bar z(s)| + r_0|\bar z(s)| + K r_0 s|\bar z(s)| \\
&\qquad\qquad\qquad \leq r_0(K_1 T + KT + 1)|\bar z(s)|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big\langle -\frac{\bar p(s)}{|\bar p(s)|}, \dot{\bar y}_\theta(s) + F_{-\bar p(s)}(\bar y_\theta(s))\Big\rangle \\
&\qquad\qquad \geq \frac{e^{3KT} - 2c_0 RT^2}{2r_0(K_1T + KT + 1)^2}|\dot{\bar y}_\theta(s) + F_{-\bar p(s)}(\bar y_\theta(s))|^2,
\end{aligned}$$

and (3.7.52) follows . The proof is complete. □

Finally, let us denote by $\mathcal{A}(x, T)$ the attainable set from x in time T for the differential inclusion in (3.7.42). One can see from Theorem 3.7.2 that there exists a time $T_0 > 0$ such that for all $0 < T < T_0$, the set $\mathcal{A}(x, T)$ has the inner ball property of radius $R(T)$ given by (3.7.43). Moreover, for any closed set $\mathcal{S} \subset \mathbb{R}^N$, let us set

$$\mathcal{A}(\mathcal{S}, T) \ = \bigcup_{x \in \mathcal{S}} \mathcal{A}(x, T).$$

Corollary 3.7.4 *Suppose that $\mathcal{S}$ is nonempty and closed. Under the assumptions in Theorem 3.7.2, there exists $T_0 > 0$ such that, for all $0 < T < T_0$, then the set $\mathcal{A}(\mathcal{S}, T)$ has the inner ball property of radius $R(T)$ given by (3.7.43).*

Applying Theorem 3.7.1 and the above results to the minimum time function for a general target, we obtain the following theorem together with useful corollaries.

Theorem 3.7.3 *Assume (F), (H) and suppose $F(x)$ has the inner ball property of radius R for some $R > 0$ and all $x \in \mathbb{R}^n$. Suppose further that $\mathcal{S}$ is nonempty, closed and $T(\cdot)$ is continuous in a open subset $\mathcal{O}$ of $\mathcal{C}$. Then, the hypograph of $T_{|\mathcal{O}}(\cdot)$ satisfies the $\rho_T(\cdot)$-exterior sphere condition for some continuous function $\rho_T(\cdot) : \mathcal{O} \to (0, \infty)$.*

Proof. We define, for any $0 < t < T_0$,

$$\mathcal{O}^t = \{x \in \mathcal{O} \mid T(x) > t\}, \quad \mathcal{S}^t = \mathcal{A}(\mathcal{S}, t),$$

and $T^t(\cdot)$ is the minimum time function for the differential inclusion (3.7.1) with the target $\mathcal{S}_t$. One can see that

$$T^t(x) = T(x) - t \quad \text{for all } x \in \mathcal{O}^t.$$

Since S^t has the inner ball property of radius $R(t) > 0$, by applying Theorem 3.7.1 to $T^t(\cdot)$, we obtain that the hypograph of $T^t_{|\mathcal{O}^t}(\cdot)$ satisfies a $\rho^t_T(\cdot)$-exterior sphere condition for some continuous function $\rho^t_T(\cdot) : \mathcal{O}^t \to (0, \infty)$. Hence, the hypograph of $T_{|\mathcal{O}^t}(\cdot)$ satisfies the $\rho^t_T(\cdot)$-exterior sphere condition. Observe that, since $\mathcal{O}^{t_1} \subseteq \mathcal{O}^{t_2} \subseteq \mathcal{O}$ for $0 < t_1 < t_2 < T_0$ and $\cup_{t \in (0,T_0)} \mathcal{O}^t = \mathcal{O}$, one can prove that $T_{|\mathcal{O}}(\cdot)$ satisfies a $\rho_T(\cdot)$-exterior sphere condition for some continuous function $\rho_T(\cdot) : \mathcal{O} \to (0, \infty)$. □

Corollary 3.7.5 *Under the assumptions of Theorem 3.7.3, $T_{|\mathcal{O}}(\cdot)$ satisfies properties* (1)-(3) *in Theorem 2.2.2.*

Corollary 3.7.6 *Under the assumptions of Theorem 3.7.3, if $T(\cdot)$ is locally Lipschitz in $\mathcal{O}$, then $T(\cdot)$ is locally semiconcave in $\mathcal{O}$.*

Chapter 4

Semiconvexity type results

In the theory of autonomous linear control systems, the assumption of *normality*, i.e., a strong controllability assumption requiring that if each control component is used separately, then Kalman rank condition is satisfied, is well known. In particular (see [44, Sections 14, 15, and 16]), normality implies that the control steering the origin to a point x in minimum time is unique and bang-bang; moreover the reachable set from the origin, $\mathcal{R}^\tau$, is a *strictly* convex body for all times $\tau > 0$. Simple examples, on the other hand, show that convexity of the reachable set is easily lost when passing to a nonlinear dynamics, even if the control covers all directions and appears linearly. In particular, the control system

$$\begin{cases} \dot{y}_1(t) & = -[y_2(t)]^2 + u_1(t), \quad -1 \leq u_1 \leq 1 \\ \dot{y}_2(t) & = u_2(t) \qquad\qquad\qquad\quad -1 \leq u_2 \leq 1 \end{cases} \tag{4.0.1}$$

which was analyzed in [21], fails to have convex (and even normally regular, i.e., "without inward corners") reachable sets from the origin at any positive time.

Similarly, the minimum time to reach the origin for a linear system is semiconvex, i.e., it is a quadratic perturbation of a convex function, provided a first order controllability assumption is satisfied (see [21, 22]), or has epigraph with positive reach, i.e., it satisfies a quite good kind of generalized convexity, provided Kalman rank condition holds (see [30, 32]). On the other hand, the same example introduced above shows that such properties may fail even for a simple nonlinear dynamics.

We are not aware of any result of semiconvexity type valid for a nonlinear dynamics, where actually results of semiconcave type are more natural and easier to obtain, provided the target (or the dynamics) satisfy an inner ball condition (see [22, 18, 20, 51, 24, 20]). Further results on the regularity of the minimum time function T, for two dimensional systems with a single input control, are described in [13, Chapter 3], where in particular, under generic assumptions, a characterization of smooth and nonsmooth points of

the level sets of T is given.
In this chapter, we give a contribution to the understanding of the behavior of the minimum time function both for linear, autonomous and nonautonomous, and nonlinear control systems. We take the origin as the target (or as the source, for the reversed dynamics), hence going outside the realm of semiconcavity, and give results, under easily verifiable assumptions, on the following topics:

- strict convexity of the reachable set from the origin
- uniqueness of the optimal control
- a nonlinear bang-bang principle
- extending backwards optimal trajectories
- positive reach of the epigraph of T.

Our method is based on new results on linear control systems satisfying the normality condition, and on linearization at the origin. The underlying idea, in fact, is requiring enough strength to such linearization, so that examples like (4.0.1) are ruled out. Our linear results hold in any space dimension, while the nonlinear part is confined to two or three dimensional spaces. Our arguments are based essentially on a careful analysis of the *switching function*, namely the function whose sign is expected to determine the optimal control, according to Pontryagin's Maximum Principle. The point is exactly showing that this sign is well defined, except for at most finitely many zeros. To this aim, the normality assumption is pivotal, as it permits to split any finite interval into finitely many sets, each one being a disjoint union of finitely many intervals, where the switching function or its derivatives are uniformly bounded away from zero (see Lemma 4.2.2). From this fact we are able to deduce a quantitative estimate on the strict convexity of the reachable set. More precisely (see Theorem 4.2.1), for a linear control system in $\mathbb{R}^N$ we show that for all $\tau > 0$ there exists a positive constant $\gamma > 0$ such that

$$\langle \zeta, y - x \rangle \leq -\gamma \|\zeta\| \, \|y - x\|^N \qquad \text{for all } x, y \in \mathcal{R}^\tau, \ \zeta \in N_{\mathcal{R}^\tau}(x)$$

(here $N_{\mathcal{R}^\tau}(x)$ denotes the normal cone to $\mathcal{R}^\tau$ at x). We show through an example that the exponent N is optimal. Section 4.2 is devoted to the above topic, together with an auxiliary study for a linear nonautonomous dynamics.
The nonlinear part starts with a nonlinear bang-bang result (Section 4.3), valid up to dimension 3. We consider a nonlinear control system which is affine with respect to the control: if the linearization at the origin is normal, then every optimal control is bang-bang. The proof is based on

Pontryagin's Maximum Principle: if the nonlinearity contains only parts which are of order larger or equal to the space dimension, then we are able to transfer to the switching function all the properties satisfied by the switching function of the dynamics linearized at the origin. This idea is at the basis also of the strict convexity of the reachable set for a nonlinear two dimensional dynamics (see Theorem 4.4.1) and of proving that all points close enough to the origin are *optimal*, i.e., any trajectory steering a point to the origin optimally can be extended backwards still remaining optimal (see Theorem 4.5.1). In this case, the difficulty is extending the optimal control: our analysis permits to predict backwards the sign of the switching function. Finally, we show under the same assumptions that the epigraph of the minimum time function T has positive reach, hence obtaining a rich bunch of regularity properties for T, as listed in Theorem 2.2.2. We also show through an example (see Example 4.4.2) that the assumptions on the nonlinear part cannot be avoided, while (4.0.1) shows that normality at the origin is essential. The restrictions on the space dimension for the nonlinear results will be explained after the relevant proofs (see Remarks 4.3.1 and 4.4.3).
To our knowledge, the results we present here as well as most of the used methods do not trace back to previous literature. In particular, the nonlinear bang-bang results of Krener [45] and Sussmann [61] seem to be of a very different nature.

4.1 Linear control system and normality

We will consider control systems linear or nonlinear with respect to the space variable and affine and symmetric with respect to the control. More precisely, we will consider the linear control system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) & \text{a.e.} \\ u(t) \in \mathcal{U} = [-1,1]^M & \text{a.e.} \\ y(0) = 0, \end{cases} \tag{4.1.1}$$

where $1 \leq M \leq N$ and $A \in \mathbb{M}_{N\times N}$, $B \in \mathbb{M}_{N\times M}$, being possibly time dependent, and $\mathcal{U} = [-1,1]^M \ni (u_1, \ldots, u_M) =: u$, together with the nonlinear control system

$$\begin{cases} \dot{y}(t) = F(y(t)) + G(y(t))u(t) & \text{a.e.} \\ u(t) \in [-1,1]^M & \text{a.e.} \\ y(0) = 0, \end{cases} \tag{4.1.2}$$

where F and G are suitable vector fields (the actual assumptions will be stated later). We will use also the notation $B = (b_1, \ldots, b_M)$ or $G = (g_1, \ldots, g_M)$, where each entry is an N-dimensional column. We denote by $\mathcal{U}_{ad}$, the set of admissible controls, i.e., all measurable functions u, such that $u(s) \in \mathcal{U}$ for a.e. s. For any $u(\cdot) \in \mathcal{U}_{ad}$, the (unique, as it will follow from the assumptions on F and G) Carathéodory solution of (4.1.1) or of (4.1.2) is denoted by $y^u(\cdot)$. In the linear case,

$$y^u(t) = \int_0^t e^{A(t-s)} Bu(s)ds,$$

so that the reachable set from 0 in time t can be described by

$$\mathcal{R}^t = \Big\{ \int_0^t e^{A(t-s)} Bu(s)ds) \mid u(\cdot) \in \mathcal{U}_{ad} \Big\}. \tag{4.1.3}$$

It is well known that in the linear case the set $\mathcal{R}^t$ is convex and compact (see, e.g., [44, Lemma 12.1]), while in the nonlinear case (4.1.2) $\mathcal{R}^t := \{y^u(t) \mid u(\cdot) \in \mathcal{U}_{ad}\}$ is compact and not necessarily convex (see, e.g., [25, Chapter 10]).

For a fixed $x \in \mathbb{R}^N$, we define

$$\theta(x, u) := \min \{t \geq 0 \mid y^{x,u}(t) = 0\},$$

where $y^{x,u}(\cdot)$ denotes the solution of $\dot{y} = F(y) + G(y)u$ such that $y(0) = x$. Of course, $\theta(x, u) \in [0, +\infty]$, and $\theta(x, u)$ is the time taken for the trajectory $y^{x,u}(\cdot)$ to reach 0, provided $\theta(x, u) < +\infty$. The *minimum time* $T(x)$ to reach 0 from x is defined by

$$T(x) := \inf \{\theta(x, u) \mid u(\cdot) \in \mathcal{U}_{\rm ad}\}. \tag{4.1.4}$$

Observe that the sublevel $\mathcal{R}_t = \{x : T(x) \leq t\}$ of $T(\cdot)$ equals the reachable set from the origin within the (same) time t for the *reversed dynamics*

$$\dot{x} = -F(x) - G(x)u, \quad u \in \mathcal{U}.$$

If $\bar{u}$ is an admissible control steering x to the origin in the minimum time $T(x)$ (i.e., an *optimal* control), then the Dynamic Programming Principle (see, e.g., Proposition 2.1, Chapter IV, in [12]) implies that $T(\cdot)$ is strictly increasing along the optimal trajectory $y^{x,\bar{u}}$. Therefore, for all $0 < t < T(x)$ the point $y^{x,\bar{u}}(t)$ belongs to the boundary of $\mathcal{R}_t$.

Pontryagin's Maximum Principle is a fundamental tool for the analysis of optimal control problems. We state it for points belonging to the boundary of reachable sets. In view of the previous remark, this will apply also to points belonging to optimal trajectories. We give first its linear version.

Theorem 4.1.1 (PMP for linear systems) *Consider the problem* (4.1.1)*, fix $T > 0$, and suppose $\bar{x} \in \mathcal{R}^T$ is realized by the control $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ (i.e., $y^{\bar{u}}(T) = \bar{x}$). Then $\bar{x} \in \operatorname{bdry} \mathcal{R}^T$ if and only if for some $\zeta \in N_{\mathcal{R}^T}(\bar{x})$, $\zeta \neq 0$, it holds*

$$\bar{u}_i(t) = \operatorname{sign}\langle \zeta, e^{A(T-t)} b_i \rangle, \quad a.e.\ t \in [0, T], \tag{4.1.5}$$

for all $i = 1, 2, \ldots, M$.

A well known reference for this result is [44, Lemma 13.1].

Before stating Pontryagin's principle for the nonlinear case (4.1.2) (with $M = 1$), we need to introduce the *Maximized Hamiltonian.*
We define for every triple $(x, p, u) \in \mathbb{R}^N \times \mathbb{R}^N \times [-1, 1]$

$$\mathcal{H}(x, p, u) = \langle p, F(x)\rangle + u\langle p, G(x)\rangle$$

and

$$H(x, p) = \max\{\mathcal{H}(x, p, u) : u \in [-1, 1]\}.$$

Then Pontryagin's principle reads as follows (see, e.g., [13, Section 2.1] or [27, Theorem 3.5.4]).

Theorem 4.1.2 (PMP for nonlinear systems) *Fix $T > 0$ and let y^u be a trajectory of (4.1.2) such that $y^u(T)$ belongs to the boundary of $\mathcal{R}^T$. Then there exist a function $\lambda : [0, T] \to \mathbb{R}^N$, never vanishing, and a constant $\lambda_0 \leq 0$ such that for a.e. $t \in [0, T]$ one has:*

i) $\dot{\lambda}(t) = -\lambda(t)\big(DF(y^u(t)) + DG(y^u(t))u(t)\big)$,

ii) $\mathcal{H}(y^u(t), \lambda(t), u(t)) + \lambda_0 = 0$,

iii) $\mathcal{H}(y^u(t), \lambda(t), u(t)) = H(y^u(t), \lambda(t))$.

Furthermore,

$$\lambda(T) \in N_{\mathcal{R}^T}(y^u(T))$$

where here $N_{\mathcal{R}^T}(y^u(T))$ denotes the Clarke normal cone.

Definition 4.1.1 *We say that a control u is essentially determined by Pontryagin's Principle if for any u_1 satisfying iii) in Theorem 4.1.2 (for the adjoint curve λ associated with the trajectory y^u) one has $u_1(t) = u(t)$ a.e. in $[0, T]$.*

In the following we will make extensive use of the classical concept of normality for linear systems, which we are now going to introduce.

Definition 4.1.2 *The system (4.1.1) is normal if and only if for every column b_i of B, $i = 1, \ldots, M$, we have*

$$\text{Rank}\big[b_i, Ab_i, \ldots, A^{N-1}b_i\big] = N.$$

The main classical result for normal linear systems is concerned with the reachable set.

Theorem 4.1.3 *Assume that the linear control system(4.1.1) is normal. Then the reachable set $\mathcal{R}^T$ is strictly convex for any $T > 0$.*

Proof. One can find a proof in [44], Sections 14 and 15. □

Remark 4.1.1 *If the system (4.1.1) is normal then* (A, B) *satisfies the Kalman rank condition. Therefore the minimum time function is everywhere finite and continuous (actually Hölder continuous with exponent* $1/N$*, see, e.g., Theorem 17.3 in [44] and Theorem 1.9, Chapter IV, in [12] and references therein).*

4.2 Quantitative strict convexity of reachable sets for the linear case

4.2.1 Autonomous systems

This subsection is devoted to improving the classical result on the strict convexity of reachable sets for normal linear control systems of the type (4.1.1). We will give an estimate for the boundary of reachable sets which implies a uniform (polynomial) strict convexity with an optimal exponent. In the first Lemma we define the switching function and begin studying its behavior under the normality assumption.

Lemma 4.2.1 *Let* $A \in \mathbb{M}_{N\times N}$ *and* $b \in \mathbb{R}^N$ *be such that*

$$\mathrm{Rank}\big[b, Ab, \ldots, A^{N-1}b\big] = N. \tag{4.2.1}$$

Take $\zeta \in \mathbb{R}^N$*, with* $\|\zeta\| = 1$*, and define, for* $s \in [0, +\infty)$

$$g(s) \;=\; \langle e^{As}b \;,\; \zeta\rangle. \tag{4.2.2}$$

Then there exists a constant $\mathcal{L}$*, depending only on* A, b, N *such that, for all* $s \in [0, +\infty)$*,*

$$\sum_{i=0}^{N-1} |g^{(i)}(s)| \geq \mathcal{L}e^{-\|A\|s}. \tag{4.2.3}$$

Proof. Set

$$K = \big(b,\;\; Ab,\;\; \ldots,\;\; A^{N-1}b\big)$$

and observe that, by (4.2.1)

$$\mathcal{L} = \min_{\|\zeta\|=1} \;\|K\zeta\| \;> 0. \tag{4.2.4}$$

Fix $\zeta \in \mathbb{R}^N$ with $\|\zeta\| = 1$ and write $\zeta_1(s) = e^{sA^T}\zeta$. Observe that $\zeta = e^{-A^Ts}\zeta_1(s)$ and $\|\zeta_1(s)\| \geq e^{-s\|A\|}$. We compute now, for $i = 0, 1, \ldots, N-1$,

$$g^{(i)}(s) = \langle e^{As}A^ib, \zeta\rangle = \langle A^ib, \zeta_1(s)\rangle. \tag{4.2.5}$$

Therefore,

$$\begin{aligned} K\zeta_1(s) &= \big(b,\ Ab,\ \ldots,\ A^{N-1}b\big)\,\zeta_1(s) \\ &= \big(g^{(0)}(s),\ g^{(1)}(s),\ \ldots,\ g^{(N-1)}(s)\big)\,. \end{aligned}$$

Using (4.2.4) we have that

$$\|K\zeta_1(s)\| \geq \mathcal{L}e^{-s\|A\|}.$$

On the other hand,

$$\|K\zeta_1(s)\| \leq \sum_{i=0}^{N-1} |g^{(i)}(s)|$$

and the proof is concluded. □

The next Lemma is crucial for estimating the number of zeros of the switching function g (corresponding to the number of switching points of the optimal control associated with g) and for studying their multiplicity. We recall that the constant $\mathcal{L}$ was defined in (4.2.4).

Lemma 4.2.2 *Let $A \in \mathbb{M}_{N\times N}$ and $b \in \mathbb{R}^N$ be satisfying (4.2.1). Take $\zeta \in \mathbb{R}^N$, $\|\zeta\| = 1$, and fix $T > 0$. Let $g(s)$, $s \in [0,T]$, be defined as in (4.2.2).*

Then there exist disjoint sets $I_0, \ldots, I_{N-1}$ and numbers $\mathcal{N}_i$, depending only on A, b, T and N such that

$$[0,T] = \bigcup_{i=0}^{N-1} I_i$$

and, for all $i = 0, 1, \ldots, N-1$, the set I_i is the disjoint union of at most $\mathcal{N}_i$ intervals. Moreover, for each $i = 0, 1, \ldots, N-1$, for all $s \in I_i$, we have

$$|g^{(i)}(s)| \geq \frac{\mathcal{L}}{N} e^{-\|A\|T}. \tag{4.2.6}$$

Proof. We proceed by induction for i from 0 to $N-1$. Set

$$c(T) = \frac{\mathcal{L}e^{-\|A\|T}}{N}, \tag{4.2.7}$$

and

$$J_0 = \{s \in (0,T) \mid |g(s)| < c(T)\}.$$

Since J_0 is open, we can write it as the disjoint union of at most countably many open intervals,

$$J_0 = \bigcup_{k=1}^{\infty} J_0^k. \tag{4.2.8}$$

We assume, without loss of generality, that there are at least N such intervals. Fix now any number $N' \geq N$, and take a subfamily of the intervals J_0^k consisting of at most N' elements. Without loss of generality, we can rearrange their indexes k so that $J_0^k = (a_{2k}, a_{2k+1})$, where $1 \leq k \leq N'$ and $0 \leq a_2 < a_3 \leq a_4 < a_5 \leq \ldots \leq a_{2k} < a_{2k+1} \ldots < a_{2N'+1} \leq T$.

Now, fix k and consider the N intervals $(a_{2k}, a_{2k+1}), \ldots,$ $(a_{2(k+N-1)}, a_{2(k+N)-1})$. Set, for $j = 0, 1, \ldots, N-1$,

$$(a_{2(k+j)}, a_{2(k+j)+1}) := I_j^-,$$

and, for $j = 0, 1, \ldots, N-2$

$$[a_{2(k+j)+1}, a_{2(k+j+1)}] := I_j^+.$$

Observe that

$$\bigcup_{j=0}^{N-1} I_j^- \cup \bigcup_{j=0}^{N-2} I_j^+ = (a_{2k}, a_{2(k+N)-1}).$$

We are going to give a lower bound on $|a_{2(k+N)-1} - a_{2k}|$ which will turn out to be independent of both k and N'. From this fact it will follow automatically that the intervals (a_{2k}, a_{2k+1}) are nonempty only for finitely many k.

Observe that for each $j = 0, 1, \ldots, N-2$, there exists at least one point $c_j^1 \in I_j^+$ such that $g'(c_j^1) = 0$. Therefore, there exist at least $N-2$ points, say c_j^2, for $j = 0, 1, \ldots, N-3$, such that

$$c_j^2 \in (c_j^1, c_{j+1}^1) \quad \text{and} \quad g''(c_j^2) = 0.$$

Proceeding by induction we see that, for each $i = 1, \ldots, N-1$, there exists at least one point $c_i \in (a_{2k+1}, a_{2(k+N)-1})$ such that $g^{(i)}(c_i) = 0$.
Pick any $s_0 \in (a_{2k}, a_{2k+1})$. We have

$$|g(s_0)| < c(T), \tag{4.2.9}$$

and, for $i = 1, \ldots, N-1$,

$$\begin{aligned} |g^{(i)}(s_0)| &= |g^{(i)}(s_0) - g^{(i)}(c_i)| = \Big| \int_{s_0}^{c_i} g^{(i+1)}(s) ds \Big| \\ &\leq \int_{a_{2k}}^{a_{2(k+N)-1}} |g^{(i+1)}(s)| ds \leq (a_{2(k+N)-1} - a_{2k}) e^{\|A\|T} \|A^{i+1} b\|, \end{aligned}$$

where the last inequality is due to (4.2.5). Therefore,

$$\sum_{i=1}^{N-1} |g^{(i)}(s_0)| \leq (a_{2(k+N)-1} - a_{2k}) e^{\|A\|T} \sum_{i=1}^{N-1} \|A^{i+1} b\|. \tag{4.2.10}$$

On the other hand, recalling (4.2.3), (4.2.7), and (4.2.9) we have

$$\sum_{i=1}^{N-1} |g^{(i)}(s_0)| \geq \mathcal{L}e^{-\|A\|T} - c(T) = \frac{N-1}{N}\mathcal{L}e^{-\|A\|s_0} \geq \frac{N-1}{N}\mathcal{L}e^{-\|A\|T} \quad (4.2.11)$$

From (4.2.10) and (4.2.11) we obtain

$$a_{2(k+N)-1} - a_{2k} \geq \frac{(N-1)\mathcal{L}e^{-2\|A\|T}}{N\sum_{i=1}^{N-1}\|A^{i+1}b\|}, \quad (4.2.12)$$

which is the desired estimate. Observe that the right hand side of (4.2.12) depends only on A, b, T and N.

We set now $\mathcal{N}_0$ to be the number of nonempty intervals contributing to the union in (4.2.8), and recall that we have just proved that $\mathcal{N}_0$ depends only on A, b, T and N, and actually

$$\mathcal{N}_0 \leq \frac{N^2}{N-1}\frac{T}{\mathcal{L}}e^{2\|A\|T}\sum_{i=1}^{N-1}\|A^{i+1}b\| + N - 1. \quad (4.2.13)$$

Set $I_0 = [0,T]\backslash J_0$ and observe that we have completed the proof of the lemma for $i = 0$.

After this step, we formulate our induction process. We are going to construct, for each $i = 0, \ldots, N-1$, two disjoint sets I_i, J_i with the following properties

(ind1) for every $s \in I_i$, $|g^{(i)}(s)| \geq c(T)$;

(ind2) for every $s \in J_i$, $\sum_{j=i+1}^{N-1} |g^{(j)}(s)| \geq (N-i-1)c(T)$;

(ind3) $J_i \bigcup I_i = J_{i-1}$;

(ind4) J_i is a finite union of open intervals whose number is at most $\mathcal{N}_i$, and $\mathcal{N}_i$ depends only on $T, \mathcal{L}, A, b, N, i$;

(ind5) I_i is the finite union of at most $\mathcal{N}_i + \mathcal{N}_{i-1}$ intervals.

For $i = 0$ the above construction was already performed (take $J_{-1} = (0,T)$). Pick any $i = 1, \ldots, N-2$ (the case $i = N-1$ will be treated separately) and assume that (ind1), ..., (ind5) hold up to $i-1$. We wish to show that the above statements hold for i as well. To this aim, consider the set

$$J_i := \{s \in J_{i-1} \mid |g^{(i)}(s)| < c(T)\}.$$

For every connected component (a, b) of J_{i-1}, we are going to prove that $J_i^{(a,b)} := J_i \cap (a, b)$ is a finite union of intervals, and give a bound on their number $\mathcal{N}_i^{(a,b)}$.

So, fix a connected component (a, b) of J_i and represent the open set $J_i^{(a,b)}$ as a disjoint union of at most countably many intervals J_i^k, $k \in \mathbb{N}$ (for simplicity of writing we drop the dependence on (a, b)). Assume without loss of generality that there are at most $N - i$ intervals J_i^k, fix any number $N'' \geq N - i$, and take any subfamily of $\{J_i^k\}$ consisting of at most N'' intervals. We can write $J_i^k = (a_{2k}, a_{2k+1})$, where $1 \leq k \leq N''$ and $a \leq a_2 < a_3 \leq a_4 < a_5 \leq \ldots < a_{2N''+1} \leq b$. Fix k and consider the intervals $(a_{2k}, a_{2k+1}), \ldots, (a_{2(k+N-i-1)}, a_{2(k+N-i)-1})$. Set, for $j = 0, \ldots, N - i - 1$,

$$(a_{2(k+j)}, a_{2(k+j)+1}) := I_j^-,$$

and, for $j = 0, \ldots, N - i - 2$,

$$[a_{2(k+j)+1}, a_{2(k+j+1)}] := I_j^+.$$

Observe that

$$\bigcup_{J=0}^{N-i-1} I_j^- \cup \bigcup_{j=1}^{N-i-2} I_j^+ = (a_{2k}, a_{2(k+N-i)-1}).$$

For each $j = 0, \ldots, N - i - 2$ there exists at least one point $c_j^i \in I_j^+$ such that $g^{(i)}(c_j^i) = 0$. Proceeding by induction we see that for each $m = 0, \ldots, N - i - 2$ there exists a point $c_m \in (a_{2k+1}, a_{2(k+N-i)-1})$ such that $g^{(i+m)}(c_m) = 0$. Pick any $s_0 \in (a_{2k}, a_{2(k+1)})$. By arguing as for $i = 0$, we obtain on one hand

$$|g^{(i)}(s_0)| < c(T) \tag{4.2.14}$$

and, for all $m = 0, \ldots N - i - 1$,

$$|g^{(i+m)}(s_0)| \leq (a_{2(k+N-i)-1} - a_{2k}) e^{\|A\|T} \|A^{i+m+1} b\|,$$

the latter inequality being due to (4.2.5). Thus

$$\sum_{m=i+1}^{N-1} |g^{(m)}(s_0)| \leq (a_{2(k+N-i)-1} - a_{2k}) e^{\|A\|T} \sum_{m=i+1}^{N-1} \|A^{m+1} b\|. \tag{4.2.15}$$

On the other hand, owing to (ind2) and (4.2.14), we obtain

$$\sum_{m=i+1}^{N-1} |g^{(m)}(s)| \geq (N - i - 1) c(T).$$

By combining the above inequality with (4.2.15) we now obtain

$$a_{2(k+N-i)-1} - a_{2k} \geq \frac{N - i - 1}{N} \mathcal{L} e^{-\|A\|T} \frac{1}{\sum_{m=i+1}^{N-1} \|A^{m+1} b\|}.$$

Therefore, $J_i^{(a,b)}$ is the union of finitely many disjoint open intervals $\left(a_{2k}^{i+1}, a_{2k+1}^{i+1}\right)$, $k = 1, \ldots, \mathcal{N}_i^{(a,b)}$, where

$$\mathcal{N}_i^{(a,b)} \le \frac{N(N-i)}{N-i-1} \frac{|b-a|}{\mathcal{L}} e^{2\|A\|T} \sum_{m=i+1}^{N-1} \|A^{m+1}b\| + N - i - 1. \qquad (4.2.16)$$

We define

$$I_i^{(a,b)} = \{s \in (a,b) \mid |g^{(i)}(s)| \ge c(T)\}$$

and observe that $I_i^{(a,b)}$ is the union of at most $\mathcal{N}_i^{(a,b)} + 1$ intervals.
We finally set I_i to be the union of the $I_i^{(a_j,b_j)}$ over all the (at most $\mathcal{N}_{i-1}$) connected components (a_j, b_j) of J_{i-1}. Therefore, I_i is the union of at most $\mathcal{N}_i + \mathcal{N}_{i-1}$ intervals, where

$$\mathcal{N}_i = \sum_{j=1}^{\mathcal{N}_{i-1}} \mathcal{N}_i^{(a_j,b_j)} \le \frac{N(N-i)}{N-i-1} \frac{T}{\mathcal{L}} e^{2\|A\|T} \sum_{m=i+1}^{N-1} \|A^{m+1}b\| + \mathcal{N}_{i-1}(N-i-1). \qquad (4.2.17)$$

Finally we observe that J_i is the union of at most $\mathcal{N}_i$ open intervals.

If $i = N-1$, we observe that for each $s \in J_{N-2}$, recalling (ind2) we have $|g^{(N-1)}(s)| \ge c(T)$. Therefore we set $J_{N-1} = \emptyset$ and $I_{N-1} = J_{N-2}$. The proof is concluded. □

We are now going to prove the main result of this subsection.

Theorem 4.2.1 *Consider the linear control system*

$$\dot{x} = Ax + Bu, \qquad (4.2.18)$$

where $1 \le M \le N$, $A \in \mathbb{M}_{N\times N}$, $B \in \mathbb{M}_{N\times M}$, *and* $u = (u_1, \ldots, u_M) \in [-1,1]^M$.
Assume that (4.2.18) is normal, i.e., for every column b_j, $j = 1, \ldots, M$, *of* B,

$$\text{rank } [b_j, Ab_j, \ldots, A^{N-1}b_j] = N.$$

Let $\mathcal{R}^T$ *be defined according to* (4.1.3). *Then for all* $T > 0$ *there exists a constant* $\gamma > 0$, *depending only on* N, M, A, B, T *such that for all* $x, y \in \mathcal{R}^T$, *for all* $\zeta \in N_{\mathcal{R}^T}(x)$, *the inequality*

$$\langle \zeta, y - x\rangle \le -\gamma \, \|\zeta\| \, \|y - x\|^N \qquad (4.2.19)$$

holds. Moreover, there exists another constant γ', *depending only on* N, M, A, B, T, *such that*

$$\textit{the ball } \; B(0, \gamma' T^N) \; \textit{ is contained in } \mathcal{R}^T. \qquad (4.2.20)$$

for all $T > 0$. *Finally, the constants* γ *and* γ' *are bounded away from zero as* $T \to 0^+$.

Proof. We consider first the case $M = 1$, so (4.2.18) reads as

$$\dot{x} = Ax + bu \quad , \ |u| \leq 1,$$

for a suitable $b \in \mathbb{R}^N$. Fix $\bar{x} \in \operatorname{bdry} \mathcal{R}^T$ together with an optimal control $\bar{u}(\cdot)$ steering 0 to $\bar{x}$ in time T and $\zeta \in N_{\mathcal{R}^T}(\bar{x})$, $\|\zeta\| = 1$. We assume first that ζ satisfies Pontryagin's maximum principle, i.e., for a.e. $t \in [0, T]$,

$$\bar{u}(t) = \operatorname{sign}\langle \zeta, e^{A(T-t)} b \rangle$$

and prove (4.2.19) for such ζ. The general case will then follow using Proposition 7.2.1.
So, we take $\bar{y} \in \mathcal{R}^T$ together with a control $u(\cdot)$ steering 0 to $\bar{y}$ and compute:

$$\begin{aligned} \langle \zeta, \bar{y} - \bar{x} \rangle &= \int_0^T \langle \zeta, e^{A(T-t)} b \rangle (u(t) - \bar{u}(t)) \, dt \\ &\qquad \text{(recalling (4.1.5))} \\ &= -\int_0^T |\langle \zeta, e^{A(T-t)} b \rangle| \, |u(t) - \bar{u}(t)| \, dt. \end{aligned}$$

Set $K(t) = \frac{1}{2}|u(t) - \bar{u}(t)|$ and observe that $0 \leq K(t) \leq 1$ for a.e. $t \in [0, T]$, and

$$\langle \zeta, \bar{y} - \bar{x} \rangle = -2 \int_0^T |\langle \zeta, e^{A(T-t)} b \rangle| K(t) dt = -2 \int_0^T |\langle \zeta, e^{At} b \rangle| K(T-t) dt. \tag{4.2.21}$$

Moreover,

$$\|\bar{y} - \bar{x}\| = \left\| \int_0^T e^{A(T-t)} b (u(t) - \bar{u}(t)) dt \right\| \leq 2 e^{T\|A\|} \|b\| \int_0^T K(t) dt. \tag{4.2.22}$$

Set, for $s \in [0, +\infty)$,

$$g(s) = \langle e^{As} b \ , \ \zeta \rangle.$$

By Lemma 4.2.2 there exist disjoint sets $I_0, I_1, \ldots, I_{N-1}$ and numbers $\mathcal{N}_i$ such that $[0, T] = \bigcup_{i=0}^{N-1} I_i$, each I_i is the disjoint union of at most $\mathcal{N}_i$ intervals and (4.2.6) holds. Observe that, in particular, it follows that g may vanish at most at finitely many times, and so, recalling (4.1.5), the control $\bar{u}$ is piecewise constant and equal to either 1 or to -1.
We rewrite

$$\langle \zeta, \bar{y} - \bar{x} \rangle = -2 \sum_{i=0}^{N-1} \int_{I_i} |g(s)| K_1(s) ds \tag{4.2.23}$$

where $K_1(s) = K(T - s)$. We are now going to estimate separately the integrals $\int_{I_i} |g(s)| K_1(s) ds$, for all $i = 0, 1, \ldots, N-1$.
For $i = 0$, we have

$$\int_{I_0} |g(s)| K_1(s) ds \geq \frac{\mathcal{L}}{N} e^{-\|A\|T} \int_{I_0} K_1(s) ds. \tag{4.2.24}$$

Fix now $i = 1, 2, \ldots, N-1$, and write, recalling Lemma 4.2.2,

$$\overline{I_i} = \bigcup_{j=1}^{\mathcal{N}_i} [a_{ij}, b_{ij}]$$

where all *open* intervals (a_{ij}, b_{ij}) are disjoint. Recalling (4.2.6), we have, for all $s \in I_i$, $|g^{(i)}(s)| \geq \frac{\mathcal{L}}{N} e^{-\|A\|T}$. Fix $j \in \{1, 2, \ldots, \mathcal{N}_i\}$. We are now going to apply inductively Lemma 7.4.2 on $[a_{ij}, b_{ij}]$ with the functions $g^{(i-k-1)}$ in place of f, for $k = 0, \ldots, i-1$. Let $k = 0$ and set $f = g^{(i-1)}$. Then the assumption (7.4.1) is satisfied with $C = \frac{\mathcal{L}}{N} e^{-\|A\|T}$, thanks to (4.2.6), and Lemma 7.4.2 yields that for some point $c^0_{ij} \in [a_{ij}, b_{ij}]$ we have

$$|g^{(i-1)}(s)| \geq C|s - c^0_{ij}| \quad \forall s \in [a_{ij}, b_{ij}].$$

Let $k = 1$. By applying Lemma 7.4.2 on each of the two (possibly degenerate) intervals a_{ij}, c^0_{ij}, c^0_{ij}, b_{ij} to the function $f = g^{(i-2)}$ with $C = \frac{\mathcal{L}}{N} e^{-\|A\|T}$, we find suitable points $c^1_{ij} \in [a_{ij}, c^0_{ij}]$ and $c^2_{ij} \in [c^0_{ij}, b_{ij}]$ such that we have both

$$|g^{(i-2)}(s)| \;\geq\; \frac{C}{2}(s - c^1_{ij})^2 \quad \forall s \in [a_{ij}, c^0_{ij}]$$

and

$$|g^{(i-2)}(s)| \;\geq\; \frac{C}{2}(s - c^2_{ij})^2 \quad \forall s \in [c^0_{ij}, b_{ij}].$$

By continuing the induction process until $k = i-1$, we split the interval $[a_{ij}, b_{ij}]$ into at most 2^i intervals $[a_{ij} = c^0_{ij}, c^1_{ij}], [c^1_{ij}, c^2_{ij}], \ldots, [c^{2^i-1}_{ij}, b_{ij} = c^{2^i}_{ij}]$ (some of them being possibly degenerate) such that for all $l = 0, 1, \ldots, 2^i - 1$ and $s \in [c^l_{ij}, c^{l+1}_{ij}]$ one has

$$\text{either} \;\; |g(s)| \geq \frac{C}{i!}(s - c^l_{ij})^i \quad \text{or} \quad |g(s)| \geq \frac{C}{i!}(c^{l+1}_{ij} - s)^i. \tag{4.2.25}$$

Recalling (4.2.23) and the above discussion, we have

$$\begin{aligned}
\langle \zeta, \bar{y} - \bar{x} \rangle &= -2 \sum_{i=0}^{N-1} \int_{I_i} |g(s)| K_1(s)\, ds \\
&= -2 \left[\int_{I_0} |g(s)| K_1(s)\, ds + \sum_{i=1}^{N-1} \sum_{j=0}^{\mathcal{N}_i - 1} \sum_{l=0}^{2^i - 1} \int_{c^l_{ij}}^{c^{l+1}_{ij}} |g(s)| K_1(s)\, ds \right].
\end{aligned}$$

Recalling (4.2.24) and (4.2.25), and $C = \frac{\mathcal{L}}{N} e^{-\|A\|T}$, we obtain from the above expression that

$$\begin{aligned}
&\langle \zeta, \bar{y} - \bar{x} \rangle \\
&\leq -\frac{2\mathcal{L}}{N} e^{-\|A\|T} \left[\int_{I_0} K_1(s)\, ds + \sum_{i=1}^{N-1} \sum_{j=0}^{\mathcal{N}_i - 1} \sum_{l=0}^{2^i - 1} \int_{c^l_{ij}}^{c^{l+1}_{ij}} \frac{|s - \bar{c}^l_{ij}|^i}{i!} K_1(s)\, ds \right]
\end{aligned} \tag{4.2.26}$$

where $\bar{c}_{ij}^l$ is either c_{ij}^l or c_{ij}^{l+1}, according to the two possibilities appearing in (4.2.25). Applying Lemma 7.4.1 to each summand of (4.2.26) we therefore obtain

$$\begin{aligned}
&\langle \zeta, \bar{y} - \bar{x} \rangle \\
&\quad \leq -2\frac{\mathcal{L}}{N} e^{-\|A\|T} \left[\int_{I_0} K_1(s)\, ds + \sum_{i=1}^{N-1} \sum_{j=0}^{\mathcal{N}_i - 1} \sum_{l=0}^{2^i - 1} \frac{\left(\int_{c_{ij}^l}^{c_{ij}^{l+1}} K_1(s)\, ds \right)^{i+1}}{(i+1)!} \right] \\
&\qquad\qquad \text{(using the convexity of } x \mapsto x^{i+1} \text{ on the positive half line)} \\
&\leq -\frac{2\mathcal{L}}{N} e^{-\|A\|T} \left[\int_{I_0} K_1(s)\, ds + \sum_{i=1}^{N-1} \sum_{j=0}^{\mathcal{N}_i - 1} \frac{1}{(i+1)!\; 2^{i^2}} \left(\int_{a_{ij}}^{b_{ij}} K_1(s)\, ds \right)^{i+1} \right] \\
&\quad \leq -\frac{2\mathcal{L}}{N} e^{-\|A\|T} \left[\int_{I_0} K_1(s)\, ds + \sum_{i=1}^{N-1} \frac{1}{(i+1)!\; 2^{i^2} \mathcal{N}_i^i} \left(\int_{I_i} K_1(s)\, ds \right)^{i+1} \right].
\end{aligned}$$

Thus, recalling that $0 \leq K_1(s) \leq 1$ a.e.,

$$\begin{aligned}
\langle \zeta, \bar{y} - \bar{x} \rangle \;\leq\; & -\frac{2\mathcal{L}}{N} e^{-\|A\|T} \left[\frac{1}{|I_0|^{N-1}} \left(\int_{I_0} K_1(s)\, ds \right)^N \right. \\
& \left. + \sum_{i=1}^{N-1} \frac{1}{(i+1)!\; 2^{i^2} \mathcal{N}_i^i |I_i|^{N-i-1}} \left(\int_{I_i} K_1(s)\, ds \right)^N \right].
\end{aligned} \tag{4.2.27}$$

Owing to (4.2.13) and (4.2.17), we see that $\mathcal{N}_0 < \mathcal{N}_1 < \ldots < \mathcal{N}_N \leq C(A, b, N) e^{2\|A\|T}$ where $C(A, b, N)$ depends only on A, b, N. Therefore, we obtain finally from (4.2.27) and the definition of K_1 that

$$\langle \zeta, \bar{y} - \bar{x} \rangle \leq -C(A, b, N, T) e^{-\|A\|T} \left(\int_0^T |u(t) - \bar{u}(t)| \right)^N, \tag{4.2.28}$$

where $C(A, b, N, T)$ is a positive constant, depending only on A, b, N, T such that

$$\liminf_{T \to 0} C(A, b, N, T) > 0. \tag{4.2.29}$$

Recalling (4.2.22), we complete the proof for the case $M = 1$ (i.e., a scalar control) by setting

$$\gamma = 2^{-N} e^{-\|A\|(N+1)T} \|b\|^{-N} C(A, b, N, T).$$

Let now $M > 1$. Take $\bar{x} \in \operatorname{bdry} \mathcal{R}^T$ together with an optimal control $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \ldots, \bar{u}_M(\cdot))$ steering the origin to $\bar{x}$ in the optimal time T, and $\bar{y} \in \mathcal{R}^T$ with a control $u(\cdot) = (u_1(\cdot), \ldots, u_M(\cdot))$ steering the origin to $\bar{y}$ in time T. Then, for each $\zeta \in N_{\mathcal{R}^T}(\bar{x})$, $\|\zeta\| = 1$, we can write

$$\begin{aligned}\langle \zeta, \bar{y} - \bar{x} \rangle &\leq \int_0^T \big\langle \zeta, e^{A(T-s)} B w(s) \big\rangle ds \\ &= \sum_{i=1}^{M} \int_0^T \big\langle \zeta, e^{A(T-s)} b_i \big\rangle (u_i(s) - \bar{u}_i(s))\, ds, \quad (4.2.30)\end{aligned}$$

where $B = \big(b_i\big)_{i=1,2,\ldots,M}$. Recalling (4.1.5) we have also

$$\langle \zeta, \bar{y} - \bar{x} \rangle = - \sum_{i=1}^{M} \int_0^T \big| \big\langle \zeta, e^{A(T-s) b_i} \big\rangle \big| \, \big| u_i(s) - \bar{u}_i(s) \big| \, ds. \qquad (4.2.31)$$

Moreover,

$$\|\bar{y} - \bar{x}\| \leq e^{\|A\|T} \sum_{i=1}^{M} \|b_i\| \int_0^T |u_i(s) - \bar{u}_i(s)| ds. \qquad (4.2.32)$$

We now apply the same argument leading to (4.2.28) to each summand of the right hand side of (4.2.30). Therefore we obtain, using (4.2.31), that

$$\langle \zeta, \bar{y} - \bar{x} \rangle \leq -C'(A,B,T,N,M) e^{-\|A\|T} \sum_{i=1}^{M} \Big(\int_0^T |u_i(s) - \bar{u}_i(s)| \, ds \Big)^N, \qquad (4.2.33)$$

where the positive constant C' depends only on A, B, T, N, M and

$$\liminf_{T \to 0} C'(A,B,T,N,M) > 0.$$

We conclude the proof of (4.2.19) by applying (4.2.32) and setting

$$\gamma = 2^N e^{-(N+1)\|A\|T} C''(A,B,T,N,M),$$

where C'' is a constant enjoying the same properties as C'.

In order to prove the statement concerning the ball contained in $\mathcal{R}^T$, observe that the inequality (4.2.33) with $u \equiv 0$ and $\bar{y} = 0$, taking into account that the control $\bar{u}$ is necessarily bang-bang, becomes

$$\langle \zeta, \bar{x} \rangle \geq C'(A,B,T,N,M) e^{-\|A\|T} M T^N,$$

from which (taking $\|\zeta\| = 1$) we obtain

$$\|\bar{x}\| \geq C'(A,B,T,N,M) e^{-\|A\|T} M T^N := \gamma' T^N. \qquad (4.2.34)$$

The above inequality yields in particular that 0 belongs to the interior of $\mathcal{R}^T$. Since $\mathcal{R}^T$ is convex and (4.2.34) holds for all $\bar{x} \in \operatorname{bdry} \mathcal{R}^T$, (4.2.20) follows.

The last statement follows from (4.2.29) and the explicit expressions for γ and γ'. The proof is concluded. □

Remark 4.2.1 *The exponent N in (4.2.19) is optimal.*

In fact, consider the dynamics

$$x^{(N)} = u, \qquad u \in [-1, 1].$$

Let $x_1(\cdot)$ be the solution corresponding to the control $u \equiv 1$. Fix $s > 0$ and let $x_s(\cdot)$ be the solution corresponding to the control

$$u_s(t) = \begin{cases} 1 & 0 < t < s \\ -1 & s < t. \end{cases}$$

Fix any $T > 0$ and observe that $x_1(T) = T^N/N!$, while, for $0 < s < T$,

$$x_s(T) - x_1(T) = \frac{-2}{N!}(T - s)^N.$$

Observe also that $|x_s^{(N-1)}(T) - x_1^{(N-1)}(T)| = 2|T - s|$, so that setting $X_i = (x_i, x_i', \dots, x_i^{(N-1)})$, $i = 1, s$, one obtains

$$\langle e_1, X_s(T) - X_1(T)\rangle = \frac{-2}{N!}(T - s)^N \geq -\gamma \big\| X_s(T) - X_1(T)\big\|^N,$$

for a suitable positive constant γ. If $s \to T$, then $X_s(T) \to X_1(T)$, and this shows that N is the smallest exponent allowed in (4.2.19). □

4.2.2 Nonautonomous systems

The following Lemma is a first step for studying the reachable sets in the case of nonlinear control systems by using the linearization approach which we design in this paper. We will prove that under the rank condition (normality) at 0 of the linear nonautonomous control system (4.2.35), the strict convexity of the reachable sets is preserved up to a sufficiently small time, provided $A(t)$ and $B(t)$ are not too far from $A(0)$, $B(0)$. Sufficient conditions for the validity of the assumptions if $N = 2$ will be given below. In Section 4.3 sufficient conditions in the case where A and B come from a linearization around a trajectory will also be given.

Let $A : \mathbb{R}^+ \to \mathbb{M}_{N\times N}$ and $B : \mathbb{R}^+ \to \mathbb{M}_{N\times M}$, $1 \leq M \leq N$, be measurable and consider the linear nonautonomous control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{4.2.35}$$

together with

$$\dot{x}(t) = A(0)x(t) + B(0)u(t), \tag{4.2.36}$$

where $u = (u_1, \dots, u_M) \in [-1,1]^M$. We denote by $M^\top(\cdot,\cdot)$ the matrix solution of

$$\begin{cases} \frac{\partial}{\partial t} X(t,s) = X(t,s)A(t) \text{ for } t,s \geq 0 \\ M(s,s) = \mathbb{I} \end{cases} \tag{4.2.37}$$

and by $M_0^\top(\cdot,\cdot)$ the matrix solution of

$$\begin{cases} \frac{\partial}{\partial t} X(t,s) = X(t,s)A(0) \text{ for } t,s \geq 0 \\ M_0(s,s) = \mathbb{I}. \end{cases} \tag{4.2.38}$$

Let $b(\cdot)$ be a column of $B(\cdot)$, let $\mathcal{T} > 0$ and define, for any $\zeta \in \mathbb{R}^N$,

$$g(t) = \langle \zeta M^\top(\mathcal{T}, t), b(t) \rangle, \tag{4.2.39}$$

and

$$g_0(t) = \langle \zeta M_0^\top(\mathcal{T}, t), b(0) \rangle. \tag{4.2.40}$$

Observe that g and g_0 can be seen as the switching functions related to (4.2.35) and (4.2.36), respectively.
We state now an abstract result which permits to transfer to g some properties of g_0 and to establish the quantitative strict convexity estimate for the reachable set from the origin of (4.2.35). Sufficient conditions in order to apply the following lemma to suitable linearizations in dimension 2 and 3 will be given in Section 4.3.

Lemma 4.2.3 *Fix $\zeta \in \mathbb{R}^N$, $\|\zeta\| = 1$, and let g and g_0 be defined according to (4.2.39) and* (4.2.40)*, respectively. Assume that*

(i) $\operatorname{rank}[b(0), A(0)b(0), \dots, A^{N-1}(0)b(0)] = N$*;*

(ii) g is $N-2$ times differentiable and $g^{(N-2)}$ is absolutely continuous;

(iii) there exists a constant $K = K(A,B)$ and a time $\mathcal{T}'$ such that for all $i = 0, \dots, N-1$ one has

$$|g^{(i)}(t) - g_0^{(i)}(t)| \leq Kt \qquad \text{for all } 0 \leq t \leq \mathcal{T}'.$$

Let $\mathcal{L}$ be defined by (4.2.4)*, with $A(0)$, $b(0)$ in place of A, b, respectively. Then there exists $\mathcal{T} = \mathcal{T}(A,B,N,\mathcal{T}')$ with the following property:
for every $0 < \tau < \mathcal{T}$ there exist disjoint sets I_i and numbers $\mathcal{N}_i$, $i = 0, \dots, N-1$, depending only on $A(0), B(0), \tau$, such that*

(a) $[0, \mathcal{T}] = \bigcup_{i=0}^{N-1} I_i$*;*

(b) each I_i, $i = 0, \dots, N-1$ is the disjoint union of at most $\mathcal{N}_i$ intervals;

(c) for each $i = 0, \dots, N-2$ and each $s \in I_i$

$$|g^{(i)}(s)| \geq \frac{\mathcal{L}}{2N} e^{-\|A(0)\|s};$$

(d) $g^{(N-1)}$ *has constant sign in every connected component of* I_{N-1} *and, for each* $s \in I_{N-1}$

$$|g^{(N-1)}(s)| \geq \frac{\mathcal{L}}{2N} e^{-\|A(0)\|s}.$$

Remark 4.2.2 $I_0, \ldots, I_{N-1}$ *are exactly the intervals constructed in Lemma 4.2.2 with* g_0 *in place of* g.

Proof of Lemma 4.2.3. Let $I_0, \ldots, I_{N-1}$ be the sets appearing in the statement of Lemma 4.2.2, with g_0 in place of g, so that, in particular, (a) and (b) hold. Moreover, for each $s \in I_i$ we have

$$|g_0^{(i)}(s)| \geq \frac{\mathcal{L}}{N} e^{-\|A(0)\|\tau}.$$

Therefore, if $K\mathcal{T} < \frac{\mathcal{L}}{2N} e^{-\|A(0)\|s}$, also (c) and (d) hold, owing to (iii). □

Remark 4.2.3 *Sufficient conditions for the validity of the assumptions of Lemma 4.2.3 in the case* $N = 2$.

Let $N = 2$. Sufficient conditions for the validity of assumptions (ii) and (iii) in the above Lemma are the following:

(C_0) $A(\cdot)$ is measurable and

$$\|A(t) - A(0)\| \leq Lt \text{ for all } t \geq 0;$$

(C_1) $B(\cdot)$ is absolutely continuous and

$$\left\| \frac{d}{dt} B(t) \right\| \leq 2Lt \text{ for all } t \geq 0,$$

where L is a positive constant.

Observe that condition (C_0) implies that $t = 0$ is a continuity point for $A(\cdot)$, so that $A(0)$ in (C_0) is meaningful. □

As an immediate corollary of Lemma 4.2.3 we obtain the following

Theorem 4.2.2 *Consider the linear nonautonomous control system* (4.2.35) *under the assumptions of Lemma* (4.2.3). *Let* $\mathcal{R}^\tau$ *denote the reachable set at time* $\tau > 0$ *from the origin for* (4.2.35). *Then there exist a time* $\mathcal{T} = \mathcal{T}(A, B, N) > 0$ *and a constant* $\gamma = \gamma(A, B, N) > 0$ *such that for every* $0 \leq \tau \leq \mathcal{T}$, *for every* $x, y \in \mathcal{R}^\tau$, *for every* $\zeta \in N_{\mathcal{R}^\tau}(x)$, *we have*

$$\langle \zeta, y - x \rangle \leq -\gamma \|\zeta\| \|y - x\|^N.$$

Moreover, there exists another constant $\gamma' = \gamma'(A, B, N) > 0$ *such that for every* $0 < \tau \leq \mathcal{T}$

$$\text{the ball } B(0, \gamma' \tau^N) \text{ is contained in } \mathcal{R}^\tau. \qquad (4.2.41)$$

Proof. The argument developed in the proof of Theorem 4.2.1 can be used also in this case. Indeed, fix $\bar{x} \in \mathrm{bdry}\,\mathcal{R}^\tau$ together with a control $\bar{u}(\cdot)$ steering 0 to $\bar{x}$ in time τ and let $\zeta \in N_{\mathcal{R}^\tau}(\bar{x})$, $\|\zeta\| = 1$, be such that

$$\bar{u}(t) = \mathrm{sign}\langle \zeta, b(t)M(\tau, t)\rangle \qquad \text{for a.e. } \ t \in (0, \tau) \tag{4.2.42}$$

(here, as at the beginning of the proof of Theorem 4.2.1, we assume that $B = b$ is a vector, i.e., the control is scalar).
Then the proof proceeds exactly as for Theorem 4.2.1, provided that g is given by

$$g(t) = \langle \zeta, b(t)M(\tau, t)\rangle,$$

so that for all $\bar{y} \in \mathcal{R}^\tau$ one has

$$\langle \zeta, \bar{y} - \bar{x}\rangle = -2 \sum_{i=0}^{N-1} \int_{I_i} |g(s)| K_1(s)\, ds,$$

where $K_1(s) = \frac{1}{2}|u(\tau - s) - \bar{u}(\tau - s)|$, $u(\cdot)$ is the control which steers the origin to $\bar{y}$, and the sets I_i are those appearing in the statement of Lemma 4.2.3. □

4.3 A nonlinear bang bang principle in dimensions 2 and 3

Starting from the present section we will deal with nonlinear control systems, which are affine and symmetric with respect to the control. This section is devoted to giving sufficient conditions so that controls steering the origin to the boundary of the reachable set are always bang-bang, provided the final time is sufficiently small. More precisely, the following result holds.

Theorem 4.3.1 *Let $N = 2$ or $N = 3$. Consider the control system*

$$\begin{cases} \dot{x}(t) = & F(x(t)) + G(x(t))u(t), \\ x(0) = & 0, \end{cases} \tag{4.3.1}$$

where $1 \le M \le N$, $u(\cdot) = (u_1(\cdot), \dots, u_M(\cdot)) \in [-1, 1]^M$ a.e., and $F : \mathbb{R}^N \to \mathbb{R}^N$ and $G = (G_1, \dots, G_M) : \mathbb{R}^N \to \mathbb{M}_{N\times M}$ satisfy the following assumptions:

(i) F and G are of class $\mathcal{C}^{N-1}$ and all partial derivatives are Lipschitz with constant L;

(ii) $F(0) = 0$;

(iii) $\mathrm{rank}[G_i(0), DF(0)G_i(0), \dots, (DF(0))^{N-1}G_i(0)] = N$ for $i = 1, \dots, M$;

(iv) if $N = 3$*, then* $DG(0) = 0$ *and* $D^2F(0) = 0$.

Let $\mathcal{R}^\tau$ *denote the reachable set of* (4.3.1) *at time* $\tau > 0$*. Then there exists* $\mathcal{T} > 0$*, depending only on* $DF(0), G(0), L, N$*, such that for every* $0 < \tau < \mathcal{T}$ *the following properties hold:*

(a) every admissible control $u(\cdot)$ *such that the corresponding trajectory* $y^u(\cdot)$ *of* (4.3.1) *at time* τ *belongs to the boundary of* $\mathcal{R}^\tau$ *is essentially determined by the curve* $\lambda(\cdot)$*, the solution of the adjoint equation*

$$\begin{cases} \dot\lambda(t) &= -\lambda(t)\big(DF(y^u(t)) + DG(y^u(t))u(t)\big) \\ \lambda(\tau) &= \zeta \in N_{\mathcal{R}^\tau}(y^u(\tau)), \quad \|\zeta\| = 1, \end{cases} \tag{4.3.2}$$

through the identity

$$u(t) = \operatorname{sign}\langle \lambda(t), G(y^u(t))\rangle \qquad a.e.; \tag{4.3.3}$$

(b) u *is bang-bang, i.e.,* $u(t) \in \{-1, 1\}^M$ *a.e., and is piecewise constant;*

(c) the maximum number of discontinuities of u *depends only on* $DF(0)$*,* $G(0)$*,* L*, and* N*.*

Proof. We consider first the case where $M = 1$, i.e., $G(x)$ is a vector and the control u is scalar.

Fix $\tau > 0$ and an admissible control u such that $\bar x := y^u(\tau) \in \operatorname{bdry}\mathcal{R}^\tau$. By the Maximum Principle (see Theorem 4.1.2) there exist $\zeta \in N_{\mathcal{R}^\tau}(\bar x)$, $\|\zeta\| = 1$, and an adjoint curve $\lambda(\cdot)$, a solution of (4.3.2), such that u satisfies (4.3.3). Proving (a), (b), and (c) amounts to showing that the switching function $\langle \lambda(t), G(y^u(t))\rangle$ vanishes at most finitely many times in $[0, \tau]$ and the number of its zeros depends only on $DF(0)$, $G(0)$, L, and N.

For convenience we rewrite the switching function in the following way. Let $M^\top(\cdot,\cdot)$ denote the matrix solution of (4.2.37), with $A(t) = DF(y^u(t)) + DG(y^u(t))u(t)$, and set $b(t) = G^\top(y^u(t))$, $t \in [0, \tau]$. Then

$$g(t) := \langle \lambda(t), b(t)\rangle = \langle \zeta, b(t)M(\tau, t)\rangle.$$

Let also $M_0^\top(\cdot,\cdot)$ be defined by (4.2.38), where $A(0) = DF(0)$, and let g_0 be defined according to (4.2.40). We wish to apply Lemma 4.2.3 to the above introduced mappings g and g_0.

First of all, we compute g' and observe that it is continuous. Indeed,

$$\begin{aligned} g'(t) &= \big\langle \zeta, b'(t)M(\tau,t)\rangle + \langle \zeta, b(t)\frac{\partial}{\partial t}M(\tau,t)\big\rangle \\ &= \big\langle \zeta, DG^\top(y^u(t))\dot y^u(t)M(\tau,t)\big\rangle + \big\langle \zeta, -b(t)A^\top(t)M(\tau,t)\big\rangle \\ &= \big\langle \zeta, DG^\top(y^u(t))\big(F(y^u(t)) + G(y^u(t))u(t)\big)^\top M(\tau,t)\big\rangle \\ &\qquad + \big\langle \zeta, -G^\top(y^u(t))\big(DF(y^u(t)) + DG(y^u(t))u(t)\big)^\top M(\tau,t)\big\rangle \\ &= \big\langle \zeta, \big(DG^\top(y^u(t))F^\top(y^u(t)) - DF^\top(y^u(t))G^\top(y^u(t))\big)M(\tau,t)\big\rangle \\ &= \big\langle \lambda(t), [F^\top, G^\top](y^u(t))\big\rangle, \end{aligned}$$

where $[F^\top, G^\top](x) := DG^\top(x)F^\top(x) - DF^\top(x)G^\top(x)$ denotes the Lie bracket. Moreover, if $N = 3$ g' is a.e. differentiable and we have

$$g''(t) = \big\langle \dot{\lambda}(t), [F^\top, G^\top](y^u(t))\big\rangle + \big\langle \lambda(t), \frac{d}{dt}[F^\top, G^\top](y^u(t))\big\rangle.$$

Finally, observing that

$$g_0'(t) = \big\langle \zeta, -DF^\top(0)G^\top(0)M_0(\tau,t)\big\rangle = \big\langle \zeta M_0^\top(\tau,t), [F^\top, G^\top](0)\big\rangle,$$

we have

$$\begin{aligned} g'(t) - g_0'(t) &= \big\langle \zeta\big(M^\top(\tau,t) - M_0^\top(\tau,t)\big), [F^\top, G^\top](y^u(t))\big\rangle \\ &\quad + \big\langle \zeta M_0^\top(\tau,t), DG^\top(y^u(t))F^\top(y^u(t))\big\rangle \\ &\quad + \big\langle \zeta, \big(DF^\top(0) - DF^\top(y^u(t))\big)G^\top(y^u(t))\big\rangle \\ &\quad + \big\langle \zeta, DF^\top(0)\big(G^\top(0) - G^\top(y^u(t))\big)\big\rangle \\ &= I + II + III + IV. \end{aligned}$$

We are now going to estimate separately each summand of the above expression.
First, we observe that, thanks to the assumptions (i) and (ii), we have

$$\begin{aligned} &\|M(\tau,t) - M_0(\tau,t)\| \\ &\quad \le \int_0^t \Big(\|A(s)\|\,\|M_0(\tau,s) - M(\tau,s)\| + \|M_0(\tau,s)\|\,\|A(0) - A(s)\|\Big)\,ds \\ &\qquad\qquad \le K_1 \int_0^t \|M(\tau,s) - M_0(\tau,s)\|\,ds + K_2 t^2, \end{aligned}$$

where K_1 and K_2 are suitable constants depending only on $DF(0)$, $G(0)$, L, and τ. Gronwall's lemma therefore yields

$$\|M(\tau,t) - M_0(\tau,t)\| \le K_3 t^2 \qquad \text{for all } t \in [0,\tau],$$

where the constant K_3 depends only on K_1, K_2. Therefore, there exists a constant K_I, depending only on $DF(0)$, $G(0)$, L, and τ such that

$$|I| \le K_I t^2 \qquad \text{for all } t \in [0,\tau]. \tag{4.3.4}$$

Assumptions (i) and (ii) yield in turn

$$|II| \le K_{II} t \qquad \text{for all } t \in [0,\tau], \tag{4.3.5}$$

for a suitable constant K_{II} depending only on $DF(0)$, L, and τ, and

$$|III| \le K_{III} t \qquad \text{for all } t \in [0,\tau], \tag{4.3.6}$$

$$|IV| \leq K_{IV} t \qquad \text{for all } t \in [0, \tau], \tag{4.3.7}$$

where again K_{III} and K_{IV} depend only on $DF(0)$, $G(0)$, L, and τ. Therefore, summing (4.3.4), (4.3.5), (4.3.6), and (4.3.7), we obtain that there exist K and $\mathcal{T}' > 0$, depending only on $DF(0)$, $G(0)$, and L, such that

$$|g'(t) - g_0'(t)| \leq Kt \qquad \text{for all } 0 \leq t \leq \mathcal{T}'. \tag{4.3.8}$$

Let now $N = 3$, and observe that owing to assumption (iv) each summand I, II, III, IV, divided by t^2, is bounded and a.e. differentiable, so that

$$|g''(t) - g_0''(t)| \leq K' t \qquad \text{for a.e. } t \in [0, \mathcal{T}'],$$

where the constant K' depends only on $DF(0)$, $G(0)$, and L and so does $\mathcal{T}'$. Observe that all the above constants do not depend on ζ. Therefore, invoking assumption (iii), we can apply Lemma 4.2.3 (for $N = 2, 3$), thus obtaining that there exists $\mathcal{T} > 0$, depending only on $G(0)$, $DF(0)$, L, and N with all the properties (a), (b), (c), and (d), which are exactly those required to complete the proof.
In the general case (i.e., $1 < M \leq N$), it suffices to apply the above argument to each column of G. The proof is concluded. □

Remark 4.3.1 *On the assumption $N = 2$ or $N = 3$.*

The restriction $N \leq 3$ depends on our method for comparing the switching function g for the nonautonomous system coming from the linearization along an optimal trajectory, with the switching function for autonomous system obtained by linearizing at the origin. This comparison requires higher order derivatives of g, whose existence we are not able to insure if $N > 3$.□

4.4 Quantitative strict convexity of reachable sets and uniqueness of optimal controls for the nonlinear two dimensional case

In this subsection, we will show that, provided the linearization at 0 satisfies the normality condition and the nonlinear part is smooth and small enough, the reachable set is strictly convex up to a sufficiently small time.

Theorem 4.4.1 *Consider the control system*

$$\begin{cases} \dot{x}(t) = F(x(t)) + G(x(t))u(t), \\ x(0) = 0, \end{cases} \tag{4.4.1}$$

under the following assumptions (in the following M is either 1 or 2):
$u(\cdot) = (u_1(\cdot), u_M(\cdot)) \in [-1, 1]^M$ *a.e.,* $F : \mathbb{R}^2 \to \mathbb{R}^2$, $G : \mathbb{R}^2 \to \mathbb{M}_{2\times M}$ *are of class* $\mathcal{C}^{1,1}$ *(with Lipschitz constant L) and*

(i) $F(0) = 0$,

(ii) Rank $[G_i(0), DF(0)G_i(0)] = 2$ *for* $i = 1, M$ *where* $G = (G_1, G_M)$,

(iii) $DG(0) = 0$.

Let $\mathcal{R}^\tau$ *denote the reachable set at time* $\tau > 0$ *for* (4.4.1)*. Then there exists* $\mathcal{T} > 0$*, depending only on* $L, DF(0), G(0)$*, with the following properties:*

(a) for every $\tau \leq \mathcal{T}$ *and every* $x \in \operatorname{bdry} \mathcal{R}^\tau$ *there exists one and only one admissible control* u *steering the origin to* x *in time* τ *(and* u *is bang-bang with finitely many switchings).*

(b) For every $0 < \tau < \mathcal{T}$ *the reachable set* $\mathcal{R}^\tau$ *is strictly convex. More precisely, for every* $x_1 \in \operatorname{bdry} \mathcal{R}^\tau$ *and* $x_2 \in \mathcal{R}^\tau$*, for every* $\zeta \in N^P_{\mathcal{R}^\tau}(x_1)$*, one has*

$$\langle \zeta, x_2 - x_1 \rangle \leq -\gamma \|\zeta\| \|x_2 - x_1\|^2. \tag{4.4.2}$$

where γ *is a positive constant depending only on* L*,* $DF(0)$*,* $G(0)$*.*

(c) There exist another time $\mathcal{T}' > \mathcal{T}$*, depending only on* L*,* $DF(0)$*,* $G(0)$*, such that for every* $0 < \tau < \mathcal{T}'$ *the reachable set* $\mathcal{R}^\tau$ *has positive reach. More precisely, for every* $x_1 \in \operatorname{bdry} \mathcal{R}^\tau$ *and* $x_2 \in \mathcal{R}^\tau$*, for every* $\zeta \in N^P_{\mathcal{R}^\tau}(x_1)$*, one has*

$$\langle \zeta, x_2 - x_1 \rangle \leq \gamma' \|\zeta\| \|x_2 - x_1\|^2, \tag{4.4.3}$$

where γ' *is a nonnegative constant depending only on* L*,* $DF(0)$*,* $G(0)$*.*

(d) There exist another positive constant γ'' *and a positive time* $\mathcal{T}'' \leq \mathcal{T}$*, depending only on* L*,* $DF(0)$*,* $G(0)$*, such that the ball* $B(0, \gamma''\tau^2)$ *is contained in* $\mathcal{R}^\tau$ *for all* $0 < \tau < \mathcal{T}''$*.*

(e) The minimum time function is continuous in $\mathcal{R}^\tau$*, for all* $0 < \tau < \mathcal{T}''$*.*

Proof. We begin proving the result for $M = 1$, i.e., for a scalar control.

Fix $\tau > 0$ and $x_1 \in \operatorname{bdry} \mathcal{R}^\tau$, together with an optimal control $u_1(\cdot)$ steering 0 to x_1 and the associate trajectory $x_1(\cdot)$. Take any $x_2 \in \mathcal{R}^\tau$ together with $u_2(\cdot)$ steering 0 to x_2 and the associate trajectory $x_2(\cdot)$, and set $x(t) = x_2(t) - x_1(t)$. Then, for a.e. $t \in [0, \tau]$,

$$\dot{x}(t) = A_1(t)x(t) + G(x_1(t))w(t), \tag{4.4.4}$$

where $w(t) = u_2(t) - u_1(t)$ and

$$A_1(t) = \int_0^1 DF(x_1(t) + \tau x(t))d\tau + u_2(t) \int_0^1 DG(x_1(t) + \tau x(t))d\tau.$$

Let $z(\cdot)$ be the solution of the linear system which is defined by linearizing along the optimal trajectory $x_1(\cdot)$:

$$\begin{cases} \dot{z}(t) = A(t)z(t) + G(x_1(t))w(t), \\ z(0) = 0, \end{cases} \tag{4.4.5}$$

where $A(t) = DF(x_1(t)) + DG(x_1(t))u_1(t)$.
We have

$$\begin{aligned} \frac{d}{dt}\|x(t) - z(t)\| &\leq \|A_1(t)x(t) - A(t)z(t)\| \\ &\leq \|A(t)\|\|x(t) - z(t)\| + \|A_1(t) - A(t)\|\|x(t)\| \\ &\leq L_1\|x(t) - z(t)\| + L\|x(t)\|^2, \end{aligned}$$

where $L_1 = \|DF(0)\| + 2Le^{2L\tau}$.
Thus, by Gronwall's inequality we get

$$\|x(t) - z(t)\| \leq e^{L_1 t} L \int_0^t \|x(s)\|^2 \, ds. \tag{4.4.6}$$

On the other hand, observing that

$$\frac{d}{dt}\|x(t)\| \leq L_2\|x(t)\| + L_3|w(t)|,$$

(where $L_2 = \|DF(0)\| + 4Le^{2L\tau}$ and $L_3 = |G(0)| + e^{2L\tau}$) we also have

$$\|x(t)\| \leq L_3 e^{L_2 t} \int_0^t |w(s)| ds. \tag{4.4.7}$$

From (4.4.6) and (4.4.7), one obtains

$$\|x(t) - z(t)\| \leq L_4 t \Big(\int_0^t |w(s)| ds \Big)^2, \tag{4.4.8}$$

where $L_4 = LL_3^2 e^{(L_1+2L_2)\tau}$.

Since $\bar{x}_1 \in \operatorname{bdry} \mathcal{R}^\tau$, by Pontryagin's maximum principle there exists an absolutely continuous function $\lambda : [0, \tau] \to \mathbb{R}^2$ with the following properties

$$\dot{\lambda}(t) = -\lambda(t)A(t), \quad \lambda(\tau) = \zeta \in N_{\mathcal{R}^\tau}(\bar{x}_1), \ \zeta \neq 0,$$

$$u_1(t) = \operatorname{sign}\langle \lambda(t), G(x_1(t)) \rangle. \tag{4.4.9}$$

We set now $b(t) = G(x_1(t))$ and consider the linear nonautonomous control system

$$\begin{cases} \dot{y}(t) = A(t)y(t) + b(t)u(t), \\ y(0) = 0, \end{cases} \tag{4.4.10}$$

together with the trajectory $y_1(\cdot)$, corresponding to the control $u_1(\cdot)$. Observe first that, thanks to (4.4.9), x_1 belongs to the boundary of the reachable set at time τ for (4.4.10). Observe moreover that $A(\cdot)$ is measurable, and, since both F and G are Lipschitz with constant L and $DG(0) = 0$, we have

$$\begin{aligned}\|A(t) - A(0)\| &= \|DF(x_1(t)) + DG(x_1(t))u_1(t) - DF(0)\| \\ &\leq 2L\|x_1(t)\| \leq 2L\|G(0)\|e^{2Lt}t.\end{aligned}$$

Finally, $b'(t) = DG(x_1(t))\dot{x}_1(t)$ so that

$$\|b'(t) - b(0)\| = \|b'(t)\| \leq L\|x_1(t)\|\big(2L\|x_1(t)\| + \|G(0)\|\big) \leq Kt,$$

where $K = Le^{2L\tau}\big(2Le^{2L\tau} + 1\big)\|G(0)\|^2$.
Therefore, all assumptions of Theorem 4.2.2 are satisfied by (4.4.10), so that there exists a time $\mathcal{T}_0 > 0$ depending only on $L, DF(0), G(0)$ such that if $0 \leq \tau < \mathcal{T}_0$ the following properties hold:

i) $u_1(t)$ is uniquely determined by (4.4.9) on $(0, \tau)$,

ii) there exists a constant $\gamma(\tau) > 0$, depending only on $L, DF(0), G(0), \tau$ such that, for all trajectories $y(\cdot)$ of (4.4.10), $\langle \zeta, y(\tau) - y_1(\tau)\rangle \leq -\gamma(\tau)\|y(\tau) - y_1(\tau)\|^2$. More precisely, recalling (4.2.28),

$$\langle \zeta, y_2(\tau) - y_1(\tau)\rangle \leq -\gamma_1(\tau)\Big(\int_0^\tau |w(s)|ds\Big)^2, \tag{4.4.11}$$

where $y_2(\cdot)$ is the trajectory of (4.4.10) associated with the control $u_2(\cdot)$ and γ_1 enjoys the same properties of γ.

We remark (see (4.2.29)) that $\gamma_1(\tau)$ is bounded away from 0 as $\tau \to 0^+$. Moreover, one can see that $z(t) = y_2(t) - y_1(t)$. Therefore

$$\begin{aligned}\langle \zeta, x_2 - x_1\rangle &= \langle \zeta, x(\tau) - z(\tau)\rangle + \langle \zeta, z(\tau)\rangle \\ &\leq \|x(\tau) - z(\tau)\| + \langle \zeta, z(\tau)\rangle.\end{aligned} \tag{4.4.12}$$

Recalling (4.4.8) and (4.4.11), we obtain

$$\langle \zeta, x_2 - x_1\rangle \leq (L_4\tau - \gamma_1(\tau))\Big(\int_0^\tau |w(s)|ds\Big)^2. \tag{4.4.13}$$

Thus if $\tau \leq \frac{1}{2L_4}\liminf_{\tau\to 0^+}\gamma(\tau) =: \mathcal{T}_1$ then

$$\langle \zeta, x_2 - x_1\rangle \leq -\frac{\gamma_1(\tau)}{2}\Big(\int_0^\tau |w(s)|ds\Big)^2.$$

From this inequality the uniqueness of the control steering the origin to x_1 in time τ follows immediately by contradiction. Setting $\mathcal{T} = \min\{\mathcal{T}_0, \mathcal{T}_1\}$ and recalling (4.4.7) we obtain (4.4.2). The proof of the strict convexity is completed by applying Proposition 7.2.1.

The proof of (4.4.3) is entirely analogous, where it suffices to take $\mathcal{T}' =$

$\mathcal{T}_0$.

We consider now the statement concerning the ball contained in the reachable set. To this aim, take $u_2 \equiv 0$ and set y_1 to be the solution of (4.4.5) with u_1 in place of w. Then (4.4.8) yields, for all $t > 0$,

$$\|x_1(t) - y_1(t)\| \leq L_4 t^3.$$

Recalling (4.2.41), we obtain from the previous inequality that

$$\|x_1(t)\| \geq \|y_1(t)\| - L_4 t^3 \geq \tilde{\gamma} t^2 - L_4 t^3,$$

for a suitable constant $\tilde{\gamma}$, which yields in particular that 0 belongs to the interior of $\mathcal{R}^t$, $0 \leq t \leq \mathcal{T}$. Since the above argument can be repeated for every point in the boundary of $\mathcal{R}^t$ and the constant L_4 is independent of the reference point, the statement follows by recalling that we already proved that $\mathcal{R}^t$ is convex for all $0 \leq t \leq \mathcal{T}$.

The continuity of T follows easily from the the fact that reachable sets contain a ball (see, e.g., Propositions IV.1.2 and IV.1.6 in [12]).

In the case $M = 2$, it suffices to apply the above arguments to each control. □

The following Remark follows immediately from the proof of Theorem 4.4.1.

Remark 4.4.1 *Let $x_1(\cdot)$ and $\lambda(\cdot)$ be as in the proof of Theorem 4.4.1. For all $0 < t \leq \tau = T(x_1)$, one has $\lambda(t) \in N_{\mathcal{R}^t}(x_1(T(x_1 - t)))$. More precisely,*

$$\langle \lambda(t), y - x_1(T(x_1) - t) \rangle \leq -\gamma \|\lambda(t)\| \|y - x_1(T(x_1) - t)\|^2$$

for all $y \in \mathcal{R}^t$, where γ is the constant appearing in (4.4.2)*.*

In fact, put $\lambda(t)$ in place of ζ in (4.4.12). Then the first summand can be estimated in the same way, while the upper bound on the second summand, namely the analogue of (4.4.11), can be obtained through the same arguments leading to (4.4.11). □

We conclude this section with a counterexample showing the sharpness of assumption (iii) in Theorem 4.4.1.

Remark 4.4.2 *An example of a two dimensional nonlinear control system satisfying assumptions (i) and (ii) of Theorem 4.4.1 such that the reachable set $\mathcal{R}^\tau$ is not convex for all $\tau > 0$.*

Consider the control system

$$\begin{cases} \dot{x}_1 &= x_2(1+u) \\ \dot{x}_2 &= u, \qquad u \in [-1,1]. \end{cases} \tag{4.4.14}$$

Set $F(x_1, x_2) = (x_2, 0)$, $G(x_1, x_2) = (x_2, 1)$, and observe that the assumptions (i) and (ii) of Theorem 4.4.1 are satisfied, while (iii) is not. The Hamiltonian for this system is

$$\mathcal{H}\big((x_1, x_2), (p_1, p_2), u\big) = x_2 p_1 + u(p_1 x_2 + p_2),$$

and Pontryagin's Maximum Principle states that if $\bar{x}(\cdot)$ is an optimal trajectory corresponding to the control $\bar{u}(\cdot)$, then there exists a function $\lambda = (\lambda_1, \lambda_2)$, never vanishing, and a constant $\lambda_0 \leq 0$ such that, for a.e. t,

$$\begin{aligned}
\lambda_1(t) &\equiv \lambda_1 \neq 0 \\
\dot{\lambda}_2(t) &= -(1 + \bar{u}(t))\lambda_1 \\
\bar{x}_2(t)\lambda_1 + \bar{u}(t)(\lambda_1 \bar{x}_2(t) + \lambda_2(t)) + \lambda_0 &= 0 \\
\bar{u}(t)(\lambda_1 \bar{x}_2(t) + \lambda_2(t)) &= \max_{|u| \leq 1} u(\lambda_1 \bar{x}_2(t) + \lambda_2(t)).
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\lambda_1 \bar{x}_2(t) + \lambda_2(t) &= \lambda_1 \int_0^t \bar{u}(s)\, ds - \lambda_1 t - \lambda_1 \int_0^t \bar{u}(s)\, ds + \lambda_2(0) \\
&= -\lambda_1 t + \lambda_2(0).
\end{aligned}$$

Therefore the function $\lambda_1 \bar{x}_2(t) + \lambda_2(t)$ has at most one zero, so that the optimal control $\bar{u}$ is unique, bang-bang, and has at most one switching. Fix now $\tau > 0$ and $0 < s < 1$ and consider the control

$$u_s(t) = \begin{cases} 1 & 0 < t < s\tau \\ -1 & s\tau < t < \tau. \end{cases}$$

The trajectory of (4.4.14) emanating from the origin and corresponding to the control u_2 is, at time τ,

$$\bar{x}_s^1 := x_s^1(\tau) = s^2\tau^2, \quad \bar{x}_s^2 := x_s^2(\tau) = \tau(2s - 1).$$

Simple computations show that any other bang-bang control with at most one switching cannot reach $(\bar{x}_s^1, \bar{x}_s^2)$ at a time $\tau' < \tau$, for all $0 < s < 1$. Thus u_s is optimal. In particular, the curve $\gamma(s) := (\bar{x}_s^1, \bar{x}_s^2)$ belongs to the boundary of the reachable set $\mathcal{R}^\tau$. The unique unit normal to the curve $\gamma(s)$ at $s = 1/2$ which points outside $\mathcal{R}^\tau$ is

$$\zeta = \frac{(2, -\tau)}{\sqrt{4 + \tau^2}}.$$

We compute:

$$\big\langle \zeta, \gamma(s) - \gamma(\tfrac{1}{2}) \big\rangle = \frac{2\tau^2(s - \frac{1}{2})^2}{\sqrt{4 + \tau^2}},$$

which implies that $\mathcal{R}^\tau$ is not convex. Observe however that $\mathcal{R}^\tau$ has positive reach at $\gamma(\frac{1}{2})$ for every $\tau > 0$, since

$$\left\langle \zeta, \gamma(s) - \gamma(\frac{1}{2}) \right\rangle \leq \frac{8}{(16+\tau^4)\sqrt{4+\tau^2}} \left\| \gamma(s) - \gamma(\frac{1}{2}) \right\|^2.$$

□

Remark 4.4.3 *On the assumption $N = 2$.*

Motivations for the restriction $N = 2$ are twofold. First of all, our analysis is based on the switching function of the nonautonomous system obtained by linearizing around an optimal trajectory, and this method requires $N \leq 3$ (see Remark 4.3.1). Second, the distance between trajectories of the nonlinear system (4.4.1) and of the linearized system (4.4.5) is of order two with respect to the control (see (4.4.8)), and this quadratic perturbation can be balanced by the strict convexity of the reachable set of the linearized system only if $N = 2$ (see (4.4.13)). □

4.5 Further results for the nonlinear two dimensional case

This section is devoted to proving that the epigraph of the minimum time function has positive reach, under the assumptions of Theorem 4.4.1. To this aim, a results of optimal points, i.e., on points which are *crossed* by an optimal trajectory, is needed.

4.5.1 Optimal points

The classical definition of optimal point reads as follows.

Definition 4.5.1 *Let $x \in \mathbb{R}^N \setminus \{0\}$. We say that x is optimal if and only if there exists a point x_1 such that $T(x_1) > T(x)$ and a control u with the property that $y^{x_1,u}(\cdot)$ steers x_1 to 0 in the optimal time $T(x_1)$ and $x = y^{x_1,u}(T(x_1) - T(x))$.*

The following is the result on optimal points which will be used in the next subsection in order to ensure the positive reach of the epigraph of the minimum time function. It is based on the same estimates which lead to the strict convexity of the reachable set, and so it is restricted to two dimensional control systems.

Theorem 4.5.1 *Let $N = 2$ and let the assumptions of Theorem 4.4.1 be satisfied. Let $\mathcal{T} > 0$ be such that, according to Theorem 4.4.1, for all $0 \leq \tau < \mathcal{T}$, the reachable set $\mathcal{R}^\tau$ of (4.4.1) satisfies (4.4.2) for all $0 < \tau < \mathcal{T}$. Let $\bar{x}$ be such that $T(\bar{x}) < \mathcal{T}$. Then $\bar{x}$ is an optimal point.*

Proof. We consider first the case where G is a vector and the control u is one-dimensional. Set $\tau = T(\bar{x})$ and let $\bar{u}(\cdot)$ be the admissible control steering $\bar{x}$ to 0 in the optimal time τ, together with the associate trajectory $\bar{x}(\cdot)$. Set, for all $t \in [0, \tau]$,

$$A(t) = DF(\bar{x}(t)) + DG(\bar{x}(t))\bar{u}(t), \qquad b(t) = G(\bar{x}(t))$$

We assume preliminarily that $\bar{x}$ belongs to the boundary of $\mathcal{R}^\tau$ and let, by the Maximum Principle, λ be a solution of

$$\begin{cases} \dot{\lambda}(t) = -\lambda(t)A(t), \\ \lambda(\tau) = \zeta, \end{cases} \tag{4.5.1}$$

where $\zeta \in N_{\mathcal{R}^\tau}(\bar{x})$, $\zeta \neq 0$, and, for a.e. $t \in [0, \tau]$,

$$\bar{u}(t) = \text{sign}\langle \lambda(t), b(t)\rangle. \tag{4.5.2}$$

Set, for $t \in [0, \tau]$,

$$g(t) = \langle \lambda(t), b(t)\rangle.$$

We are now going to extend $\bar{u}(\cdot)$ in an interval $[\tau, \tau + \delta]$ for a suitable $\delta > 0$, with the property that the extended control and its associate trajectory satisfy the Maximum Principle.

Three cases may occur:

(i) $g(\tau) > 0$,

(ii) $g(\tau) < 0$,

(iii) $g(\tau) = 0$.

In the first case, we set $\bar{u}(t) = 1$ for all $t > \tau$ and let $\bar{x}(\cdot)$ be the associate trajectory satisfying $\bar{x}(\tau) = \bar{x}$. We extend analogously $A(\cdot)$, $b(\cdot)$, $\lambda(\cdot)$ and $g(\cdot)$ for $t > \tau$. Set $g(\tau) := \delta_1$. Observe that g is locally Lipschitz, so that, for $t > \tau$,

$$g(t) = g(\tau) + g(t) - g(\tau) > \delta_1 - L_1(t - \tau)$$

for a suitable constant L_1. Therefore we can find $\delta > 0$ such that $0 \leq \tau + \delta < \mathcal{T}$ and $g(t) > 0$ for all $t \in [\tau, \tau + \delta]$, i.e.,

$$\bar{u}(t) = \text{sign } g(t) \quad \forall t \in [\tau, \tau + \delta].$$

The second case is entirely analogous, by substituting 1 with -1.
We consider now the third case. Let the I_0, I_1 be given by Lemma 4.2.3 for the function g in the interval $[0, \tau]$. Observe that necessarily $\tau \in I_1$, so that, in particular, $g'(\tau) \neq 0$. We set, for $t > \tau$

$$\bar{u}(t) = 1 \quad \text{if} \quad g'(\tau) > 0$$

or

$$\bar{u}(t) = -1 \quad \text{if} \quad g'(\tau) < 0$$

and let $\bar{x}(\cdot)$ be the associate trajectory satisfying $\bar{x}(\tau) = \bar{x}$. We extend analogously $A(\cdot), b(\cdot), \lambda(\cdot)$ and $g(\cdot)$ for $t > \tau$. Observe that

$$\dot{g}(t) = \langle \lambda(t), [F, G](\bar{x}(t)) \rangle,$$

where $[F, G](x) = DG(x)F(x) - DF(x)G(x)$ denotes the Lie bracket of F and G. Therefore $\dot{g}$ is continuous, so that there exists $\delta > 0$ such that the sign of $\dot{g}(t)$ equals the sign of $\dot{g}(\tau)$ for all $t \in [\tau - \delta, \tau + \delta]$. Therefore our construction of $\bar{u}(\cdot)$ on $[0, \tau + \delta]$ is such that for a.e. $t \in [0, \tau + \delta]$,

$$\bar{u}(t) = \text{sign } g(t).$$

Consequently, all conclusions of Theorem 4.4.1 hold up to the time $\tau + \delta$. In particular, for all $t \in [0, \tau + \delta]$, $\bar{x}(t) \in \text{bdry}\,\mathcal{R}^t$. Since $T(\cdot)$ is continuous in a neighborhood of the trajectory $\bar{x}(\cdot)$, we obtain that $\bar{u}(\cdot)$ steers the origin optimally to $\bar{x}(\tau + \delta)$ in time $\tau + \delta$. Since the above argument can be applied also to the reversed dynamics $\dot{x} = -F(x) + G(x)u$, $u \in [-1, 1]$, then $T(\bar{x}(\tau + \delta)) = \tau + \delta$.
Let us now drop the assumption $\bar{x} \in \text{bdry}\,\mathcal{R}^\tau$. Since T is strictly decreasing along the optimal trajectory $\bar{x}(\cdot)$, and so $\bar{x}(t) \in \text{bdry}\,\mathcal{R}^{\tau - t}$ for all $0 < t < \tau$, there exists a nontrivial adjoint vector $\lambda(\cdot)$ which uniquely determines $\bar{u}(t)$ as in (4.5.2) up to the time τ. Thus the above argument can be applied also to $\bar{x}$.
If G is a 2×2 matrix, it suffices to perform the above construction for each column of G. The proof is concluded. □

We conclude the section with two corollaries. The first one is an immediate consequence of the proof of Theorem 4.5.1.

Corollary 4.5.1 *Under the same assumptions of Theorem 4.5.1, let* $\tau = T(\bar{x}) < \tau_1 < \mathcal{T}$. *Then there exists* $x_1 \in \text{bdry}\,\mathcal{R}^{\tau_1}$ *and a control* $u_1 : [\tau, \tau_1] \to [-1, 1]$ *such that the trajectory* $\tilde{x}(\cdot)$ *corresponding to the control*

$$\tilde{u}(t) = \begin{cases} \bar{u}(t) & 0 \le t \le \tau, \\ u_1(t) & \tau < t \le \tau_1. \end{cases}$$

and such that $\tilde{x}(0) = x_1$ *reaches* 0 *in the optimal time* τ_1 *and moreover* $\tilde{x}(\tau_1 - \tau) = \bar{x}$.

From Corollary 4.5.1 we obtain that $\bar{x}(\cdot)$ and $\lambda(\cdot)$ in the proof of Theorem 4.5.1 can be extended up to the time $\mathcal{T}$. Therefore, we obtain the following further corollary.

Corollary 4.5.2 *Under the assumptions of Theorem 4.4.1, the maximized Hamiltonian along* $\bar{x}(\cdot)$ *associated with* $\lambda(\cdot)$ *is constant in* $[0, \mathcal{T})$*, i.e.,*

$$H(\bar{x}(t), \lambda(t)) = C \quad \forall t \in [0, \mathcal{T}).$$

Proof. Let G be a vector and so the control u be scalar. Then

$$H(\bar{x}(t), \lambda(t)) = \langle \lambda(t), F(\bar{x}(t)) \rangle + |\langle \lambda(t), G(\bar{x}(t)) \rangle|.$$

Observe that the switching function $g(t) = \langle \lambda(t), G(\bar{x}(t)) \rangle$ vanishes at most on a countable subset of $[0, \mathcal{T})$. Therefore, for a.e. $t \in [0, \mathcal{T})$, we have

$$\frac{d}{dt} H(\bar{x}(t), \lambda(t)) = \Big(- \langle \lambda(t), [F, G](\bar{x}(t)) \rangle + \langle \lambda(t), [F, G](\bar{x}(t)) \rangle \Big) \operatorname{sign} g(t) = 0.$$

If G is a 2×2 matrix, it suffices to perform the above computation for each column. The proof is concluded. □

4.5.2 The epigraph of the minimum time function has positive reach

The present section is devoted to studying the "convexity type" of the minimum time function $T(\cdot)$, in the case where the dynamics satisfies a weak controllability condition, i.e., the function $T(\cdot)$ is merely continuous. The statement is two dimensional, since it is based on Theorem 4.5.1.

Theorem 4.5.2 *Let* $N = 2$ *and let the assumptions of Theorem 4.4.1 hold. Let* $\mathcal{T}$ *be given by Theorem 4.4.1. Then for every* $0 < \tau < \mathcal{T}$ *the epigraph of the minimum time function* $T(\cdot)$ *on* $\mathcal{R}^\tau$ *has positive reach.*

Corollary 4.5.3 *Under the same assumptions of Theorem 4.5.2 the minimum time function* T *satisfies all the properties listed in Theorem 2.2.2.*

Before beginning the proof of Theorem 4.5.2 we introduce the minimized Hamiltonian and study its sign.

Definition 4.5.2 *Let* $x, \zeta \in \mathbb{R}^N$*. We define the minimized Hamiltonian for the control system in* (4.4.1) *as*

$$h(x, \zeta) = \langle \zeta, F(x) \rangle + \min_{u \in \mathcal{U}} \langle \zeta, G(x) u \rangle.$$

Proposition 4.5.1 *Let* x *belong to the boundary of the reachable set* $\mathcal{R}^\tau$ *for* (4.4.1) *for some* $\tau > 0$*. Let* $\zeta \in N^F_{\mathcal{R}^\tau}(x)$[1]*. Then* $h(x, \zeta) \leq 0$*.*

[1] here $N^F_{\mathcal{R}^\tau}(x)$ denotes the Fréchet normal cone to $\mathcal{R}^\tau$ at x, i.e., all vectors v such that $\limsup_{\mathcal{R}^\tau \ni y \to x} \langle v, (y - x)/\|y - x\| \rangle \leq 0$

Proof. Let $\bar{u}(\cdot)$ be an admissible control steering x to 0 in time τ, together with the associate trajectory $\bar{x}(\cdot)$. Then, for all $0 \le t \le \tau$ the point $\bar{x}(t)$ belongs to $\mathcal{R}^\tau$, so that, by definition of Fréchet normal we have

$$\limsup_{t\to 0^+} \left\langle \zeta, \frac{\bar{x}(t)-x}{\|\bar{x}(t)-x\|} \right\rangle \le 0.$$

Observing that $\|x(t)-x\| \le Kt$ for a suitable constant K, we have

$$\limsup_{t\to 0} \left\langle \zeta, \frac{\bar{x}(t)-x}{t} \right\rangle \le 0.$$

In other words,

$$\begin{aligned} 0 \;&\ge\; \limsup_{t\to 0} \Big\langle \zeta, \frac{1}{t}\int_0^t \big(F(\bar{x}(s)) + G(\bar{x}(s))\bar{u}(s)\big)ds \Big\rangle \\ &=\; \langle \zeta, F(x)\rangle + \limsup_{t\to 0} \Big\langle \zeta, G(x)\frac{\int_0^t \bar{u}(s)ds}{t} \Big\rangle. \end{aligned}$$

Let $t_n \to 0$ be a sequence such that $\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \bar{u}(s)ds := \tilde{u}$ exists. By the convexity of $\mathcal{U}$, $\tilde{u} \in \mathcal{U}$, and so $h(x,\zeta) \le \langle \zeta, F(x)\rangle + \langle \zeta, G(x)\tilde{u}\rangle \le 0$. □

We are now ready to prove Theorem 4.5.2.

Proof of Theorem 4.5.2. Let $x \neq 0$ be such that $T(x) < \mathcal{T}$ and let $(\bar{u}(\cdot), \bar{x}(\cdot))$ be an optimal pair for x. By Maximum Principle[2] there exists $0 \neq \zeta \in N_{\mathcal{R}^{T(x)}}(x)$ such that the adjoint arc λ, with

$$\begin{cases} \dot{\lambda}(t) = \lambda(t)\big(DF(\bar{x}(t)) + DG(\bar{x}(t))\bar{u}(t)\big), \\ \lambda(T(x)) = \zeta \end{cases}$$

satisfies

$$\big\langle \lambda(t), F(\bar{x}(t)) + G(\bar{x}(t))\bar{u}(t) \big\rangle = h(\bar{x}(t), \lambda) \quad \text{for a.e. } t \in (0, T(x)).$$

We claim that

$$(\zeta, h(x,\zeta)) \in N^P_{\mathrm{epi}(T)}(x, T(x)), \tag{4.5.3}$$

i.e., there exists a constant $\sigma > 0$ such that, for all $y \in \mathbb{R}^N$ with $0 < T(y) < \mathcal{T}$ and for all $\beta \ge T(y)$, we have

$$\big\langle (\zeta,\theta), (y,\beta) - (x, T(x)) \big\rangle \le \sigma \|(\zeta,\theta)\| \Big(\|y-x\|^2 + |\beta - T(x)| \Big), \tag{4.5.4}$$

where $\theta = h(x,\zeta)$, and, moreover,

$$\sigma \text{ is independent of } x \text{ and } \zeta. \tag{4.5.5}$$

Indeed, we consider two cases:

[2]observe that we are applying Theorem 4.1.2 for the *reversed dynamics* $\dot{x} = -F(x) - G(x)u$

(a) $T(y) \leq T(x)$;

(b) $T(y) > T(x)$.

In the first case, $y \in \mathcal{R}^{T(x)}$, so that by Theorem 4.4.1

$$\langle \zeta, y - x \rangle \leq 0.$$

If $\beta \geq T(x)$ then (4.5.4) is automatically satisfied, since $\theta \leq 0$ by Proposition 4.5.1. If instead $\beta < T(x)$, we set $x_1 = \bar{x}(\beta)$.

We estimate first $\langle \zeta, y - x_1 \rangle$. Since $y \in \mathcal{R}^{\beta}$, recalling Remark 4.4.1, we have for suitable constants K_1, K_2 given by Gronwall's Lemma,

$$\begin{aligned}
\langle \zeta, y - x_1 \rangle &= \langle \lambda(\beta), y - x_1 \rangle + \langle \lambda(T(x)) - \lambda(\beta), y - x_1 \rangle \\
&\leq \langle \lambda(T(x)) - \lambda(\beta), y - x_1 \rangle \leq K_1 \|\lambda(T(x))\| \, |T(x) - \beta| \, \|y - x_1\| \\
&\leq K_1 \|\lambda(T(x))\| \, |T(x) - \beta| \big(\|y - x\| + \|x_1 - x\| \big) \\
&\leq K_1 \|\zeta\| \, |T(x) - \beta| \, \big(\|y - x\| + K_2 |T(x) - \beta| \big) \\
&\leq K_3 \|\zeta\| \big(\|y - x\|^2 + |T(x) - \beta|^2 \big)
\end{aligned}$$

for another suitable constant K_3.

Second, we estimate $\langle \zeta, x_1 - x \rangle$. We have

$$\begin{aligned}
\langle \zeta, x_1 - x \rangle &= \int_{\beta}^{T(x)} \big\langle \lambda(T(x)), F(\bar{x}(s)) + G(\bar{x}(s)) \bar{u}(s) \big\rangle \, ds \\
&= \int_{\beta}^{T(x)} \big\langle \lambda(s), F(\bar{x}(s)) + G(\bar{x}(s)) \bar{u}(s) \big\rangle \, ds \\
&+ \int_{\beta}^{T(x)} \big\langle \lambda(T(x)) - \lambda(s), F(\bar{x}(s)) + G(\bar{x}(x)) \bar{u}(s) \big\rangle \, ds \\
&\leq (T(x) - \beta) h(x, \zeta) + K_4 \|\zeta\| |T(x) - \beta|^2,
\end{aligned}$$

for a suitable constant K_4, owing to Corollary 4.5.2 (applied to the reversed dynamics $\dot{x} = -F(x) - G(x)u$. Therefore, since $h(x\zeta) \leq 0$,

$$\langle (\zeta, \theta), (y, \beta) - (x, T(x)) \rangle \leq (K_3 + K_4) \|\zeta\| \big(\|y - x\|^2 + |T(x) - \beta|^2 \big),$$

and the proof for the case (a) is concluded by observing that K_3 and K_4 are independent of ζ and x.

In the second case we need to use the optimality of x. We observe first that, since $\theta \leq 0$, we only need to prove (4.5.4) for $\beta = T(y)$. Recalling Corollary 4.5.1, we can extend the control $\bar{u}$ up to the time $T(y)$ so that the associated trajectory (still denoted by $\bar{x}(\cdot)$) remains optimal. Let also λ be the extended adjoint vector and denote by $\tilde{x}(\cdot)$ the trajectory of the reversed dynamics associated with the extended control $\bar{u}$, i.e.,

$$\begin{cases} \dot{\tilde{x}}(t) = -F(\tilde{x}(t)) - G(\tilde{x}(t)) \tilde{u}(t), \\ \tilde{x}(0) = 0, \end{cases}$$

where $\tilde{u}(t) = \bar{u}(T(y) - t)$.
Set $x_1 = \tilde{x}(T(y))$. We estimate first $\langle \zeta, y - x_1 \rangle$. We have, by arguing similarly as before,

$$\begin{aligned}
\langle \zeta, y - x_1 \rangle &= \langle \lambda(T(y)), y - x_1 \rangle + \langle \lambda(T(x)) - \lambda(T(y)), y - x_1 \rangle \\
&\quad \text{(the first summand is } \leq 0 \text{ by the construction in Theorem 4.5.1)} \\
&\leq \langle \lambda(T(x)) - \lambda(T(y)), y - x_1 \rangle \\
&\leq K_5 \|\zeta\| \big(|T(y) - T(x)|^2 + \|y - x_1\|^2 \big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \zeta, x_1 - x \rangle &= \int_{T(x)}^{T(y)} \big\langle \zeta, -F(\tilde{x}(s)) - G(\tilde{x}(s))\tilde{u}(s) \big\rangle ds \\
&= \int_{T(x)}^{T(y)} \big\langle \lambda(s), -F(\tilde{x}(s)) - G(\tilde{x}(s))\tilde{u}(s) \big\rangle ds \\
&\quad + \int_{T(x)}^{T(y)} \big\langle \lambda(T(x)) - \lambda(s), -F(\tilde{x}(s)) - G(\tilde{x}(s))\tilde{u}(s) \big\rangle ds \\
&\leq \int_{T(x)}^{T(y)} \max_{u \in \mathcal{U}} \big\langle \lambda(s), -F(\tilde{x}(s)) - G(\tilde{x}(s))u \big\rangle ds + K_6 \|\zeta\| (T(y) - T(x))^2 \\
&\quad \text{(for a suitable constant } K_6 \text{ given by Gronwall's Lemma).}
\end{aligned}$$

Recalling Corollary 4.5.2, the maximized Hamiltonian in the integral of the first summand is constant. Therefore we obtain

$$\langle \zeta, x_1 - x \rangle \leq -h(x, \zeta)(T(y) - T(x)) + K_6 \|\zeta\| \, |T(y) - T(x)|^2.$$

Combining the above estimates we obtain finally

$$\langle (\zeta, \theta), (y, T(y)) - (x, T(x)) \rangle \leq (K_5 + K_6) \|\zeta\| \big(\|y - x\|^2 + |T(y) - T(x)|^2 \big),$$

and the proof of the claim is concluded, by observing, again, that K_5, K_6 are independent of x and ζ.

In order to conclude the proof we observe that $N^P_{\text{epi}(T)}$ is pointed at every point $(x, T(x))$, $x \in \mathcal{R}^\tau$, since it is easy to see that the projection of every $(\zeta, \theta) \in N^P_{\text{epi}(T)}(x, T(x))$ onto $\mathbb{R}^N$ is normal to the strictly convex set $\mathcal{R}^\tau$. Therefore, we can apply Corollary 3.1 in [51], with $\Omega_P = \text{int}\mathcal{R}^\tau$, which shows that epi$(T)$ has positive reach. □

Part II

Regularity of a class of continuous functions

Chapter 5

External sphere condition and continuous functions

Our aim in this chapter is proving that if the hypograph of a continuous function $f : \Omega \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfies an external sphere condition then it has "essentially" positive reach, i.e., the hypograph of the restriction of f outside a closed set of zero measure has (locally) positive reach. Hence such a function enjoys some properties of a concave function, in particular a.e. twice differentiability.

The result is based on studying the set of *bad points* where the proximal normal cones to such points are not wedged, i.e., the set of horizontal proximal supergradients at "bad point" x, $\partial^\infty f(x)$, contains a nontrivial subspace. Since the hypograph of f satisfies an external sphere condition, we can construct a special subspace of $\partial^\infty f(x)$ which is convex combination of vectors v such that $(-v, 0)$ is a proximal normal vector to hypo(f) at $(x, f(x))$ realized by a ball of uniform radius $\theta > 0$. Thus, the set of *bad points* is closed in Ω. Finally, we prove that the density Lebesgue measure of a bad point is zero by using our special subspace and the inductive method. Therefore, the set of "bad points" has zero Lebesgue measure.

The chapter is organized as follows: Section 5.1 is devoted to definitions and basic facts, while Section 5.2 contains statements of main results. The same section contains also an outline of the proof of Theorem 5.2.1, which is a localized version of the main result and where all the basic arguments appear. Detailed arguments begin in Section 5.3, which contains several lemmas concerning the set of *bad points* (i.e., the normal cone to the hypograph of the function at those points contains at least one line). Section 5.4 is finally devoted to proof of Theorem 5.2.1. On the basis of Theorem 5.2.1, our main theorem will be proved in the same section together with its corollaries.

5.1 Notation

We first rewrite quickly some basic notations which concern in the chapter.

Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. The hypograph of f is denoted by

$$\text{hypo}(f) \;=\; \{(x,\beta) \mid x \in \Omega, \beta \le f(x)\}. \tag{5.1.1}$$

The vector $(-v,\lambda) \in \mathbb{R}^N \times \mathbb{R}$ is a *proximal normal vector* to hypo(f) (we will denote this fact that $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$) at $(x, f(x))$ iff there exists a constant $\sigma > 0$ such that for all $y \in \Omega$ and for all $\beta \le f(y)$, it holds

$$\langle(-v,\lambda)\ ,\ (y,\beta) - (x, f(x))\rangle \;\le\; \sigma\,(\|y-x\|^2 + |\beta - f(x)|^2). \tag{5.1.2}$$

Equivalently, $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ iff there exists a constant $\gamma > 0$ such that

$$B_{N+1}((x, f(x)) + \gamma(-v,\lambda), \gamma\|(-v,\lambda)\|) \cap \text{hypo}(f) = \varnothing \tag{5.1.3}$$

where

$$B_k(a,r) \;=\; \{z \in \mathbb{R}^k \mid \|z-a\| < r\}$$

is the open ball with center a and radius r in $\mathbb{R}^k$.
Moreover, the vector $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ is *realized by a ball of radius* $\rho > 0$ if $(-v,\lambda) \ne 0$ and (5.1.2) is satisfied for $\sigma = \frac{\|(-v,\lambda)\|}{2\rho}$.

Remark 5.1.1 *If* $(-v,\lambda) \in N^P_{\text{hypo}(f)}(x, f(x))$ *then* $\lambda \ge 0$.

Associated with hypo(f), we define that

1. $\partial^P f(x) = \{v \mid (-v,1) \in N^P_{\text{hypo}(f)}(x, f(x))\}$ is set of *proximal supergradients* of f at x .
2. $\partial^\infty f(x) \;=\; \{v \mid (-v,0) \in N^P_{\text{hypo}(f)}(x, f(x))\}$ v is the set a *proximal horizon supergradients* of f at x.

We are now giving some new notations. These notations are concerned with the set of *bad points* where the proximal normal cone of hypo(f) contains at least one line (i.e., it is not wedged). First we introduce two special types of normal vectors, namely

1. Normal vectors which are limit of unique normals at nearby points

$$\begin{aligned} N^L(x) \;=\; \{\ \xi \in \mathbb{R}^{N+1} \mid \exists\ \{x_n\} \to x \text{ such that} \\ i)\ f \text{ is Fréchet differentiable at } x_n \text{ and} \\ ii)\ \xi = \lim_{n\to\infty} \frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|} \quad \}. \end{aligned}$$

2. Among them we select the horizontal ones

$$N_0^L(x) \;=\; N^L(x) \cap (-\partial^\infty f(x), 0).$$

We also denote the subspace which is generated by $N_0^L(x)$ as

$$H_0(x) \;= \operatorname{span}\{N_0^L(x)\} \;=\; \{\sum_{i=1}^{k} \alpha_i \xi_i \mid \xi_i \in N_0^L(x) \text{ and } \alpha_i \in \mathbb{R}\},$$

and the positive cone which is generated by $N_0^L(x)$ as

$$H_0^+(x) \;= \operatorname{span}^+\{N_0^L(x)\} \;=\; \{\sum_{i=1}^{k} \alpha_i \xi_i \mid \xi_i \in N_0^L(x) \text{ and } \alpha_i \geq 0\}.$$

3. The largest vector subspace contained in $N^P_{\text{hypo}(f)}(x, f(x))$ will be denoted by

$$NL(x) \;=\; \{\, \xi \in N^P_{\text{hypo}(f)}(x, f(x)) \mid -\xi \in N^P_{\text{hypo}(f)}(x, f(x))\}.$$

From Remark 5.1.1, one can see that $NL(x) \subseteq (-\partial^\infty f(x), 0)$.

4. We denote the set of *bad points* of f by

$$BP_f \;=\; \{x \in \Omega \mid NL(x) \neq 0\} \tag{5.1.4}$$

At each point $x \in BP_f$, we write BP_f as the union of the two sets

$$BP_f^+(x) \;=\; \{y \in BP_f \mid f(y) \geq f(x)\}$$

$$BP_f^-(x) \;=\; \{y \in BP_f \mid f(y) \leq f(x)\}.$$

5.2 Main results

5.2.1 Statement of main results

Our results are the following theorem, together with several corollaries. We recall that the notation BP_f was defined in (5.1.4).

Theorem 5.2.1 *Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(f) satisfies the $\theta -$ external sphere condition, where $\theta : \Omega \longrightarrow (0, \infty)$ is continuous. Then*
i) $\Omega_P := \Omega \backslash BP_f$ *is open.*
ii) $\mathcal{L}^N(\Omega \backslash \Omega_P) = 0$.

Corollary 5.2.1 *Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(f) satisfies the $\theta -$ external sphere condition where $\theta : \Omega \longrightarrow [0, \infty)$ is continuous. Then the hypograph of $f_{|\Omega_P}$ has positive reach.*

Corollary 5.2.2 *Let $f : \Omega \longrightarrow \mathbb{R}$ be as in the statement of Theorem 5.2.1 then f satisfies properties (1)-(3) of Theorem 2.2.2.*

In view of Proposition 3.2.1 in Chapter 3, we can apply the previous results to the minimum time function.

Corollary 5.2.3 *Let $(f,\mathcal{U})$ be the control system 3.1.1 and $\mathcal{S}$ be a target in Chapter 3 . Under the conditions (H1), (H2) , (H3), (H4), there exists an open set $\mathcal{S}^c_P \subset \mathcal{S}^c$ such that $\mathcal{L}^N(\mathcal{S}^c \backslash \mathcal{S}^c_P) = 0$ and the restricted continuous function $T_{|\mathcal{S}^c_P} : \mathcal{S}^c_P \longrightarrow [0, +\infty)$ has the hypograph with positive reach.*

Corollary 5.2.4 *Under the conditions (H1), (H2) , (H3), (H4), the minimum time function is twice differentiable a.e. in S^c.*

In order to make our proof more clear, we prefer to state our main theorem in a particular case (local case). The arguments are used in the proof of the main part of the proof of Theorem 5.2.1.

Theorem 5.2.2 *Let $f : B_N(0,1) \longrightarrow \mathbb{R}$ be continuous and let $\rho > 0$. Assume that hypo(f) satisfies the $\rho-$ external sphere condition. Then*
i) $BP_f \cup \partial B_N(0,1)$ *is closed.*
ii) $\mathcal{L}^N(BP_f) = 0$.

5.2.2 Outline of proof of Theorem 5.2.2

The part (i) is precisely Lemma 5.3.4.
To prove the part (ii) we will use induction.

For the case $N = 1$. By using Lemma 5.4.1 and Corollary 5.3.5 we obtain that the $\mathcal{L}^1$-density of BP_f at x, $D^1_{BP_f}(x) = \lim_{\sigma\to\infty} \frac{\mathcal{L}^1(BP_f \cap B_1(x,\sigma))}{\mathcal{L}^1(B_1(x,\sigma))} = 0$ for all $x \in BP_f$. Therefore, the proof is completed by the Lebesgue theorem.

In order to get the conclusion for $N = k+1$ from the inductive assumption for $N = k \geq 1$. We divide the set BP_f into two parts:

The first part is $BP_f^{\zeta^+} \cup BP_f^{\zeta^+}$ (see the definition of BP_f^{ζ} near Lemma 5.3.7) where $\zeta^+ = (0,1)$ and $\zeta^- = (0,-1)$. Using Lemma 5.3.7, we get $\mathcal{L}^N(BP_f^{\zeta^+} \cup BP_f^{\zeta^+}) = 0$.

To prove $\mathcal{L}^N[BP_f \backslash (BP_f^{\zeta^+} \cup BP_f^{\zeta^+})] = 0$, we notice that Lemma 5.3.6 can be used at every point in the open set $B_N(0,1)\backslash(BP_f^{\zeta^+} \cup BP_f^{\zeta^+})$. We need to prove that for all $B_N(x,r_x) \subset B_N(0,1)\backslash(BP_f^{\zeta^+} \cup BP_f^{\zeta^+})$, it holds $\mathcal{L}^N(BP_f \cap B_N(x,r_x)) = 0$. Three small steps are considered

Step 1: Let $\bar{f} = f_{|B_N(x,r_x)}$. By Lemma 5.3.6, the hypo$(\bar{f}_{x_2})$ (See the definition of $\bar{f}_{x_2}$ near Lemma 5.3.6) satisfies the $\theta-$ *external sphere condition.*

Step 2: From Lemma 7.2.3 and the inductive assumption, we obtain that $\mathcal{L}^{N-1}(BP_{\bar{f}_{x_2}}) = 0$.

Step 3: We use Fubini's theorem to complete the proof.

5.3 Preparatory Lemmas

This section is devoted to several partial results which are needed to prove our main theorem. To simplify our statements, we agree that the continuous function f in this section is defined on $B_N(0,1)$ and hypo(f) satisfies the $\rho-$ *external sphere condition* for a given constant $\rho > 0$

5.3.1 Closedness of the set of bad points

The first lemma shows that the proximal normal unit vector to the hypograph of f at $(x, f(x))$ where f is differentiable is unique and is realized by a ball of radius ρ.

Lemma 5.3.1 *Let x be in $B_N(0,1)$ such that $f(.)$ is differentiable at x. Then $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is the unique proximal normal unit vector to hypo(f) at $(x, f(x))$. Moreover, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is realized by a ball of radius ρ, i.e, for all $y \in B_N(0,1)$ and for all $\beta \leq f(y)$, it holds:*

$$\langle \frac{(-Df(x),1)}{\|(-Df(x),1)\|} \ , \ (y,\beta) - (x, f(x)) \rangle \ \leq \ \frac{1}{2\rho} \ (\|y-x\|^2 + |\beta - f(x)|^2).$$

Proof. Since $f(.)$ is differentiable at x, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is unique Fréchet normal unit vector to the hypograph of $f(.)$ at $(x, f(x))$. Therefore, since hypo(f) satisfies the $\rho-$*external sphere condition*, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|}$ is the unique proximal normal unit vector to hypo(f) at $(x, f(x))$. Thus, $\frac{(-Df(x),1)}{\|(-Df(x),1)\|} \in N^P_{hypo(f)}(x, f(x))$ is realized by a ball of radius ρ. □

From this lemma and the continuity of f, three corollaries follow.

Corollary 5.3.1 *Let $x \in B_N(0,1)$. Then*

$$N^L(x) \ \subseteq \ N^P_{\text{hypo}(f)}(x, f(x)).$$

More precisely, for each $0 \neq \xi \in N^L(x)$ we have that ξ is a unit proximal normal vector to hypo(f) at $(x, f(x))$ realized by a ball of radius ρ.

Proof. Let $\xi \in N^L(x)$, and take a sequence $\{x_n\}$ converging to x such that f is differentiable at x_n and $\{\frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|}\}$ converges to ξ. By Lemma 5.3.1, $\frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|} \in N^P_{\text{hypo}(f)}(x_n, f(x_n))$ is realized by a ball of radius ρ, i.e., for all $y \in B_N(0,1)$ and for all $\beta \leq f(y)$, we have

$$\langle \frac{(-Df(x_n),1)}{\|(-Df(x_n),1)\|} \ , \ (y,\beta) - (x_n, f(x_n)) \rangle \ \leq \ \frac{1}{2\rho} \ (\|y-x\|^2 + |\beta - f(x_n)|^2). \tag{5.3.1}$$

By letting n approach to ∞ in (5.3.1), the inequality

$$\langle \xi \ , \ (y,\beta)-(x,f(x))\rangle \ \leq \ \frac{1}{2\rho}\ (\|y-x\|^2+|\beta-f(x)|^2)$$

holds for all $y\in B_N(0,1)$ and for all $\beta\leq f(y)$.
The proof is completed. $\square$

Corollary 5.3.2 *$N^L_0(x)$ is closed for all $x\in B_N(0,1)$. Moreover, if $\xi\in N^L_0(x)$ then ξ is a proximal normal unit vector to hypo(f) at $(x,f(x))$ realized by a ball of radius ρ.*

Proof. Let $\{\xi_n\}\subseteq N^L_0(x)$ converge to $\bar{\xi}$. We need to prove that $\bar{\xi}\in N^L_0(x)$. Indeed, for each n, there exists a sequence $\{x^k_n\}$ converging to x such that f is differentiable at x^k_n and $\{\frac{(-Df(x^k_n),1)}{\|(-Df(x^k_n),1)\|}\}$ converges to a unit vector $\xi_n\in(-\partial^\infty f(x),0)$. For each n we can take a point $y_n\in\{x^k_n\}$ such that $\|y_n-x\|\leq\frac{1}{n}$ and $\|\frac{(-Df(y_n),1)}{\|(-Df(y_n),1)\|}-\bar{\xi}\|\leq\frac{1}{n}$. Therefore $\{y_n\}$ and $\{\frac{(-Df(y_n),1)}{\|(-Df(y_n),1)\|}\}$ converge respectively to x and $\bar{\xi}$. This implies that $\bar{\xi}\in N^L(x)$. On the other hand, since $\{\xi_n\}\subseteq N^L_0(x)$ converges to $\bar{\xi}$ we have $\bar{\xi}\in(-\partial^\infty f(x),0)$. The proof is completed. $\square$

With a similar proof, we get the third corollary.

Corollary 5.3.3 *Let $\{x_n\}\in B_N(0,1)$ converge to $x\in B_N(0,1)$ and let $\xi_n\in N^L_0(x_n)$ converge to $\bar{\xi}$, then $\bar{\xi}\in N^L_0(x)$.*

The next lemma says that if there exists a vector $0\neq p_0\in(-\partial^\infty f(x))$ then we can find a vector in $N^L_0(x)$. This vector is found by considering a sequence which converges to x along the ray $\{x+tp_0 \mid t>0\}$ such that f is differentiable at each point of this sequence. This idea is inspiredly the proof of Lemma 4.7 in [20].

Lemma 5.3.2 *Let $x\in B_N(0,1)$ such that $\partial^\infty f(x)\neq 0$. Then $N^L_0(x)$ is nonempty.*

Proof. Let $0\neq -p_0\in\partial^\infty f(x)$. By the definition of $\partial^\infty f(x)$, $(p_0,0)\in N^P_{\text{hypo}(f)}(x,f(x))$, i.e. there exists a constant $\sigma_0>0$ such that

$$\langle (p_0,0)\ , \ (y,\beta)-(x,f(x))\rangle \ \leq \ \sigma_0\ (\|y-x\|^2+|\beta-f(x)|^2) \qquad (5.3.2)$$

for all $y\in B_N(0,1)$ and for all $\beta\leq f(y)$.
Set $x_n=x+\frac{p_0}{n}$. By the density theorem (see Theorem 1.3.1 in [28]), for each n there exists z_n such that

$$\partial_P f(z_n) \ \neq \ \emptyset \qquad (5.3.3)$$

$$\|z_n-x_n\| \ \leq \ \frac{1}{n^2} \qquad (5.3.4)$$

(5.3.3) implies that there exists a vector $(\zeta_n, -1)$ which is a proximal normal vector to the epigraph of $f(.)$ at $(z_n, f(z_n))$. Therefore, since hypo(f) satisfies the $\rho-$ *external sphere*
condition we obtain that $f(.)$ is differentiable at z_n. Recalling Lemma 5.3.1, for all $z \in B_N(0,1)$ and for all $\beta \leq f(z)$, we have

$$\langle(-Df(z_n),1)\ ,\ (z,\beta)-(z_n,f(z_n))\rangle \leq \frac{1}{2\rho}\ \|(-Df(z_n),1)\|\ (\|z_-z_n\|^2+|\beta-f(z_n)|^2). \quad (5.3.5)$$

Recalling (5.3.4), $z_n \in B_N(0,1)$ for n large enough. Thus by taking $y = z_n$ in (5.3.2), we obtain

$$\langle p_0\ ,\ z_n - x\rangle\ \leq\ \sigma_0\ (\|z_n-x\|^2+|\beta-f(x)|^2) \quad (5.3.6)$$

for all $\beta \leq f(z_n)$.
We have

$$\begin{aligned}\langle p_0\ ,\ z_n - x\rangle &= \langle p_0\ ,\ \frac{p_0}{n}\rangle + \langle p_0\ ,\ z_n - x_n\rangle \\ &= \frac{\|p_0\|^2}{n} + \langle p_0\ ,\ z_n - x_n\rangle.\end{aligned}$$

Combining the above inequality with (5.3.4), we get

$$\langle p_0\ ,\ z_n - x\rangle\ \geq\ \frac{\|p_0\|^2}{n} - \frac{\|p_0\|}{n^2}. \quad (5.3.7)$$

Moreover, from (5.3.4) we get

$$\|z_n - x\|\ = 0(\frac{1}{n}). \quad (5.3.8)$$

Recalling (5.3.6), (5.3.7) and (5.3.8), for n large enough, the estimate

$$\frac{\|p_0\|^2}{n}\ \leq 0(\frac{1}{n^2}) + |\beta - f(x)|^2 \quad (5.3.9)$$

holds for all $\beta \leq f(z_n)$.
Therefore, there exists a constant $C > 0$ such that

$$f(x) - f(z_n)\ \geq\ \frac{C}{\sqrt{n}}. \quad (5.3.10)$$

for n large enough.

We are now going to prove that : $\limsup_{n\to\infty} \|(-Df(z_n),1)\| = +\infty$. Assume by contradiction that there exists a constant $K > 0$ such that

$$\|(-Df(z_n),1)\|\ \leq\ K \qquad \text{for all } n. \quad (5.3.11)$$

By taking $z = x$ and $\beta = f(x)$ in (5.3.5) and by recalling (5.3.11) we have

$$(f(x)-f(z_n))(1-\frac{K}{2\rho}(f(x)-f(z_n))) \leq K(1+\frac{\|x-z_n\|}{2\rho})\|x-z_n\|. \quad (5.3.12)$$

for n large enough. Therefore, by (5.3.10) and (5.3.8), we get from the above inequality that there exists a constant $C_1 > 0$ such that

$$\frac{1}{\sqrt{n}} \leq C_1\frac{1}{n}.$$

for n large enough.
This is a contradiction.

We now assume without of loss of generality that $\lim_{n\to\infty}\frac{(-Df(z_n),1)}{\|(-Df(z_n),1)\|} = (-\overline{\zeta}_0, 0)$. Since $\{z_n\}$ converges to x, we have $(-\overline{\zeta}_0, 0) \in N_0^L(x)$. The proof is completed. □

Corollary 5.3.4 *If $x \in BP_f$ then $N_0^L(x)$ is nonempty.*

The following lemma is a crucial observation. At every *bad point*, we can extract a line from $H_0^+(x) \subseteq NL(x) \subseteq N^P_{\text{hypo}(f)}(x, f(x))$. It is also pivotal to prove Lemma 5.3.4 and Theorem 5.4.1. The difference between the proof of this lemma and the proof of the previous lemma is the way of choosing a sequence which allows us to get a vector in $N_0^L(x)$.

Lemma 5.3.3 *If $x \in BP_f$ then $H_0^+(x)$ contains at least one line.*

Proof.. We recall that by Corollary 5.3.4, $N_0^L(x)$ is nonempty. Assume by contradiction that $H_0^+(x)$ does not contain lines. From Corollary 5.3.2, $N_0^L(x)$ is compact and does not contain 0. Thus by applying Lemma 7.2.1 for $C = N_0^L(x)$, there exists a constant $\delta_0 > 0$ such that for all $0 \neq \xi_1, \xi_2 \in H_0^+(x)$, one has

$$\langle\frac{\xi_1}{\|\xi_1\|}, \frac{\xi_2}{\|\xi_2\|}\rangle > -1 + \delta_0.$$

Therefore, there exist a vector $(v_0, 0) \in H_0(x)$ and a constant $\delta_1 > 0$ such that $v_0 \in \mathbb{R}^N$, $\|v_0\| = 1$ and

$$\langle -(v_0, 0), \frac{\xi}{\|\xi\|}\rangle \geq \delta_1 \quad \text{for all } 0 \neq \xi \in H_0^+(x). \quad (5.3.13)$$

Since $x \in BP_f$ (namely, $NL(x)$ contains at least one line) there exists a unit vector $p_0 \in \mathbb{R}^N$ such that $(p_0, 0) \in NL(x)$ and $\langle p_0, v_0\rangle \geq 0$.
Setting $v_1 = v_0 + \frac{\delta_1}{2}p_0$, one can easily get from (5.3.13) that:

$$\langle -(v_1, 0), \frac{\xi}{\|\xi\|}\rangle \geq \frac{\delta_1}{2} \quad \text{for all } 0 \neq \xi \in H_0^+(x). \quad (5.3.14)$$

Setting $x_n = x + \frac{v_1}{n}$. By the density theorem (see Theorem 1.3.1 in [28]), for each n there exists z_n such that

$$\begin{aligned} \partial_P f(z_n) &\neq \emptyset && (5.3.15) \\ \|z_n - x_n\| &\leq \frac{1}{n^2} && (5.3.16) \end{aligned}$$

(5.3.15) implies that there exists a vector $(\zeta_n, -1)$ which is a proximal normal vector to the epigraph of $f(\cdot)$ at $(z_n, f(z_n))$. Therefore, since hypo(f) satisfies the *ρ-external sphere*
condition we obtain that $f(\cdot)$ is differentiable at z_n (see Proposition 3.15, p.51, [22]). Recalling Lemma 5.3.1, for all $z \in B_N(0,1)$ and for all $\beta \leq f(z)$, we have

$$\begin{aligned} &\langle (-Df(z_n), 1)\ ,\ (z, \beta) - (z_n, f(z_n)) \rangle \\ &\qquad \leq \frac{1}{2\rho} \|(-Df(z_n), 1)\| \ (\|z_- z_n\|^2 + |\beta - f(z_n)|^2). \quad (5.3.17) \end{aligned}$$

On the other hand, since $(p_0, 0) \in NL(x)$, there exists a constant $\sigma_0 > 0$ such that

$$\langle (p_0, 0)\ ,\ (y, \beta) - (x, f(x)) \rangle \ \leq \ \sigma_0 \ (\|y - x\|^2 + |\beta - f(x)|^2) \qquad (5.3.18)$$

for all $y \in B_N(0,1)$ and for all $\beta \leq f(y)$.
Recalling (5.3.16), $z_n \in B_N(0,1)$ for n large enough. Thus by taking $y = z_n$ in (5.3.18), we have

$$\langle p_0\ ,\ z_n - x \rangle \ \leq \ \sigma_0 \ (\|z_n - x\|^2 + |\beta - f(x)|^2) \qquad (5.3.19)$$

for all $\beta \leq z_n$.
We have

$$\begin{aligned} \langle p_0\ ,\ z_n - x \rangle &= \langle p_0\ ,\ \frac{v_0}{n} \rangle + \langle p_0\ ,\ \frac{\delta_1}{2n} p_0 \rangle + \langle p_0\ ,\ z_n - x_n \rangle \\ &\geq \frac{\delta_1}{2n} + \langle p_0\ ,\ z_n - x_n \rangle. \end{aligned}$$

Combining the above inequality with (5.3.16), we get

$$\langle p_0\ ,\ z_n - x \rangle \ \geq \ \frac{\delta_1}{2n} - \frac{1}{n^2}. \qquad (5.3.20)$$

Moreover, from (5.3.16) we get

$$\|z_n - x\| \ = 0(\frac{1}{n}). \qquad (5.3.21)$$

Recalling (5.3.19), (5.3.20) and (5.3.21), for n large enough, the estimate holds

$$\frac{\delta_1}{2n} \ \leq 0(\frac{1}{n^2}) + |\beta - f(x)|^2 \qquad (5.3.22)$$

for all $\beta \leq f(z_n)$.
Therefore, there exists a constant $C > 0$ such that

$$f(x) - f(z_n) \;\geq\; \frac{C}{\sqrt{n}}. \tag{5.3.23}$$

for n large enough.

We are now going to prove that : $\limsup_{n\to\infty} \|(-Df(z_n), 1)\| = +\infty$. Assume by contradiction that there exists a constant $K > 0$ such that

$$\|(-Df(z_n), 1)\| \;\leq\; K \qquad \text{for all } n. \tag{5.3.24}$$

By taking $z = x$ and $\beta = f(x)$ in (5.3.17) and by recalling (5.3.24) we have

$$(f(x) - f(z_n))(1 - \frac{K}{2\rho}(f(x) - f(z_n))) \;\leq K(1 + \frac{\|x - z_n\|}{2\rho})\|x - z_n\|. \tag{5.3.25}$$

for n large enough. Therefore, by (5.3.23) and (5.3.21), we get from the above inequality that there exists a constant $C_1 > 0$ such that

$$\frac{1}{\sqrt{n}} \;\leq\; C_1 \frac{1}{n}.$$

for n large enough.
This is a contradiction.

We now assume without of loss of generality that $\lim_{n\to\infty} \frac{(-Df(z_n),1)}{\|(-Df(z_n),1)\|} = (-\overline{\zeta}_0, 0)$. Moreover, since $\{z_n\}$ converges to x, we have $(-\overline{\zeta}_0, 0) \in N_0^L(x)$. On the other hand, by (5.3.23), we can take $z = x$ and $\beta = f(z_n)$ in (5.3.17) to get

$$\langle \frac{(-Df(z_n), 1)}{\|(-Df(z_n), 1)\|} \,,\, \frac{(x - z_n, 0)}{\|x - z_n\|} \rangle \;\leq\; \frac{\|x - z_n\|}{2\rho}. \tag{5.3.26}$$

Let n tend to $+\infty$. Recalling (5.3.21), (5.3.16) we obtain

$$\langle (-\overline{\zeta_0}, 0) \,,\, (-v_1, 0) \rangle \;\leq\; 0. \tag{5.3.27}$$

Since $(-\overline{\zeta}_0, 0) \in N_0^L(x)$, we get a contradiction from (5.3.27) and (5.3.14). $\square$

Lemma 5.3.4 *$BP_f \cup \partial B_N(0,1)$ is closed.*

Proof. Letting $\{x_n\} \subseteq BP_f \cup \partial B_N(0,1)$ converge to x, we need to prove that $x \in BP_f \cup \partial B_N(0,1) \subseteq \overline{B}_N(0,1)$.
If $x \in \partial B_N(0,1)$, there is nothing to prove.
If $x \in B_N(0,1)$, we will prove that $x \in BP_f$, namely, $NL(x)$ contains at least one line.
Assume by contradiction that $NL(x) = 0$. In particular, $H_0^+(x)$ does not

contain lines. Similarly by the previous proof, there exist a vector $(v_0, 0) \in H_0(x)$ and a constant $\delta_1 > 0$ such that $v_0 \in \mathbb{R}^N$, $\|v_0\| = 1$ and

$$\langle -(v_0, 0)\ ,\ \frac{\xi}{\|\xi\|} \rangle \ \geq\ \delta_1 \quad \text{for all}\ \ 0 \neq \xi \in H_0^+(x). \tag{5.3.28}$$

On the other hand, since $x \in B_N(0,1)$ we have $x_n \in B_N(0,1)$ for n large enough. Thus $x_n \in BP_f$. From Lemma 5.3.3, for n large enough, $H_0^+(x_n)$ contains at least one line. Therefore, for each n large enough, there exists a vector $\xi_n \in N_0^L(x_n)$ such that

$$\langle -(v_0, 0)\ ,\ \xi_n \rangle\ \leq 0. \tag{5.3.29}$$

By Corollary 5.3.2, $\|\xi_n\| = 1$. We assume without of loss of generality that $\lim_{n\to\infty} \xi_n = \bar{\xi}$. Recalling Corollary 5.3.3, we have that $\bar{\xi} \in N_0^L(x)$. Moreover, by taking $n \to \infty$ in (5.3.29) we get

$$\langle -(v_0, 0)\ ,\ \bar{\xi} \rangle\ \leq 0. \tag{5.3.30}$$

Recalling (5.3.28), we get a contradiction. □

5.3.2 Zero Lebesgue measure of some special subsets of the set of bad points

The below lemma is the first step to prove that the $\mathcal{L}^N$-density of BP_f at $x \in BP_f$ has zero value.

Lemma 5.3.5 *Define, for $x \in BP_f$, $F^+(x) = \{y \in B(0,1)\ | f(y) \geq f(x)\}$. Then the $\mathcal{L}^N$-density of $F^+(x)$ at x is zero, i.e.,*

$$D^N_{F^+(x)}(x) := \lim_{\delta \to 0} \frac{\mathcal{L}^N(B_N(x,\delta) \cap F^+(x))}{\mathcal{L}^N(B_N(x,\delta))} = 0.$$

Proof. Since $x \in BP_f$ (i.e., $NL(x)$ contains at least one line), there exists $(\zeta_0, 0) \in N^P_{\text{hypo}(f)}(x, f(x))$ such that $(-\zeta_0, 0) \in N^P_{\text{hypo}(f)}(x, f(x))$ and $\|\zeta_0\| = 1$. Thus there exists a constant $\sigma_0 > 0$ such that for all $y \in B_N(0,1)$ and for all $\beta \leq f(y)$, it holds

$$\begin{cases} \langle (\zeta_0, 0)\ ,\ (y - x, \beta - f(x)) \rangle & \leq\ \sigma_0\ (\|y - x\|^2 + |\beta - f(x)|^2), \\ \langle (-\zeta_0, 0)\ ,\ (y - x, \beta - f(x)) \rangle & \leq\ \sigma_0\ (\|y - x\|^2 + |\beta - f(x)|^2). \end{cases} \tag{5.3.31}$$

Therefore, for all $y \in F^+(x) \cap B_N(x,\delta)$, by taking $\beta = f(x)$ in (5.3.31) we obtain

$$\begin{cases} \langle \zeta_0\ ,\ y - x \rangle & \leq\ \sigma_0 \|y - x\|^2\ \leq\ \sigma_0 \delta^2, \\ \langle -\zeta_0\ ,\ y - x \rangle & \leq\ \sigma_0 \|y - x\|^2\ \leq\ \sigma_0 \delta^2. \end{cases} \tag{5.3.32}$$

From (5.3.32), the set

$$F^+(x) \cap B_N(x,\delta) \subseteq x + \{t\zeta_0 + v \mid t \in [-\sigma_0\delta^2, \sigma_0\delta^2],\ v \in B_N(0,\delta) \cap \zeta_0^\perp\}$$

where $\zeta_0^{\perp} = \{w \in \mathbb{R}^N | \langle w, \zeta_0 \rangle = 0\}$. Therefore,

$$D^N_{F^+(x)}(x) := \lim_{\delta\to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta)\cap F^+(x))}{\mathcal{L}^N(B_N(x,\delta))} \leq \lim_{\delta\to 0^+} \frac{\sigma_0\delta^{N+1}}{\omega_N\delta^N} = \lim_{\delta\to 0^+} \frac{\sigma_0\delta}{\omega_N} = 0$$

where $\omega_N = \mathcal{L}^N(B_N(0,1))$. The proof is completed. □

Since $BP_f^+(x) \subseteq F^+(x)$, the next corollary follows immediately

Corollary 5.3.5 *If $x \in BP_f$ then the $\mathcal{L}^N$-density of $BP_f^+(x)$ at x*

$$D^N_{BP_f^+(x)}(x) := \lim_{\delta\to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta)\cap BP_f^+(x))}{\mathcal{L}^N(B_N(x,\delta))} = 0.$$

In order to use induction in the proof of Theorem 5.2.2, we need the following two lemmas. In the first lemma, we are working on the cases $N \geq 2$. For every vector $x \in \mathbb{R}^N$ we rewrite $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^{N-1}$ and $x_2 \in \mathbb{R}$. For every $x_2 \in (-1,1)$, the function restricted to the first $n-1$ variables, $f_{x_2} : B_{N-1}(0, \sqrt{1-x_2^2}) \longrightarrow \mathbb{R}$, is denoted by $f_{x_2}(x_1) = f(x_1,x_2)$ for all $x_1 \in B_{N-1}(0,\sqrt{1-x_2^2})$.

Lemma 5.3.6 *Let $(x_1,x_2) \in B_N(0,1)$ and let (ξ_1,ξ_2,λ) be a proximal normal vector to hypo(f) at $(x_1,x_2,f(x_1,x_2))$ realized by a ball of radius ρ. If $(\xi_1,\lambda) \neq 0$ then (ξ_1,λ) is also a proximal vector to hypo(f_{x_2}) at $(x_1, f_{x_2}(x_1))$ realized by a ball of radius $\frac{\|(\xi_1,\lambda)\|}{\|(\xi_1,\xi_2,\lambda)\|}\rho$.*

Proof. The vector (ξ_1,ξ_2,λ) being a proximal normal to the hypograph of f at $(x_1,x_2) \in B_N(0,1)$ realized by a ball of radius ρ means that for all $(y_1,y_2) \in \mathbb{R}^N$ and for all $\beta \leq f(y_1,y_2)$, we have

$$\begin{aligned}\langle \frac{(\xi_1,\xi_2,\lambda)}{\|(\xi_1,\xi_2,\lambda)\|}\ ,\ (y_1,y_2,\beta) - (x_1,x_2,f(x_1,x_2))\rangle \\ \leq \frac{1}{2\rho}(\|y_1-x_1\|^2 + |y_2-x_2|^2 + |\beta - f(x_1,x_2)|^2).\end{aligned} \tag{5.3.33}$$

By taking $y_2 = x_2$ in (5.3.33), and by replacing $f(x_1,x_2) = f_{x_2}(x_1)$, $f(y_1,y_2)$ $= f(y_1,x_2) = f_{x_2}(y_1)$ in (5.3.33), we obtain that for all $y_1 \in B_{N-1}(0,\sqrt{1-x_2^2})$ and for all $\beta \leq f_{x_2}(y_1)$, it holds

$$\langle \frac{(\xi_1,\lambda)}{\|(\xi_1,\xi_2,\lambda)\|}\ ,\ (y_1,\beta) - (x_1, f_{x_2}(x_1))\rangle \leq \frac{1}{2\rho}(\|y_1-x_1\|^2 + |\beta - f_{x_2}(x_1)|^2). \tag{5.3.34}$$

Since $(\xi_1,\lambda) \neq 0$, from (5.3.34) we get that for all $y_1 \in B_{N-1}(0,\sqrt{1-x_2^2})$ and for all $\beta \leq f_{x_2}(y_1)$, it holds

$$\langle \frac{(\xi_1,\lambda)}{\|(\xi_1,\lambda)\|}\ ,\ (y_1,\beta)-(x_1,f_{x_2}(x_1))\rangle \leq \frac{1}{2\rho\frac{\|(\xi_1,\lambda)\|}{\|(\xi_1,\xi_2,\lambda)\|}}(\|y_1-x_1\|^2+|\beta-f_{x_2}(x_1)|^2).$$

The proof is completed. □

The second lemma is used to treat the case $(\xi_1, \lambda) = 0$ in Lemma 5.3.6 in the proof of our main theorem. Some notations are needed in this lemma: Let ζ be a unit vector in $\mathbb{R}^N$, we denote by:

i) $N_0^\zeta \;=\; \{x \in B_N(0,1) \mid (\zeta, 0) \in N^P_{\text{hypo}(f)}(x, f(x))$ is realized by a ball of radius $\rho\}$,

ii) $BP_f^\zeta \;=\; BP_f \cap N_0^\zeta$.

Lemma 5.3.7

i) $BP_f^\zeta \cup \partial B_N(0,1)$ *is closed.*

ii) BP_f^ζ *has zero N-Lebesgue measure.*

Proof of (i). By Lemma 5.4.17, the set $BP_f \cup \partial B_N(0,1)$ is closed. Thus we only need to prove that $N_0^\zeta \cup \partial B_N(0,1)$ is closed.

Let $\{x_n\} \subseteq N_0^\zeta \cup \partial B_N(0,1)$ converge to x, we need to show that $x \in N_0^\zeta \cup \partial B_N(0,1)$.
If $x \in \partial B_N(0,1)$ there is nothing to prove.
If $x \in B_N(0,1)$ then for n large enough we have $x_n \in B_N(0,1)$. Thus $x_n \in N_0^\zeta$, namely, $(\zeta, 0) \in N^P_{\text{hypo}(f)}(x_n, f(x_n))$ is realized by a ball of radius ρ, i.e, for all $z \in B_N(0,1)$ and for all $\beta \leq f(z)$, one has

$$\langle \frac{(\zeta,0)}{\|(\zeta,0)\|}\ ,\ (z,\beta) - (x_n, f(x_n))\rangle \;\leq\; \frac{1}{2\rho}(\|z - x_n\|^2 + |\beta - f(x_n)|^2). \quad (5.3.35)$$

Since $\{x_n\}$ converges to x and $f(\cdot)$ is continuous, by taking $n \to \infty$ we have

$$\langle \frac{(\zeta,0)}{\|(\zeta,0)\|}\ ,\ (z,\beta) - (x, f(x))\rangle \;\leq\; \frac{1}{2\rho}(\|z - x\|^2 + |\beta - f(x|^2). \quad (5.3.36)$$

for all $z \in B_N(0,1)$ and for all $\beta \leq f(z)$.
Thus $x \in N_0^\zeta$. The proof is completed. □

Proof of (ii). First, we prove that for all $x \in BP_f^\zeta$, it holds

$$D^N_{BP_f^\zeta}(x) \;=\; \lim_{\delta \to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta) \cap BP_f^\zeta)}{\mathcal{L}^N(B_N(x,\delta))} \leq \frac{1}{2}. \quad (5.3.37)$$

Indeed, since $BP_f^\zeta \subseteq BP_f$, recalling Corollary 5.3.5 we obtain

$$D^N_{BP_f^\zeta \cap BP_f^+(x)}(x) = \lim_{\delta \to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta) \cap BP_f^\zeta \cap BP_f^+(x))}{\mathcal{L}^N(B_N(x,\delta))} = 0.$$

Thus the inequality (5.3.37) will hold if

$$D^N_{BP_f^\zeta \cap BP_f^-(x)}(x) \;=\; \lim_{\delta \to 0^+} \frac{\mathcal{L}^N(B_N(x,\delta) \cap BP_f^\zeta \cap BP_f^-(x))}{\mathcal{L}^N(B_N(x,\delta))} \leq \frac{1}{2}. \quad (5.3.38)$$

If $y \in BP_f^\zeta$, we have $(\zeta,0) \in N_0^\zeta(y)$. Thus for all $z \in B_N(0,1)$ and for all $\beta \le f(z)$, it holds

$$\langle (\zeta,0)\ ,\ (z-y,\beta-f(y))\rangle \ \le\ \frac{1}{2\rho}\,(\|z-y\|^2+|\beta-f(y)|^2). \tag{5.3.39}$$

Thus, if $y \in BP_f^\zeta \cap BP_f^-(x)$ we can take $z=x$ and $\beta=f(y)$ in 5.3.39 to get

$$\langle \zeta\ ,\ x-y\rangle \ \le\ \frac{1}{2\rho}\,\|x-y\|^2. \tag{5.3.40}$$

Therefore, for all $\delta>0$ small enough, we have

$$\langle \zeta\ ,\ x-y\rangle \ \le\ \frac{1}{2\rho}\delta^2 \qquad \text{for all } y \in [B_N(x,\delta)\cap BP_f^\zeta \cap BP_f^-(x)]. \tag{5.3.41}$$

(5.3.41) says that

$$[B_N(x,\delta)\cap BP_f^\zeta\cap BP_f^-(x)] \subset x+\{t\zeta+v|\ t\in[-\frac{\delta^2}{2\rho},\delta],\ v\in B_N(0,\delta)\cap\zeta^\perp\}$$

where $\zeta^\perp=\{w\in\mathbb{R}^N|\ \langle w,\zeta\rangle=0\}$. Thus, (5.3.38) follows. From (i), BP_f^ζ is a Borel set. Moreover, from (5.3.38), the $\mathcal{L}^N$-density of BP_f^ζ at every point which is in BP_f^ζ is less than $\frac{1}{2}$. Therefore, by the Lebesgue theorem we have $\mathcal{L}^N(BP_f^\zeta)=0$. □

5.4 Proof of our main results of Chapter 5

5.4.1 Proof of Theorem 5.2.2

One dimension case

In this part, we are working on $\mathbb{R}$. The function $f(\cdot)$ is defined on $B_1(0,1)=\{x\in\mathbb{R}\ |\ |x|<1\}$. Therefore the proximal normal cone $N^P_{\text{hypo}(f)}(x,f(x))\subset\mathbb{R}^2$ contains at most one line.

Lemma 5.4.1 *For all $x\in BP_f$, we have $N_0^L(x)=\{(1,0),\ (-1,0)\}$.*

Proof. Since $N_0^L(x)\subseteq(-\partial^\infty f(x),0)$, we have $N_0^L(x)\subseteq\{(t,0)\mid t\in\mathbb{R}\}$. Therefore, from the fact that $\|\xi\|=1$ for all $\xi\in N_0^L(x)$, we obtain

$$N_0^L(x)\ \subseteq\ \{(1,0),\ (-1,0)\} \tag{5.4.1}$$

Recalling Lemma 5.3.3, the set $H_0^+(x)=span^+\{N_0^L(x)\}$ contains at least one line. Thus, the proof is completed by (5.4.1).

The following statement is a one dimensional version of Theorem 5.2.2.

Theorem 5.4.1 *Let $f : B_1(0,1) \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(f) satisfies the $\rho-$ external sphere condition. Then*
i) $BP_f \cup \partial B_1(0,1)$ *is closed.*
ii) $\mathcal{L}^1(BP_f) = 0$.

(i) is the particular case (N=1) of Lemma 5.3.4 .
Proof of (ii). We prove first that, for all $x \in BP_f$, the $\mathcal{L}^1$-density of BP_f at x is zero, namely,

$$D^1_{BP_f}(x) := \lim_{\delta \to 0^+} \frac{\mathcal{L}^1(B_1(x,\delta) \cap BP_f)}{\mathcal{L}^1(B_1(x,\delta))} = 0. \qquad (5.4.2)$$

Recalling Corollary 5.3.5 for N=1, we have

$$D^1_{BP_f^+(x)}(x) = \lim_{\delta \to 0^+} \frac{\mathcal{L}^1(B_1(x,\delta) \cap BP_f^+(x))}{\mathcal{L}^1(B_1(x,\delta))} = 0.$$

Therefore, 5.4.2 follows from

$$D^1_{BP_f^-(x)}(x) = \lim_{\delta \to 0^+} \frac{\mathcal{L}^1(B_1(x,\delta) \cap BP_f^-(x))}{\mathcal{L}^1(B_1(x,\delta))} = 0. \qquad (5.4.3)$$

From Lemma 5.4.1, for every $y \in BP_f$, we have $N_0^L(y) = \{(1,0),\ (-1,0)\}$. Thus, for all $y \in BP_f$ it holds

$$\begin{cases} \langle (1,0)\ ,\ (z-y, \beta - f(y)) \rangle & \leq \frac{1}{2\rho}\ (|z-y|^2 + |\beta - f(y)|^2), \\ \langle (-1,0)\ ,\ (z-y, \beta - f(y)) \rangle & \leq \frac{1}{2\rho}\ (|z-y|^2 + |\beta - f(y)|^2). \end{cases} \qquad (5.4.4)$$

for all $z \in \overline{B_1(0,1)}$ and for all $\beta \leq f(z)$.
Since $f(y) \leq f(x)$ for all $y \in BP_f^-(x)$, we can take $z = x$ and $\beta = f(y)$ in (5.4.4) to get

$$|x-y| \ \leq \ \frac{1}{2\rho}\ |x-y|^2 \quad \text{for all } y \in BP_f^-(x). \qquad (5.4.5)$$

Thus $B_1(x,\delta) \cap BP_f^-(x) = \{x\}$ for all $0 < \delta < 2\rho$ and so 5.4.3 follows.

We are now going to complete the proof of (ii).
Since $BP_f \cup \partial B_1(0,1)$ is closed, BP_f is a Borel set. From 5.4.2, the $\mathcal{L}^1$-density of BP_f at x has zero value for all $x \in BP_f$. Therefore, by the Lebesgue theorem, we have $\mathcal{L}^1(BP_f) = 0$. □

General case

(i) of Theorem 5.2.2 is precisely Lemma 5.3.4.

We are going to prove (ii) of Theorem 5.2.2 by induction.
If $N = 1$, (ii) of Theorem 5.2.2 follows from Theorem 5.4.1.
Assume that (ii) of Theorem 5.2.2 holds for $N = k \geq 1$. We prove that (ii)

of Theorem 5.2.2 will hold for $N = k+1$.
Let $\zeta^+ = (0,1)$ and $\zeta^- = (0,-1)$ be in $\mathbb{R}^{k+1}$. Recalling Lemma 5.3.7, we obtain that $(BP_f^{\zeta^+} \cup \partial B_{k+1}(0,1))$ and $(BP_f^{\zeta^-} \cup \partial B_{k+1}(0,1))$ are closed. Moreover,

$$\mathcal{L}^{k+1}(BP_f^{\zeta^+}) = \mathcal{L}^{k+1}(BP_f^{\zeta^-}) = 0. \tag{5.4.6}$$

Set $E = B_{k+1}(0,1)\backslash[N_0^{\zeta^+} \cup N_0^{\zeta^-} \cup \partial B_{k+1}(0,1)]$. One can easily see that E is an open set in $\mathbb{R}^{k+1}$. From (5.4.6), the conclusion of (ii) of Theorem 5.2.2 follows from the equality

$$\mathcal{L}^{k+1}(E \cap BP_f) \;=\; 0. \tag{5.4.7}$$

Recalling Lemma 5.3.4, $BP_f \cap \partial B_{k+1}(0,1)$ is closed. Thus $E \cap BP_f$ is a Borel set. Therefore, by the Lebesgue theorem, (5.4.7) will follow if for every $x \in E \cap BP_f$, the $\mathcal{L}^{k+1}$-density $D^{k+1}_{E\cap BP_f}(x)$ at x has zero value.
We divide the proof into several steps:

The first step is pivotal (see the below inequality (5.4.8)) to show that the restricted functions (defined before Lemma 5.3.6) which are restricted from the function $f_{|B_{k+1}(x,r_x)}$ where $x \in E$, have the hypograph satisfying the $\rho_x -$ *external sphere condition.*
Step1: Let $x \in E$. Since E is open, there exists $r_x > 0$ such that $\overline{B_{k+1}}(x, r_x)$ $\subset E$. By the external sphere assumption on f, for each $y \in B_{k+1}(x, r_x)$, there exists $0 \neq (\xi_1^y, \xi_2^y, \lambda^y) \in N^P_{\text{hypo}(f)}(y, f(y))$ realized by a ball of radius ρ where $\xi_1^y \in \mathbb{R}^k$ and $\xi_2^y, \lambda^y \in \mathbb{R}$. We claim that there exists a constant $\alpha_x > 0$ such that

$$\frac{\|(\xi_1^y, \lambda^y)\|}{\|(\xi_1^y, \xi_2^y, \lambda^y)\|} \;\geq\; \alpha_x > 0 \qquad \text{for all } y \in B_{k+1}(x, r_x). \tag{5.4.8}$$

Assume by contradiction that there exists a sequence $\{y_n\} \subseteq B_{k+1}(x, r_x)$ such that

$$\lim_{n\to\infty} \frac{\|(\xi_1^{y_n}, \lambda^{y_n})\|}{\|(\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})\|} = 0. \tag{5.4.9}$$

Assume without loss of generality that $\lim_{n\to\infty} y_n = \bar{y} \in \overline{B_{k+1}}(x, r_x)$ and $\lim_{n\to\infty} \frac{(\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})}{\|(\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})\|} = (\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda})$. From (5.4.9), one can see that

$$(\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda}) \in \; \{(0,1,0),\; (0,-1,0)\} = \{(\zeta^+, 0),\; (\zeta^-, 0)\}. \tag{5.4.10}$$

Moreover, $(\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda})$ is a proximal normal vector to hypo(f) at $(\bar{y}, f(\bar{y}))$ realized by a ball of radius ρ. Indeed, since $0 \neq (\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})$ and $(\xi_1^{y_n}, \xi_2^{y_n}, \lambda^{y_n})$ $\in N^P_{\text{hypo}(f)}(y_n, f(y_n))$ is realized by a ball of radius ρ, we have

$$\langle \frac{(\xi_1^n, \xi_2^n, \lambda^n)}{\|(\xi_1^n, \xi_2^n, \lambda^n)\|} \;,\; (z, \beta) - (y_n, f(y_n))\rangle \;\leq\; \frac{1}{2\rho}(\|z - y_n\|^2 + |\beta - f(y_n)|^2)$$

for all $z \in B_{k+1}(0,1)$ and for all $\beta \leq f(z)$.
By taking $n \to \infty$, we obtain that

$$\langle (\bar{\xi}_1, \bar{\xi}_2, \bar{\lambda}) \ , \ (z,\beta) - (\bar{y}, f(\bar{y})) \rangle \ \leq \ \frac{1}{2\rho}(\|z-\bar{y}\|^2 + |\beta - f(\bar{y})|^2)$$

for all $z \in B_{k+1}(0,1)$ and for all $\beta \leq f(z)$.
Therefore, by (5.4.10), we get $\bar{y} \in N_0^{\zeta^+} \cup N_0^{\zeta^-}$. This is a contradiction because $\bar{y} \in \overline{B_{k+1}}(x,r_x) \subset E = B_{k+1}(0,1) \backslash [N_0^{\zeta^+} \cup N_0^{\zeta^-} \cup \partial B_{k+1}(0,1)]$.

The second step allows us to make a connection between the set of *bad points* of f and the set of *bad points* of restricted functions of f.
Step 2: Let $x \in E \cap BP_f$. We claim that there exists a line $\{t\xi_x \mid t \in \mathbb{R}\} \subseteq N^P_{\text{hypo}(f)}(x)$ such that $\{t\xi_x \mid t \in \mathbb{R}\} \neq \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$.
Assume by contraction, since $x \in BP_f$, i.e. $NL(x) \neq 0$, we have $NL(x) = \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$. Recalling Lemma 5.3.3, the set $H_0^+(x) \subseteq NL(x)$ contains at least one line. Therefore $H_0^+(x) = \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$ which implies that $(\zeta^+, 0) \in N_0^L(x)$. Recalling Corollary 5.3.2, $(\zeta^+, 0) \in N^P_{\text{hypo}(f)}(x, f(x))$ is realized by a ball of radius ρ. Thus $x \in N_0^{\zeta^+}$ and this is a contradiction because $x \in E$.

In the next step, we are going to prove that $\mathcal{L}^{k+1}(B_{k+1}(x,r_x) \cap BP_f) = 0$ by our inductive assumption .
Step 3: Let $\bar{f} = f_{|B_{k+1}(x,r_x)} : B_{k+1}(x,r_x) \longrightarrow \mathbb{R}$ be the restricted function of f on $B_{k+1}(x,r_x)$. From Lemma 7.2.2, the continuous function $\bar{f}$ has hypo$(\bar{f})$ satisfying the $\rho-$*external sphere condition*, and

$$BP_f \cap B_{k+1}(x,r_x) = BP_{\bar{f}}. \tag{5.4.11}$$

Moreover, two properties which we claimed in Step 1 and Step 2 still hold for the function $\bar{f}$.

Since (5.4.11) holds, we only need to prove $\mathcal{L}^{k+1}(BP_{\bar{f}}) = 0$.

In order to make the proof more clear, we restate our above problem by replacing $x = 0$, $r_x = 1$ and $\bar{f} = f$. The statement is that

Let $f : B_{k+1}(0,1) \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(f) satisfies $\rho-$ *external*
sphere condition. Moreover,
i) For all $y \in B_{k+1}(0,1)$, there exists a non zero vector $(\xi_1^y, \xi_2^y, \lambda^y) \in N^P_{\text{hypo}(f)}(y, f(y))$ realized by a ball of radius ρ such that

$$\frac{\|(\xi_1^y, \lambda^y)\|}{\|(\xi_1^y, \xi_2^y, \lambda^y)\|} \ \geq \ \alpha_0 > 0. \tag{5.4.12}$$

ii) For all $x \in BP_f$, there exists a line $\{t\xi_x \mid t \in \mathbb{R}\} \subseteq NL(x)$ such that $\{t\xi_x \mid t \in \mathbb{R}\} \neq \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$.
Then $\mathcal{L}^{k+1}(BP_f) = 0$.
Proof. Since $k \geq 1$, for every $x \in \mathbb{R}^{k+1}$, we write $x = (x_1, x_2)$ where

$x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}$. For each $x_2 \in (-1,1)$, the restricted function $f_{x_2} : B_k(0,\sqrt{1-x_2^2}) \longrightarrow \mathbb{R}$ is denoted by $f_{x_2}(x_1) = f(x_1,x_2)$ for all $x_1 \in B_k(0,\sqrt{1-x_2^2})$.

First, we claim that hypo(f_{x_2}) satisfies $\rho\alpha_0$-*external sphere condition.* Indeed by assumption (i) of the above statement we have that , for each $x_1 \in B_k(0,\sqrt{1-x_2^2})$, or $(x_1,x_2) \in B_{k+1}(0,1)$, there exists a vector

$$0 \neq (\xi_1^{(x_1,x_2)}, \xi_2^{(x_1,x_2)}, \lambda^{(x_1,x_2)}) \in N^P_{\text{hypo}(f)}((x_1,x_2), f(x_1,x_2))$$

realized by a ball of radius ρ such that

$$\frac{\|(\xi_1^{(x_1,x_2)}, \lambda^{(x_1,x_2)})\|}{\|(\xi_1^{(x_1,x_2)}, \xi_2^{(x_1,x_2)}, \lambda^{(x_1,x_2)})\|} \geq \alpha_0 > 0. \tag{5.4.13}$$

Recalling Lemma 5.3.6 for $N = k+1 \geq 2$ and observing that $(\xi_1, \xi_2, \lambda) = (\xi_1^{(x_1,x_2)}, \xi_2^{(x_1,x_2)}, \lambda^{(x_1,x_2)})$, and by (5.4.13) we obtain that $(\xi_1^{(x_1,x_2)}, \lambda^{(x_1,x_2)})$ is also a proximal normal vector to hypo(f_{x_2}) at $(x_1, f_{x_2}(x_1))$ realized by a ball of radius $\rho\alpha_0$.

Second, we claim that

$$\mathcal{L}^k(BP_{f_{x_2}}) = 0 \quad \text{ for all } x_2 \in (-1,1). \tag{5.4.14}$$

Indeed, set $\gamma(x_2) = \frac{1}{\sqrt{1-x_2^2}}$ and let $h_{x_2} = f_{x_2}^{\gamma(x_2)}$ be the $\gamma(x_2)$-*stretched function* of f_{x_2} (see Lemma 7.2.3). By Lemma 7.2.3 and by the first step, the continuous function $h_{x_2} : B_k(0,1) \longrightarrow \mathbb{R}$ has hypograph satisfying the ρ_1-external sphere condition where $\rho_1 = \rho\alpha_0 \frac{(1-x_2^2)^{\frac{1}{2}}}{(2-x_2^2)^{\frac{3}{2}}}$. Therefore, by the inductive assumption, we have

$$\mathcal{L}^k(BP_{h_{x_2}}) = 0. \tag{5.4.15}$$

Moreover, recalling Corollary 7.2.1 for $g = f_{x_2}$ and $\gamma = \gamma(x_2)$ we get

$$BP_{h_{x_2}} = (1-x_2^2)^{-\frac{1}{2}} BP_{f_{x_2}}. \tag{5.4.16}$$

Combining (5.4.15) and (5.4.16), we get (5.4.14).

Thirdly, we claim that

$$BP_f \subseteq \bigcup_{x_2 \in (-1,1)} BP_{f_{x_2}} \times \{x_2\}. \tag{5.4.17}$$

Assume $x = (x_1,x_2) \in BP_f$. By (ii) there exists a line $\{t\xi_x \mid t \in \mathbb{R}\} \subseteq NL(x) \subseteq (-\partial^\infty f(x), 0)$ such that $\{t\xi_x \mid t \in \mathbb{R}\} \neq \{t(\zeta^+, 0) \mid t \in \mathbb{R}\}$ and $\|\xi_x\| = 1$. Therefore, $\xi_x = (\xi_1, \xi_2, 0)$ and $-\xi_x = (-\xi_1, -\xi_2, 0)$ are proximal normal vectors to hypo(f) at $(x, f(x))$ realized by a ball of radius σ where $\sigma > 0$, $0 \neq \xi_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}$ and $\|(\xi_1, \xi_2)\| = 1$. Recalling Lemma 5.3.6, we obtain that $(\xi_1, 0)$ and $(-\xi_1, 0)$ are proximal normal vectors to the hypograph of f_{x_2} at $(x_1, f_{x_2}(x_1)$. This implies that $N^P_{\text{hypo} f_{x_2}}(x_1, f_{x_2}(x_1))$ contains the line $\{t(\xi_1, 0) \mid t \in \mathbb{R}\}$. Thus, $x_1 \in BP_{f_{x_2}}$ or $(x_1, x_2) \in (BP_{f_{x_1}}, x_2)$.

Finally, since BP_f is a Borel set contained in $B_{k+1}(0,1)$, the indicator function $\mathbf{1}_{BP_f}$ is in $\mathbb{L}^{k+1}(B_{k+1}(0,1))$. From Fubini's theorem, we have

$$\mathcal{L}^{k+1}(BP_f) = \int_{B_{k+1}(0,1)} \mathbf{1}_{BP_f} dx = \int_{-1}^{1} \int_{B_k(0,\sqrt{1-x_2^2})} \mathbf{1}_{BP_f} dx_1 dx_2. \tag{5.4.18}$$

Combining the above equality and (5.4.17), we get

$$\mathcal{L}^{k+1}(BP_f) \leq \int_{-1}^{1} \int_{B_k(0,\sqrt{1-x_2^2})} \mathbf{1}_{BP_{f_{x_2}}} dx_1 dx_2 = \int_{-1}^{1} \mathcal{L}^k(BP_{f_{x_2}}) dx_1. \tag{5.4.19}$$

The proof is completed using (5.4.19) and (5.4.14). □

5.4.2 Proof of Theorem 5.2.1

Proof of (i). It is equivalent to prove that $BP_f \cup \partial\Omega \subset \overline{\Omega}$ is closed. Let $\{x_n\} \subseteq BP_f \cup \partial\Omega$ converge to x. We need to show that $x \in BP_f \cup \partial\Omega$.
If $x \in \partial\Omega$, there is nothing to prove.
If $x \in \Omega$, we will prove $x \in BP_f$. Indeed, there exist $r_x > 0$ and $M > 0$ such that $x_n \in B_N(x, r_x) \subset \overline{B}_N(x, r_x) \subset \Omega$ for all $n > M$. From Lemma 7.2.2, we have $x_n \in BP_{f_{|B_N(x,r_x)}}$ for all $n > M$. On the other hand, from Corollary 7.2.1, and (i) of Theorem 5.2.2, one can easily see that the set $BP_{f_{|B_N(x,r_x)}} \cup \partial B_N(x, r_x)$ is closed. Therefore, the sequence $\{x_n\}$ converge to $x \in BP_{f_{|B_N(x,r_x)}} \cup \partial B_N(x, r_x)$. Recalling again Lemma 7.2.2, we obtain $x \in BP_f$.
Proof of (ii). Since $BP_f \cup \partial\Omega$ is closed, BP_f is a Borel set. Therefore, it is sufficient to prove that for all $x \in BP_f$, the $\mathcal{L}^N$-density of BP_f at x has zero value, i.e, for all $x \in BP_f$

$$D^N_{BP_f}(x) = \lim_{\delta \to 0} \frac{\mathcal{L}^N(BP_f \cap B_N(x, \delta))}{\mathcal{L}^N(B_N(x, \delta))} = 0. \tag{5.4.20}$$

Indeed, for all $x \in BP_f \subseteq \Omega$, there exists $r_x > 0$ such that $\overline{B}_N(x, r_x) \subset \Omega$. From Lemma 7.2.2 , Lemma 7.2.3, Corollary 7.2.1 and Theorem 5.2.2, one can easily get

$$\mathcal{L}^N(BP_f \cap B(x, r_x)) = \mathcal{L}^N(BP_{f_{|B_N(x,r_x)}}) = 0, \tag{5.4.21}$$

and (5.4.20) follows. □

5.4.3 Proof of Colloraries

Proof of Corollary 5.2.1. From Theorem 5.2.1 we have
The set Ω_P is open. The function $f_{|\Omega_P} :\longrightarrow \mathbb{R}$ is a continuous function and
i) The set $\text{hypo}(f_{|\Omega_P})$ satisfies the $\theta -$ *external sphere condition.*
ii) For every $x \in \Omega_P$, the set $N^P_{\text{hypo}(f_{|\Omega_P})}(x, f_{|\Omega_P}(x))$ is pointed.
The remainder of the proof is done by the argument in [20]. More precisely, one can prove that $N^P_{\text{hypo}(f_{|\Omega_P})}(x, f_{|\Omega_P}(x)) = \text{Co}\{tN^L(x) \mid t \geq 0\}$ (see Lemma 4.7, Theorem 4.1, Theorem 3.1, Theorem 3.2 in [20]). From Corollary 5.3.1 in this paper, if $\xi \in N^L(x)$ then $\xi \in N^P_{\text{hypo}(f_{|\Omega_P})}(x, f_{|\Omega_P}(x))$ is realized by a ball of radius $\theta(x)$, the proof is completed by following the proof of Theorem 3.3 in [20]. □

Proof of Corollary 5.2.3. Using the Proposition (3.1) in [20], the $\text{hypo}(T)$ satisfies the $\theta -$ *external sphere condition.* Applying Corollary 5.2.1 for $f = T(\cdot)$, we get the conclusion. □

Chapter 6

Rectifiability of the set of bad points

We prove here some rectifiability properties of the set of bad points BP_f of f where the hypograph of f doesn't require an external sphere condition. The set BP_f will be also defined more generally (see Section 6.1 for the definition). We partition the set BP_f (see (6.1.6)) into sets $BP_{f,k}, k = 1, \ldots, N$, where, roughly speaking, the suffix k corresponds to the dimension of the largest vector space contained in the set $\partial^{F,\infty} f$ of Fréchet horizon supergradients of f (see Section 6.1 for the definition). We are able to prove that $BP_{f,k}$ is countably $(N-k)$-rectifiable.

Theorem 6.0.2 *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be upper semi-continuous. Then the set $BP_{f,k}$ is countably $(N-k)$-rectifiable.*

Moreover, under an external sphere condition on the hypograph of f, the definition of BP_f here will coincide with the one in the previous chapter. Therefore, we also refine Theorem 5.2.2 as follows:

Theorem 6.0.3 *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be continuous. If the hypograph of f satisfies the θ-exterior sphere condition for some $\theta > 0$, then the set of bad points BP_f is locally $(N-1)$-rectifiable. In particular, $\mathcal{H}^{N-1}(BP_f \cap K)$ is finite for any compact set $K \subset \mathbb{R}^N$.*

Finally, in Section 6.3 we provide an example showing that, in general, the set $BP_{f,k}, k \geq 2$ may not have finite $(N-k)$-Hausdorff measure even under the exterior sphere condition.

6.1 Notations

To make the reader easy to follow, we prefer to rewrite shortly some basic notations.

Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be upper semi-continuous. The hypograph of f is denoted by

$$\text{hypo}(f) \;=\; \{(x, \beta) \mid x \in \Omega, \beta \leq f(x)\}. \qquad (6.1.1)$$

The vector $(-v, \lambda) \in \mathbb{R}^N \times \mathbb{R}$ is a Fréchet normal vector to hypo(f) at $(x, f(x))$ iff

$$\limsup_{\text{hypo}(f) \ni (y,\beta) \to (x, f(x))} \left\langle (-v, \lambda)\ ,\ \frac{(y, \beta) - (x, f(x))}{|y - x| + |\beta - f(x)|} \right\rangle \;\leq\; 0. \qquad (6.1.2)$$

We denote by $N^F_{\text{hypo}(f)}(x, f(x))$ the set of Fréchet normal vectors to hypo(f) at $(x, f(x))$.

Remark 6.1.1 If $(-v, \lambda) \in N^F_{\text{hypo}(f)}(x, f(x))$ then $\lambda \geq 0$.

Recalling that $N^P_{\text{hypo}(f)}(x, f(x))$ is the set of proximal normal vectors to hypo(f) at $(x, f(x))$, we have:

Remark 6.1.2 $N^P_{\text{hypo}(f)}(x, f(x)) \subseteq N^F_{\text{hypo}(f)}(x, f(x))$ for all $x \in \Omega$.

Associated with hypo(f), we define that

1. $\partial^F f(x) = \{v \mid (-v, 1) \in N^F_{\text{hypo}(f)}(x, f(x))\}$ is set of Fréchet supergradients of f at x .
2. $\partial^{F,\infty} f(x) \;=\; \{v \mid (-v, 0) \in N^P_{\text{hypo}(f)}(x, f(x))\}$ v is the set a Fréchet horizon supergradients of f at x.

The largest vector subspace contained in $N^F_{hypo(f)}(x, f(x))$ will be denoted by

$$NL(x) = \{\ \xi \in N^F_{hypo(f)}(x, f(x)) \mid -\xi \in N^F_{hypo(f)}(x, f(x))\}. \qquad (6.1.3)$$

From Remark 6.1.1, one can see that $NL(x) \subseteq \{(v, 0) \mid -v \in \partial^\infty f(x)\}$. Let us define

$$V_x := \{v \in \mathbb{R}^N \mid (v, 0) \in NL(x)\}; \qquad (6.1.4)$$

clearly, V_x is the largest vector space contained in $\partial^\infty f(x)$ and $\dim V_x = \dim NL(x)$. We say that $v \in V_x$ is realized by a ball of radius θ if $(v, 0) \in N^P_{\text{hypo}(f)}(x, f(x))$ is realized by a ball of radius θ.

The set of *bad points* BP_f of f is defined by

$$BP_f = \{x \in \Omega \mid NL(x) \neq \{0\}\}. \qquad (6.1.5)$$

According to the dimension of $NL(x)$, for $k = 1, \ldots, N$ we introduce

$$BP_{f,k} = \{x \in BP_f \mid \dim NL(x) = k\} = \{x \in BP_f \mid \dim V_x = k\}. \quad (6.1.6)$$

It is clear that $BP_f = \bigcup_{k=1}^N BP_{f,k}$.

Now, let $k \geq 0$ and $A, B \subset \mathbb{R}^N$ be fixed. We recall that:

(i) $\mathcal{H}_k(A)$ is the *k-dimensional Hausdorff measure* of A;

(ii) $d_{\mathcal{H}}(A, B)$ is the *Hausdorff distance* between A and B.

Finally, we will denote by $G(N, k)$ the Grassmann manifold of all $k-$ dimensional vector subspaces of $\mathbb{R}^N$; we endow $G(N, k)$ with the distance

$$d_G(V_1, V_2) := d_H(V_1 \cap S^{N-1}, V_2 \cap S^{N-1}).$$

The metric space $(G(N, k), d_G)$ is separable and, in particular, the following property holds:

$$\forall\, R > 0\ \exists\, V_1, \ldots, V_m \in G(N, k) \text{ s.t. } G(N, k) \subset \bigcup_{i=1}^{m} B_G(V_i, R) \tag{6.1.7}$$

where $B_G(V_i, R)$ denote the open ball (with respect to d_G) with center V_i and radius R.

6.2 Rectifiability results for the set of bad points

6.2.1 Preparatory Lemmas

Let $V \in G(N, k)$ be fixed; each $z \in \mathbb{R}^N$ can be written in a unique way as $z = z_V + z_{V^\perp}$ where $z_V \in V$ and $z_{V^\perp} \in V^\perp$. For $\alpha \in (0, 1)$ we denote by $C_\alpha(V)$ the open cone along V of aperture $1/\alpha$ defined by

$$C_\alpha(V) := \{z \in \mathbb{R}^N \mid \|z_V\| > \alpha \|z\|\}.$$

If $x \in \mathbb{R}^N$ we set

$$C_\alpha(x, V) := x + C_\alpha(V) = \{z \in \mathbb{R}^N \mid \|(z - x)_V\| > \alpha \|z - x\|\};$$

It is easily seen that

$$z \in C_\alpha(x, V) \iff \exists v \in V \cap S^{N-1} \text{ such that } \langle v, z - x\rangle > \alpha \|z - x\|. \tag{6.2.1}$$

We also point out the following implication:

$$d_G(V_1, V_2) < R \implies C_{\alpha+R}(x, V_1) \subset C_\alpha(x, V_2) \tag{6.2.2}$$

which holds provided $\alpha + R < 1$. To prove (6.2.2) it is enough to notice that for any $z \in C_{\alpha+R}(x, V_1)$

there exists $v_1 \in V_1 \cap S^{N-1}$ such that $\langle v_1, z - x\rangle > (\alpha + R) \|z - x\|$
there exists $v_2 \in V_2 \cap S^{N-1}$ such that $\|v_1 - v_2\| \leq R$

whence

$$\langle v_2, z - x\rangle = \langle v_1, z - x\rangle - \langle v_1 - v_2, z - x\rangle > \alpha \|z - x\|,$$

i.e., $z \in C_\alpha(x, V_2)$.

For any fixed $\rho > 0$, let us introduce the sets

$$BP^{\rho}_{f,k} = \Big\{ x \in BP_{f,k} \mid \Big\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \Big\rangle \le \frac{\|v_x\|}{8} \quad \forall v_x \in V_x, y \in B(x,\rho), \beta < f(y) \Big\}. \tag{6.2.3}$$

Remark 6.2.1 If $\rho_1 > \rho_2 > 0$ then $BP^{\rho_1}_{f,k} \subseteq BP^{\rho_2}_{f,k}$.

As the following Lemma shows, the sets $BP^{\rho}_{f,k}$ give a partition of $BP_{f,k}$.

Lemma 6.2.1 *We have*

$$BP_{f,k} = \cup_{\rho>0} BP^{\rho}_{f,k}. \tag{6.2.4}$$

In particular, from Remark 6.2.1 it holds

$$BP_{f,k} = \cup_{i \in \mathbb{N}\setminus\{0\}} BP^{1/i}_{f,k}. \tag{6.2.5}$$

Proof. Fix $x \in BP_{f,k}$ and let $v_1, v_2, ..., v_k$ be an orthonormal basis for V_x. By the definition of V_x we have $-v_i \in V_x$ for all $i \in \{1, 2, ..., k\}$. Recalling (6.1.4), (6.1.3) and (6.1.2), there exists a constant $\rho_x > 0$ such that $B(x, \rho_x) \subset \Omega$ and for all $i \in \{1, 2, ..., k\}$ one has

$$\Big\langle v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \Big\rangle \le \frac{1}{8\sqrt{k}} \text{ and } \Big\langle -v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \Big\rangle \le \frac{1}{8\sqrt{k}}$$

for all $y \in B(x, \rho_x)$ and $\beta \le f(y)$. Thus

$$\Big| \Big\langle v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \Big\rangle \Big| \le \frac{1}{8\sqrt{k}} \tag{6.2.6}$$

for all $y \in B(x, \rho_x)$ and $\beta \le f(y)$.
Fix $v_x \in V_x$; we have $v_x = \sum_{i=1}^k \alpha_i v_i$ for suitable $\alpha_i \in \mathbb{R}$. From (6.2.6), we get

$$\Big\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \Big\rangle \le \frac{\sum_{i=1}^k |\alpha_i|}{8\sqrt{k}}$$

for all $y \in B(x, \rho_x)$ and $\beta \le f(y)$. On the other hand,

$$\|v_x\| = \left(\sum_{i=1}^k \alpha_i^2 \right)^{1/2} \ge \frac{\sum_{i=1}^k |\alpha_i|}{\sqrt{k}}.$$

Therefore

$$\Big\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \Big\rangle \le \frac{\|v_x\|}{8}$$

for all $y \in B(x, \rho_x)$ and $\beta \le f(y)$. Thus $x \in BP^{\rho_x}_{f,k}$ and the proof is accomplished. □

In view of a rectifiability result for the sets $BP_{f,k}$, we begin with a technical result.

Lemma 6.2.2 *Let $a \in \mathbb{R}^N$, $\rho > 0$ and $x, y \in BP^{\rho}_{f,k} \cap B(a, \frac{\rho}{2})$ be such that $d_G(V_x, V_y) < \frac{1}{8}$; then*

$$y \in \mathbb{R}^N \setminus C_{\frac{1}{4}}(x, V_x).$$

Proof. Since $x, y \in B(a, \frac{\rho}{2})$, we have $x \in B(y, \rho)$ and $y \in B(x, \rho)$. Therefore, from (6.2.3) if $v_x \in V_x \cap \mathcal{S}^{N-1}$ we have

$$\langle v_x, y - x \rangle \;\leq\; \frac{1}{8}(\|y - x\| + |\beta - f(x)|) \quad \text{for all } \beta \leq f(y). \tag{6.2.7}$$

Similarly, for any $v_y \in V_y \cap \mathcal{S}^{N-1}$ we obtain

$$\langle v_y, y - x \rangle \;\leq\; \frac{1}{8}(\|y - x\| + |\beta - f(y)|) \quad \text{for all } \beta \leq f(x). \tag{6.2.8}$$

We have to distinguish two cases: if $f(y) \geq f(x)$, we choose $\beta = f(x)$ in (6.2.7) to get

$$\langle v_x, y - x \rangle \leq \frac{1}{8}\|y - x\| \qquad \forall\, v_x \in V_x \cap S^{N-1}.$$

Recalling (6.2.1), this implies that $y \notin C_{\frac{1}{4}}(x, V_x)$, as desired.
If $f(y) \leq f(x)$, we choose $\beta = f(y)$ in (6.2.8) to get

$$\langle v_y, y - x \rangle \leq \frac{1}{8}\|y - x\| \qquad \forall\, v_y \in V_y \cap S^{N-1}.$$

Since $d_G(V_x, V_y) < \frac{1}{8}$, for any $v_x \in V_x \cap S^{N-1}$ there exists $v_y = v_y(v_x) \in V_y \cap S^{N-1}$ such that $\|v_x - v_y\| < \frac{1}{8}$. Therefore, for any $v_x \in V_x \cap S^{N-1}$ it holds

$$\langle v_x, y - x \rangle \leq \langle v_y, y - x \rangle + |\langle v_x - v_y, y - x \rangle| \leq \frac{1}{4}\|y - x\| \tag{6.2.9}$$

i.e. $y \notin C_{\frac{1}{4}}(x, V_x)$, as desired. □

We now fix $R := 1/16$ and let $V_1, \ldots, V_m \in G(N, k)$ be given by (6.1.7). We thus divide $BP^{\rho}_{f,k}$ into m sets

$$BP^{\rho}_{f,k} = \bigcup_{j=1}^{m} BP^{\rho,j}_{f,k} \tag{6.2.10}$$

where

$$BP^{\rho,j}_{f,k} = \{x \in BP^{\rho}_{f,k} \mid d_G(V_x, V_j) < 1/16\}.$$

For $j = 1, \ldots, m$ we denote by π_j the orthogonal projection $\mathbb{R}^n \to V_j^{\perp}$; clearly, $\pi_j(z) = z_{V_j^{\perp}} = z - z_{V_j}$.

Lemma 6.2.3 *The projection* $\pi_j : BP^{\rho,j}_{f,k} \cap B(a,\rho/2) \to \pi_j(BP^{\rho,j}_{f,k} \cap B(a,\rho/2))$ *is invertible and its inverse map is Lipschitz continuous with Lipschitz constant at most 2.*

Proof. Let $x, y \in BP^{\rho,j}_{f,k} \cap B(a,\rho/2)$ be fixed. We have $d_G(V_x, V_y) < 1/8$ and Lemma 6.2.2 ensures that $y \notin C_{1/4}(x, V_x)$. Since $d_G(V_x, V_j) < 1/16$, by (6.2.2) we deduce that $C_{1/2}(x, V_j) \subseteq C_{5/16}(x, V_j) \subseteq C_{1/4}(x, V_x)$ and, in particular, that $y \notin C_{1/2}(x, V_j)$. This implies that $\|(y-x)_{V_j}\| \leq \frac{1}{2}\|y-x\|$, whence

$$\|\pi_j(y) - \pi_j(x)\| = \|\pi_j(y-x)\| = \left\|(y-x) - (y-x)_{V_j}\right\| \geq \tfrac{1}{2}\|y-x\| \,.$$

This is enough to conclude. □

The rectifiability of the sets $BP^{\rho}_{f,k}$ is now a consequence of Lemma 6.2.3.

6.2.2 Proof of main results

Theorem 6.2.1 *The set* $BP^{\rho}_{f,k} \cap K$ *is* $(N-k)$*-rectifiable for any* $\rho > 0$ *and any compact set* $K \subset \mathbb{R}^N$*; in particular*

$$\mathcal{H}^{N-k}(BP^{\rho}_{f,k} \cap K) < +\infty. \tag{6.2.11}$$

Proof. It will be sufficient to show that for any $j = 1, \ldots, m$ the set $BP^{\rho,j}_{f,k} \cap K$ is k-rectifiable. Since K is compact, there exist $a_1, \ldots, a_h \in \mathbb{R}^N$ such that

$$BP^{\rho,j}_{f,k} \cap K \subset \bigcup_{i=1}^{h} \left(BP^{\rho,j}_{f,k} \cap B(a_i,\rho/2)\right).$$

By Lemma 6.2.3, for any $i = 1, \ldots, h$ the set $BP^{\rho,j}_{f,k} \cap B(a_i,\rho/2)$ is the image of

$$\pi_j^{-1} : \pi_j\left(BP^{\rho,j}_{f,k} \cap B(a_i,\rho/2)\right) \to \mathbb{R}^N,$$

i.e. of a Lipschitz map defined on a bounded subset of $V_j^{\perp} \equiv \mathbb{R}^{N-k}$ with Lipschitz constant at most 2. In particular, $BP^{\rho,j}_{f,k} \cap B(a_i,\rho/2)$ is $(N-k)$-rectifiable and this allows to conclude. □

We can finally pass to the proof of our main results.

Proof of Theorem 6.0.2. It is an easy consequence of Lemma 6.2.1 and Theorem 6.2.1. □

Before passing to the proof of Theorem 6.0.3, we would like to discuss the relation between BP_f and the set of bad points BP^P_f considered in [51], namely,

$$BP^P_f := \{x \in \Omega \ |NL^P(x) \neq \{0\}\},$$

where $NL^P(x) = \{\ \xi \in N^P_{hypo(f)}(x, f(x)) \mid -\xi \in N^P_{hypo(f)}(x, f(x))\}$. From Remark 6.1.2 it is clear that $BP^P_f \subseteq BP_f$, but in general the two sets do not coincide.
However, the equality $BP_f = BP^P_f$ holds under the assumptions of Theorem 6.0.3. Indeed, from Corollary 3.1 in [51] it follows that the hypograph of $f_{|\Omega_P}$ has positive reach, where Ω_P is the open set defined by $\Omega_P := \Omega \backslash BP^P_f$. Therefore (see [34, Proposition 6.2 and 4.2] and [43, Theorem 4.8 (12)]) one has

$$N^P_{\mathrm{hypo}(f_{|\Omega_P})}(x, f_{|\Omega_P}(x)) = N^F_{\mathrm{hypo}(f_{|\Omega_P})}(x, f_{|\Omega_P}(x)) \quad \text{for all } x \in \Omega_P.$$

and thus

$$N^P_{\mathrm{hypo}(f)}(x, f(x)) = N^F_{\mathrm{hypo}(f)}(x, f(x)) \quad \text{for all } x \in \Omega_P.$$

Consequently, $NL(x) = NL^P(x)$ for all $x \in \Omega_P$. By the definition of BP^P_f, we have $NL^P(x) = \{0\}$ for all $x \in \Omega_P$. This implies that $NL(x) = \{0\}$ for all $x \in \Omega_P$, i.e. $BP_f \cap \Omega_P = \varnothing$. Thus, $BP_f \subseteq BP^P_f$, as claimed. □

Proof of Theorem 6.0.3. Recalling 6.0.2, we have $\mathcal{H}^{N-1}(BP_{f,k}) = 0$ for all $k \in \{2, 3..., N\}$. Since

$$BP_f = BP_{f,1} \cup \bigcup_{k=2}^{N} BP_{f,k},$$

the proof will be accomplished after proving that the set $BP_{f,1}$ is locally $(N-1)$-rectifiable. From the definition (6.1.6), for every $x \in BP_{f,1}$ the set

$$V_x = \{tv_x \mid v_x \in \mathbb{R}^N, \|v_x\| = 1 \text{ and } t \in \mathbb{R}\}$$

is a line along v_x. Therefore by [51, Lemma 4.3], $(\pm v_x, 0) \in N^P_{hypo(f)}(x, f(x))$ is realized by a ball of radius θ, i.e.

$$\left\langle \pm v_x, y - x \right\rangle \leq \frac{1}{2\theta}(\|y - x\|^2 + |\beta - f(x)|^2) \quad \forall y \in \Omega, \beta \leq f(x).$$

From the above inequality, reasoning as in the proof of Lemma 6.2.2 one can obtain that the following holds. If $a \in \mathbb{R}^N$, $\rho \in (0, \theta/8]$, $x, y \in BP_{f,1} \cap B(a, \frac{\rho}{2})$ are such that $d_G(V_x, V_y) < \frac{1}{8}$, then

$$y \in \mathbb{R}^N \backslash C_{\frac{1}{4}}(x, V_x)\,.$$

From this fact, the local $(N-1)$-rectifiability of $BP_{f,1}$ follows (up to considering $BP_{f,1}$ instead of $BP^\rho_{f,k}$) as in the proof of Theorem 6.2.1. □

6.3 A counterexample

By virtue of Theorem 6.0.3, the set of bad points BP_f is locally $(N-1)$-rectifiable provided the θ-exterior sphere condition holds. On the contrary, an analogous $(N-k)$-rectifiability result does not hold for $BP_{f,k}$; in other words, Theorem 6.0.2 cannot be refined to show that $\mathcal{H}^{N-k}(BP_{f,k} \cap K) < \infty$ for any compact set $K \subset \mathbb{R}^N$. We are going to provide an example of a continuous function $f : (-1,1) \times (-1,1) \to \mathbb{R}$ satisfying the θ-exterior sphere condition with $\theta = 1$ and such that $\mathcal{H}^0(BP_{f,2} \cap K) = +\infty$ for any neighborhood K of the origin. It will be clear from the construction that what is missing is a uniform control on the radii of exterior balls (recall that, by Theorem 6.2.1, $BP^{\rho}_{f,k}$ is locally $(N-k)$-rectifiable for any $\rho > 0$).

Let $\Omega := (-1,1) \times (-1,1)$; for $n \in \mathbb{N},\ n \geq 0$ let us define $x_n^+, x_n^- \in \overline{\Omega}$ by

$$x_n^+ := (2^{-n}, 0), \qquad x_n^- := (-2^{-n}, 0)\,.$$

We also set

$$c_n^+ := \frac{x_n^+ + x_{n+1}^+}{2} = (3 2^{-n-2}, 0) \in \Omega\,,\ c_n^- := \frac{x_n^- + x_{n+1}^-}{2} = (-3 2^{-n-2}, 0) \in \Omega$$

and

$$r_n := \frac{\|x_n^+ - x_{n+1}\|}{2} = \frac{\left\|x_n^- - x_{n+1}^-\right\|}{2} = 2^{-n-2}\,.$$

Notice that the closed balls $\overline{B(c_n^{\pm}, r_n)}$ are pairwise disjoint except for the case of consecutive balls, which instead are tangent, i.e., for any $n \geq 1$ one has

$$\overline{B(c_n^+, r_n)} \cap \overline{B(c_{n-1}^+, r_{n-1})} = \{x_n^+\}\ ,\ \overline{B(c_n^-, r_n)} \cap \overline{B(c_{n-1}^-, r_{n-1})} = \{x_n^-\}\,.$$

Define $f_1 : \Omega \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} -\sqrt{r_n^2 - \left\|x - c_n^+\right\|^2} & \text{if } x \in B(c_n^+, r_n) \\ -\sqrt{r_n^2 - \left\|x - c_n^-\right\|^2} & \text{if } x \in B(c_n^-, r_n) \\ 0 & \text{if } x \in \Omega \setminus \left(\bigcup_n B(c_n^+, r_n) \cup \bigcup_n B(c_n^-, r_n)\right). \end{cases}$$

It is easily seen that f_1 is continuous and that $\{x_n^+, x_n^- : n \geq 1\} \subset BP_{f_1}$; more precisely

$$\begin{array}{l} (1,0) \in \partial^\infty f_1(x_n^+) \text{ is realized by a ball of radius } r_{n-1} \\ (-1,0) \in \partial^\infty f_1(x_n^+) \text{ is realized by a ball of radius } r_n \\ (1,0) \in \partial^\infty f_1(x_n^-) \text{ is realized by a ball of radius } r_n \\ (-1,0) \in \partial^\infty f_1(x_n^-) \text{ is realized by a ball of radius } r_{n-1}. \end{array} \tag{6.3.1}$$

For any $x = (\xi, \eta) \in \Omega$ we also define

$$f_2(x) = -\sqrt{\eta^2 - |\eta|} = -\sqrt{1 - (1 - |\eta|)^2}\,.$$

One can easily check that f_2 is continuous on Ω and that $BP_{f_2} = \{(\xi, 0) : \xi \in (-1,1)\}$; more precisely, for any $\xi \in (-1,1)$

$$(0,1), (-1,0) \in \partial^\infty f_2(\xi, 0) \text{ are realized by balls of radius 1.} \tag{6.3.2}$$

Notice also that $f_1(x_n^\pm) = f_2(x_n^\pm) = 0$ for any $n \geq 1$. Therefore, the function $f := \inf\{f_1, f_2\}$ is continuous on Ω and $f(x_n^\pm) = f_1(x_n^\pm) = f_2(x_n^\pm) = 0$. Taking (6.3.1) and (6.3.2) into account we obtain that

$$(1,0), (-1,0), (0,1), (0,-1) \in \partial^\infty f(x_n^\pm) \qquad \text{for any } n \geq 1$$

whence

$$\{x_n^+, x_n^- : n \geq 1\} \subset BP_{f,2}$$

which in turn implies $\mathcal{H}^0(BP_{f,2}) = \infty$, as desired. □

Chapter 7

Appendix

7.1 Appendix A

In this section, under the assumptions (H_1) and (H_2) on (3.1.1) in Chapter 3, we prove some elementary estimates which are needed in Lemma 3.3.1, Lemma 3.3.2 and Lemma 3.3.3. For future use, we set

$$\begin{aligned} K_1 &= \max_{u\in\mathcal{U}} \|f(0,u)\|, \\ K_2 &= \max_{u\in\mathcal{U}} \|D_x f(0,u)\|, \\ L_2(s,\delta) &= L_1 e^{Ls}\delta + \frac{L_1(e^{Ls}-1)K_1}{L} + K_2 \quad \text{for all} \quad s,\delta \geq 0. \end{aligned} \tag{7.1.1}$$

Lemma 7.1.1 *Let $\alpha(\cdot) := y^{x,u}(\cdot)$ be the solution of (3.1.1). The following estimates hold true for all $t > 0$:*

(i) $\|\alpha(t) - x\| \leq \frac{(L\|x\|+K_1)(e^{Lt}-1)}{L}$.

(ii) $\|\alpha(t)\| \leq e^{Lt}\|x\| + \frac{(e^{Lt}-1)K_1}{L}$.

(iii) $\|f(\alpha(t),u(t))\| \leq Le^{Lt}\|x\| + e^{Lt}K_1$.

(iv) $\|D_x f(\alpha(t),u(t))\| \leq L_2(t,\|x\|)$.

Proof. Since $\alpha(\cdot)$ is the solution of (3.1.1), for all $t > 0$ we have

$$\begin{aligned} \|\alpha(t) - x\| &= \left\| \int_0^t f(\alpha(s),u(s))ds \right\| \leq \int_0^t \|f(\alpha(s),u(s))\| \, ds \\ &\leq \int_0^t \|f(\alpha(s),u(s)) - f(x,u(s))\| \, ds \\ &\qquad + \int_0^t \|f(x,u(s)) - f(0,u(s))\| \, ds + \int_0^t \|f(0,u(s))\| \, ds \\ &\leq L\int_0^t \|\alpha(s) - x\| \, ds + L\|x\| t + K_1 t. \end{aligned}$$

Applying Gronwall's inequality we obtain

$$\|\alpha(t) - x\| \leq \frac{(L\|x\| + K_1)(e^{Lt} - 1)}{L}, \tag{7.1.2}$$

whence

$$\|\alpha(t)\| \leq e^{Lt}\|x\| + \frac{(e^{Lt} - 1)K_1}{L}. \tag{7.1.3}$$

Recalling the condition (H_2), we obtain

$$\|f(\alpha(t), u(t))\| \leq Le^{Lt}\|x\| + e^{Lt}K_1 \tag{7.1.4}$$

and also

$$\|D_x f(\alpha(t), u(t))\| \leq L_1 e^{Lt}\|x\| + \frac{L_1(e^{Lt} - 1)K_1}{L} + K_2. \tag{7.1.5}$$

The proof is concluded. □

In the next Lemma, we will give some estimates related to the limiting adjoint trajectories $M^T(\cdot)$ in Chapter 3.

Lemma 7.1.2 *Let $x \in \mathcal{S}^c$, set $r = T(x) > 0$, and take $\bar{x} \in \mathcal{S}_x$ and $M(\cdot) \in \mathcal{M}_{\bar{x}}$. Then*

(i) $\|M(t)\| \leq e^{L_2(t,\|x\|)t}$ *for all* $t \in [0, r]$,

(ii) $\left\|M(t)^{-1}\right\| \leq e^{L_2(t,\|x\|)t}$ *for all* $t \in [0, r]$.

Proof. Let $x_n \to x$, $\{\bar{u}_n\} \subset \mathcal{U}_{\rm ad}$ be such that $\{y^{x_n,\bar{u}_n}(\cdot)\} \subset \mathcal{T}_{\bar{x}}$ and $M(\cdot, x_n, \bar{u}_n)$ converges to $M(\cdot)$ uniformly on $[0, T(x)]$. By (iv) in Lemma 7.1.1 and Theorem 2.2.1, p. 23, in [14], we obtain that for all $w \in \mathbb{R}^N$

$$\|M(t, x_n, \bar{u}_n)w\| \leq e^{[L_1 e^{Lt}\|x\| + \frac{L_1(e^{Lt}-1)K_1}{L} + K_2]t}\ \|w\|.$$

Taking $n \to \infty$ we conclude the proof of (i).

The proof of (ii) proceeds exactly as the proof of (i), by replacing $M(\cdot, x_n, \bar{u}_n)$ with $M(\cdot, x_n, \bar{u}_n)^{-1}$. □

The following result is essentially Theorem 2.2.4, pp. 25, 26 in [14].

Lemma 7.1.3 *Let $A_1, A_2 : [0, T] \to \mathcal{M}^{N\times N}$ be matrices with L^∞-entries, and set $\|A_i\| = L_i$, $i = 1, 2$. Let M_1, M_2 be the fundamental solution of, respectively,*

$$\begin{aligned} \dot{p}(t) &= A_1(t)p(t), \quad p(0) = \mathbb{I}^{N\times N} \\ \dot{p}(t) &= A_2(t)p(t), \quad p(0) = \mathbb{I}^{N\times N}. \end{aligned}$$

Then, for every $t \in [0, T]$ and every unit vector $v \in \mathbb{R}^N$ we have

$$\|(M_2(t) - M_1(t))v\| \leq e^{(L_1+L_2)t}\int_0^t \|A_2(s) - A_1(s)\|\, ds.$$

7.2 Appendix B

The first Lemma is to prove that the angle of a wedged cone H_C^+ which is generated by a compact set $C \subset \mathbb{R}^N \backslash \{0\}$ is far from π.

Lemma 7.2.1 *Let $C \in \mathbb{R}^N$ be a compact set which does not contain 0. We denote the positive cone generated by C as*

$$H_C^+ = \text{span}^+(C) = \{\sum_{i=1}^{k} \alpha_i c_i \mid c_i \in C \text{ and } \alpha_\text{i} \geq 0\}.$$

Assume that H_C^+ is wedged. Then:
i) H_C^+ is closed.
ii) There exists a constant $\delta_0 > 0$ such that for all $0 \neq x_1, x_2 \in H_C^+$, it holds

$$\langle \frac{x_1}{\|x_1\|} \ , \ \frac{x_2}{\|x_2\|} \rangle > -1 + \delta_0. \tag{7.2.1}$$

Proof of (i). Let a sequence $\{x_n\} \subset H_C^+$ converge to x. We need to prove that $x \in H_C^+$. By Caratheodory theorem, we can write

$$x_n = \sum_{i=1}^{N+1} \alpha_i^n c_n^i, \text{ where } \alpha_n^i \geq 0, \ c_n^i \in C. \tag{7.2.2}$$

Assume without loss of generality that $\lim_{n\to\infty} c_n^i = \overline{c^i} \in C$ for all $i \in \{1, 2, .., N+1\}$.

If $\sum_{i=1}^{N+1} \alpha_n^i$ is unbounded, we extract subsequences $\{\alpha_{n_k}^i\} \subseteq \{\alpha_n^i\}$ such that

$$\frac{\alpha_{n_k}^i}{\sum_{i=1}^{N+1} \alpha_{n_k}^i} = \overline{\alpha^i} \geq 0 \text{ and } \lim_{n_k\to\infty} \sum_{i=1}^{N+1} \alpha_{n_k}^i = +\infty.$$

Therefore, from (7.2.2) and $\lim_{n\to\infty} x^n = x$ we get

$$\sum_{i=1}^{N+1} \overline{\alpha^i} \ \overline{c^i} = \lim_{n_k\to\infty} \frac{x_{n_k}}{\sum_{i=1}^{N+1} \alpha_{n_k}^i} = 0. \tag{7.2.3}$$

Note that $\overline{\alpha^i} \geq 0$, $\sum_{i=1}^{N+1} \overline{\alpha^i} = 1$ and $\overline{c^i} \neq 0$. We recall (7.2.3) to obtain that the cone H_C^+ contains at least one line. This is a contradiction.
Thus $\sum_{i=1}^{N+1} \alpha_n^i$ is bounded. It implies that the sequences $\{\alpha_n^i\}$ are bounded for all $i \in \{1, 2, ..., N+1\}$ since $\alpha_n^i \geq 0$. We extract subsequences $\{\alpha_{n_k}^i\} \subseteq \{\alpha_n^i\}$ such that

$$\lim_{n_k\to\infty} \alpha_{n_k}^i = \overline{\alpha^i} \geq 0 \text{ for all } i \in \{1, 2, ..., N+1\}.$$

From the above equality and (7.2.2) , we have

$$x = \lim_{n\to\infty} x_n = \lim_{n_k\to\infty} x_{n_k} = \lim_{n_k\to\infty} \sum_{i=1}^{N+1} \alpha^i_{n_k} c^i_{n_k} = \sum_{i=1}^{N+1} \overline{\alpha^i}\ \overline{c^i}.$$

This implies $x \in H_C^+$.

Proof of (ii). Assume by contradiction that there exist two sequences $\{x_1^n\}$, $\{x_2^n\}$ contained in H_C^+ such that $\|x_1^n\| = \|x_2^n\| = 1$ and

$$\lim_{n\to\infty} \langle x_1^n\ ,\ x_2^n\rangle\ = -1. \tag{7.2.4}$$

Assume without loss of generality that $\lim_{n\to\infty} x_1^n = \overline{x_1}$ and $\lim_{n\to\infty} x_1^n = \overline{x_2}$. Recalling (7.2.4), we obtain that $-\overline{x_1} = \overline{x_2}$. Moreover, since H_C^+ is closed, we have $x_1, x_2 \in H_C^+$. Therefore H_C^+ contains at least one line. This is a contradiction. □

The following Proposition provides a sufficient condition for the strict convexity of a set.

Proposition 7.2.1 *Let $K \subset \mathbb{R}^N$ be compact and assume that there exist $\gamma > 0$ and $p > 1$ with the following property: for every $x \in \partial K$, there exists $\zeta \neq 0$ such that for every $y \in K$ one has*

$$\langle \zeta, y - x\rangle \leq -\gamma \|\zeta\| \|y - x\|^p. \tag{7.2.5}$$

Then K is convex (with nonempty interior) and (7.2.5) is satisfied by all $\zeta \in N_K(x)$ for each $x \in \partial K$.

Proof. We show first that K is convex. To this aim, assume by contradiction that three exist points $x_1 \neq x_2 \in K$ such that the segment $[x_1, x_2]$ is not contained in K. Let $0 < t < 1$ be such that $x_t = (1-t)x_1 + tx_2 \in \partial K$ and let $\zeta \neq 0$ be such that (7.2.5) holds with x_t in place of x. Obviously, $\zeta \perp x_2 - x_1$ and this is a contradiction. It is also easy to see that K must have nonempty interior. Since K is convex with nonempty interior, for each $x \in \partial K$ the normal cone $N_K(x)$ is pointed and so it is the convex hull of its exposed rays (see [59]).

We now see Theorem 4.6 in [30] and see that for every unit vector w belonging to our exposed ray of $N_K(x)$, there exists a sequence $x_n \to x$ such that

$$N_K(x_n) = \mathbb{R}^+ w_n, \|w_n\| = 1.$$

Of course (7.2.5) holds with x_n (resp, w_n) in place of x (resp, ζ), so that by passing to the limit, w also satisfies (7.2.5). By taking convex combinations, we conclude the proof. □

The second lemma is necessary to use Theorem 5.2.2 in the proof of the main theorem in Chapter 4.

Lemma 7.2.2 *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $g : \Omega \longrightarrow \mathbb{R}$ be continuous. Assume that hypo(g) satisfies the $\theta-$external sphere condition where $\theta : \Omega \longrightarrow [0, +\infty)$ is continuous. Let, for all $x \in \Omega$, $r_x > 0$ be such that $\overline{B}_N(x, r_x) \subset \Omega$. Then*
i) The hypograph of the restricted function $g_{|B_N(x,r_x)} : B_N(x, r_x) \to \mathbb{R}$ satisfies the θ_x-external sphere condition with $\theta_x = \max\{\theta(y) \mid y \in \bar{B}_N(x, r_x)\}$.
ii) $BP_g \cap B_N(x, r_x) = BP_{g_{|B_N(x,r_x)}}$.

Proof of (i). Let $z \in B_N(x, r_x)$, there exists a vector $0 \neq \xi \in N^P_{\text{hypo}(g)}(z, g(z))$ realized by a ball of radius $\theta(z)$, i.e, for all $y \in \Omega$ and for $\beta \leq g(y)$, it holds

$$\langle \frac{\xi}{\|\xi\|} \, , \, (y, \beta) - (z, g(z)) \rangle \;\leq\; \theta(z) \; (\|y - z\|^2 + |\beta - g(z)|^2). \qquad (7.2.6)$$

Thus, for all $y \in B_N(x, r_x)$ and for all $\beta \leq g_{|B_N(x,r_x)}(y)$, we have

$$\langle \frac{\xi}{\|\xi\|} \, , \, (y, \beta) - (z, g_{|B_N(x,r_x)}(z)) \rangle \;\leq\; \theta_x \; (\|y - z\|^2 + |\beta - g_{|B_N(x,r_x)}(z)|^2). \qquad (7.2.7)$$

The proof is completed. □

Proof of (ii). It is similar to the previous proof. Indeed, if $0 \neq \xi \in N^P_{\text{hypo}(g)}(z, g(z))$ then $0 \neq \xi \in N^P_{\text{hypo}(g_{|B_N(x,r_x)})}(z, g_{|B_N(x,r_x)}(z))$. Therefore, $BP_g \cap B_N(x, r_x) \subseteq BP_{g_{|B_N(x,r_x)}}$.

We are going now to prove $BP_{g_{|B_N(x,r_x)}} \subseteq BP_g$. It is sufficient to prove that if $0 \neq \xi \in N^P_{\text{hypo}(g_{|B_N(x,r_x)})}(z, g_{|B_N(x,r_x)}(z))$ then $0 \neq \xi \in N^P_{\text{hypo}(g)}(z, g(z))$. Indeed, $0 \neq \xi \in N^P_{\text{hypo}(g_{|B_N(x,r_x)})}(z, g_{|B_N(x,r_x)}(z))$, i.e, there exists a constant $\sigma > 0$ such that for all $y \in B_N(x, r_x)$ and for all $\beta \leq g_{|B_N(x,r_x)}(y)$, it holds

$$\langle \frac{\xi}{\|\xi\|} \, , \, (y, \beta) - (z, g_{|B_N(x,r_x)}(z)) \rangle \;\leq\; \sigma \; (\|y - z\|^2 + |\beta - g_{|B_N(x,r_x)}(z)|^2). \qquad (7.2.8)$$

Therefore, for all $y \in B_N(x, r_x)$ and for all $\beta \leq g(y)$, one has

$$\langle \frac{\xi}{\|\xi\|} \, , \, (y, \beta) - (z, g_(z)) \rangle \;\leq\; \sigma \; (\|y - z\|^2 + |\beta - g(z)|^2). \qquad (7.2.9)$$

Since $z \in B_N(x, r_x)$, one can easily get from (7.2.9) that there exists a constant $\sigma_1 > 0$ such that the inequality

$$\langle \frac{\xi}{\|\xi\|} \, , \, (y, \beta) - (z, g_(z)) \rangle \;\leq\; \sigma_1 \; (\|y - z\|^2 + |\beta - g(z)|^2)$$

holds for all $y \in \Omega$ and for all $\beta \leq g(y)$.
It means that $\xi \in N^P_{\text{hypo}(g)}(z, g(z))$. The proof is completed. □

The last one is a technical lemma which is used in Chapter 4.

Lemma 7.2.3 *Let $g : \Omega \longrightarrow \mathbb{R}$ be continuous and let $\gamma > 0$. We denote by $g^\gamma : \gamma\Omega \longrightarrow \mathbb{R}$, the $\gamma-$stretched function of g, as follows:*

$$g^\gamma(y) = g(\frac{y}{\gamma}) \quad \text{for all } y \in \gamma\Omega.$$

Assume that (ξ, λ) is a proximal normal vector to hypo(g) at $(x, g(x))$ realized by a ball of radius ρ. Then $(\frac{\xi}{\gamma}, \lambda)$ is a proximal normal vector to hypo(g^γ) at $(\gamma x, g^\gamma(\gamma x))$ realized by a ball of radius $\rho\frac{\gamma^2}{(1+\gamma^2)^{3/2}}$.

Proof. For all $z \in \Omega$ and for all $\beta \leq g(z)$, it holds

$$\langle\frac{(\xi, \lambda)}{\|(\xi, \lambda)\|}\ ,\ (z, \beta) - (x, g(x))\rangle \ \leq\ \frac{1}{2\rho}(\|z - x\|^2 + |\beta - g(x)|^2).$$

Equivalently, for all $\gamma z \in \gamma\Omega$ and for all $\beta \leq g^\gamma(\gamma z)$, it holds

$$\langle\frac{(\frac{\xi}{\gamma}, \lambda)}{\|(\xi, \lambda)\|}\ ,\ (\gamma z, \beta) - (\gamma x, g^\gamma(\gamma x))\rangle \ \leq\ \frac{1}{2\rho}(\frac{1}{\gamma^2}\|\gamma z - \gamma x\|^2 + |\beta - g^\gamma(\gamma x)|^2). \tag{7.2.10}$$

Since $\|(\xi, \lambda)\| \leq \sqrt{\gamma^2 + 1}\ \|(\frac{\xi}{\gamma}, \lambda)\|$, one can easily get from (7.2.10) that for all $\bar{z} = \gamma z \in \gamma\Omega$ and for all $\beta \leq g^\gamma(\bar{z})$, it holds

$$\langle\frac{(\frac{\xi}{\gamma}, \lambda)}{\|(\frac{\xi}{\gamma}, \lambda)\|}\ ,\ (\bar{z}, \beta) - (\gamma x, g^\gamma(\gamma x))\rangle \ \leq\ \frac{1}{2\rho\frac{\gamma^2}{(1+\gamma^2)^{3/2}}}\ (\|\bar{z} - \gamma x\|^2 + |\beta - g^\gamma(\gamma x)|^2) \quad . \tag{7.2.11}$$

The proof is completed. □

The following result is an immediate consequence of the previous lemma.

Corollary 7.2.1 *for every $\gamma > 0$, it holds*

$$BP_{g^\gamma} = \gamma BP_g.$$

7.3 Appendix C

Let $G : [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction satisfying hypotheses (SH1), (SH2) and $|v| \leq M|p|$ for every $v \in G(t, p)$.

Lemma 7.3.1 *Let $\bar{p}(.)$ a solution of the differential inclusion*

$$\begin{cases} \dot{p}(t) \ \in\ G(t, p(t)) \quad a.e. \\ p(0) \ =\ p_0, \end{cases} \tag{7.3.1}$$

Then

$$e^{-Mt}|p(0)| \ \leq\ |p(t)| \ \leq\ e^{Mt}|p(0)| \quad \forall t > 0.$$

Moreover, one also has

$$e^{-M(t_2-t_1)}|p(t_2)| \;\leq\; |p(t_1)| \;\leq\; e^{M(t_2-t_1)}|p(t_2)|$$

and

$$|p(t_2) - p(t_1)| \;\leq\; Me^{M(t_2-t_1)}(t_2 - t_1)|p(t_2)| \quad \text{for } 0 \leq t_1 \leq t_2.$$

Proof. From $p(t) = p(0) + \int_0^t \dot{p}(s)ds$, we have

$$|p(t)| \leq |p(0)| + \int_0^t |\dot{p}(s)|ds \leq |p(0)| + M\int_0^t |p(s)|ds.$$

Using Gronwall's inequality, we get: $|p(t)| \leq e^{Mt}|p(0)|$.

We are now going to prove that $e^{-Mt}|p(0)| \leq |p(t)|$ for all $t > 0$. Fixing $t > 0$, we define $g(s) := p(t-s)$ for all $s \in [0,t]$. Since $\dot{g}(s) = -\dot{p}(t-s)$ for almost $s \in [0,t]$, we have $g(s) = g(0) + \int_0^s \dot{g}(\tau)d\tau$ for all $s \in [0,t]$. Thus

$$\begin{aligned} |g(s)| \;&\leq\; |g(0)| + \int_0^s |\dot{g}(\tau)|d\tau = |g(0)| + \int_0^s |\dot{p}(t-\tau)|d\tau \\ &\leq\; |g(0)| + M\int_0^s |p(t-\tau)|d\tau = |g(0)| + M\int_0^s |g(\tau)|d\tau \end{aligned}$$

Using Gronwall's inequality, we get: $|g(s)| \leq e^{Ms}|g(0)|$ for all $s \in [0,t]$. In particular, $|g(t)| \leq e^{Mt}|g(0)|$. The proof is complete by noting that $g(t) = p(0)$, $g(0) = p(t)$. □

Corollary 7.3.1 *Let $p(\cdot)$ be a solution in (7.3.1). Then either $\bar{p}(t) = 0$ for all $t \geq 0$ or $p(t) \neq 0$ for all $t \geq 0$.*

Lemma 7.3.2 *Let $y(\cdot, x_0)$ be a solution of (3.7.1). Then for $t > 0$, the followings hold*
i) $|y(t,x_0)| \;\leq\; (|x_0|+1)e^{M_2 t} - 1$,
ii) $|y(t,x_0) - x_0| \;\leq (|x_0|+1)(e^{M_2 t} - 1) \leq M_2(|x_0|+1)e^{M_2 t}t$.

Proof. Since

$$y(t,x_0) \;=\; x_0 + \int_0^t \dot{y}(s,x_0)ds$$

and (SH3), we have

$$|y(t,x_0)| \;\leq\; |x_0| + M_2 t + M_2\int_0^t |y(s,x_0)|ds$$

By using Gronwall's inequality, one can prove that

$$|y(t,x_0)| \;\leq\; (|x_0|+1)e^{M_2 t} - 1.$$

So (i) is proved.
Observing that

$$|y(t,x_0)-x_0| = \Big|\int_0^t F(y(s,x_0))ds\Big| \le M_2\int_0^t (1+|y(s,x_0)|)ds.$$

(ii) follows using (i) in the above inequality. □

7.4 Appendix D

We will give here some inequalities on one variable functions which are used in Chapter 4.

Lemma 7.4.1 *Let $K:(a,b)\to[0,1]$ be measurable and let $k\in\mathbb{N}$. Then*

$$\int_a^b (t-a)^k K(t)dt \ge \frac{1}{k+1}\Big(\int_a^b K(t)dt\Big)^{k+1},$$

and

$$\int_a^b (b-t)^k K(t)dt \ge \frac{1}{k+1}\Big(\int_a^b K(t)dt\Big)^{k+1}.$$

Proof. Indeed,

$$\int_a^b K(t)dt = k!\int_a^b\int_{t_k}^b ... \int_{t_1}^b K(t_0)dt_0...dt_k$$

Since $K(t)\in[0,1]$ for a.e. $t\in[0,1]$, we obtain that

$$\int_a^b K(t)dt = k!\int_a^b K(t_k)\int_{t_k}^b K(t_{k-1})...\int_{t_1}^b K(t_0)dt_0...dt_k$$

By using induction, one can easily prove that

$$\int_a^b K(t_k)\int_{t_k}^b K(t_{k-1})...\int_{t_1}^b K(t_0)dt_0...dt_k = \frac{1}{k+1}\Big(\int_a^b K(t)dt\Big)^{k+1}$$

The proof is completed. □

Lemma 7.4.2 *Let $f:[a,b]\to\mathbb{R}$ be of class C^1 and fix $k\ge 0$. Assume that there exists $C\in\mathbb{R}$ such that*

$$|f'(s)| \ge C(s-a)^k \quad \forall s\in[a,b]. \tag{7.4.1}$$

Then, either f has no zeros in (a,b) and then, for all $s\in(a,b)$ either

$$|f(s)| \ge \frac{C}{k+1}(s-a)^{k+1} \quad \text{or} \quad |f(s)| \ge \frac{C}{k+1}(b-s)^{k+1}$$

or there exists $c \in (a,b)$ such that $g(c) = 0$ and then, for all $s \in [a,c]$,

$$|f(s)| \geq \frac{C}{k+1}(c-s)^{k+1}$$

and for all $s \in [c,b]$

$$|f(s)| \geq \frac{C}{k+1}(s-c)^{k+1}.$$

The same conclusion hold if (7.4.1) is substituted by

$$|f'(s)| \geq C(b-s)^k \quad \forall s \in [a,b].$$

Proof. Observe that by our assumptions f' has constant sign on $[a,b]$. We treat the case $f' \geq 0$, while the other one can be obtained by taking $-f$. So (7.4.1) now reads as

$$f'(s) \geq C(s-a)^k \quad \forall s \in [a,b].$$

If f has no zeros, we have two cases, namely $f(s) > 0$ for all $s \in (a,b)$ or $f(s) < 0$ for all $s \in (a,b)$. For the first case

$$f(s) - f(a) = \int_a^s f'(t)dt \geq C \int_a^s (t-a)dt = \frac{C}{k+1}(s-a)^{k+1},$$

therefore,

$$f(s) \geq \frac{C}{k+1}(s-a)^{k+1}.$$

In the second case,

$$\begin{aligned} f(b) - f(s) &= \int_s^b f'(t)dt \geq C \int_s^b (t-a)^k dt \\ &= \frac{C}{k+1}\left[(b-a)^{k+1} - (s-a)^{k+1}\right] \geq \frac{C}{k+1}(b-s)^{k+1}. \end{aligned}$$

Therefore,

$$f(s) \leq f(b) - \frac{C}{k+1}(b-s)^{k+1} \leq -\frac{C}{k+1}(b-s)^{k+1}.$$

Assume now that there exists $c \in (a,b)$ such that $f(c) = 0$. Then, for all $c \in [a,c]$ we have

$$-f(s) = f(c) - f(s) = \int_s^c f'(t)dt \geq \frac{C}{k+1}(c-s)^{k+1},$$

while for all $s \in [c,b]$ we have

$$\begin{aligned} f(s) &= f(s) - f(c) = \int_c^s f'(t)dt \geq \frac{C}{k+1}\left[(s-a)^{k+1} - (c-a)^{k+1}\right] \\ &\geq \frac{C}{k+1}(s-c)^{k+1}. \end{aligned}$$

and the proof is completed. □

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