Examples and Conjectures on the Regularity of Solutions to Balance Laws

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Abstract

The paper discusses various regularity properties for solutions to a scalar, 1-dimensional conservation law with strictly convex flux and integrable source. In turn, these yield compactness estimates on the solution set. Similar properties are conjectured to hold for 2×2 genuinely nonlinear systems.

1 Introduction

Consider a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x \ = \ 0. \tag{1.1}$$

It is well known that (1.1) generates a Lipschitz continuous semigroup of entropy weak solutions [4, 6, 7, 8, 10, 11, 14, 22, 27, 30], on a domain of suitably small BV functions. The later papers [34, 35] constructed a semigroup on a domain of functions with large, but finite total variation. In essence, these results show that the Cauchy problem has a unique solution, which depends continuously on the initial data as long as the total variation remains bounded.

Unfortunately, no general result is known about the global existence of BV solutions with large data. On one hand, a counterexample by Jenssen shows that, for some strictly hyperbolic systems, the total variation can blow up in finite time [32]. On the other hand, no such example is known for any physical system endowed with a strictly convex entropy. By the analysis in [2], the total variation of approximate solutions constructed by the Godunov scheme can become arbitrarily large. More recently in [9] an example was constructed of a piecewise Lipschitz approximate solution to the 2×2 system of isentropic gas dynamics where:

- wave strengths across interactions are the same as in exact solutions,
- rarefaction waves decay, due to genuine nonlinearity,
- the only error is due to slightly wrong wave speeds,
- and yet, the total variation blows up in finite time.

This indicates that there is no fundamental obstruction to the finite time blow-up for such system. Indeed, the issue of global boundedness vs. finite time blow-up of the total variation seems to hinge on the particular order in which various waves can interact with each other.

In view of the above remarks, one may try to study solutions to conservation laws in a wider space of $\mathbf{L}^1$ functions, without restrictions on the total variation. In this direction, a major goal is to understand under which conditions the semigroup generated by a system such as (1.1) can be extended to a domain of $\mathbf{L}^\infty$ functions. At present, this is known only in the scalar case [18, 33], and for some special systems of Temple class [12], or in triangular form [13]. We remark that, even in the case of 2×2 systems with initial data having small oscillation, studied in the classical memoir by Glimm and Lax [28] (see [5] for a shorter existence proof based on front-tracking approximations) the uniqueness of solutions remains an elusive open problem.

For 2×2 systems, the main tool for constructing weak solutions with large data is provided by compensated compactness, introduced by DiPerna in his famous paper [26]. While other existence theorems based on compactness rely on quantitative estimates on the regularity of solutions (say, an a priori bound on a Hölder norm, a Sobolev norm, or on the total variation), in the author's view compensated compactness remains like a "black box". Arguing by contradiction, one establishes the existence of a solution, but without further information on its uniqueness or qualitative properties. See [17] for some of the few results in this direction.

Aim of the present note is to discuss the possible regularity properties of $\mathbf{L}^\infty$ solutions to hyperbolic conservation laws (1.1). Two main cases will be considered:

(i) Scalar balance laws with convex flux and integrable source:

$$u_t + f(u)_x \;=\; g(t,x). \tag{1.2}$$

(ii) Strictly hyperbolic, genuinely nonlinear 2×2 hyperbolic systems of conservation laws.

For such systems, choosing coordinates (w_1, w_2) consisting of Riemann invariants, we observe that solutions to the system (1.1) satisfy the non-conservative system in diagonal form

$$\begin{cases} w_{1,t} + \lambda_1(w_1,w_2)w_{1,x} \;=\; \mu_1, \\ w_{2,t} + \lambda_2(w_1,w_2)w_{2,x} \;=\; \mu_2, \end{cases} \tag{1.3}$$

where μ_1, μ_2 are bounded measures, concentrated on the set of curves where w_1, w_2 have jumps. By genuine nonlinearity, the characteristic speeds satisfy $\lambda_{1,w_1} > 0$, $\lambda_{2,w_2} > 0$.

We show that, when $\mu_1 = \mu_2 = 0$, solutions to (1.3) can be constructed so that each component satisfies a one-sided Lipschitz estimate. This suggests that solutions to (1.3) share similar regularity properties as the solutions to the scalar balance law (1.2).

The remainder of the paper is organized as follows. In Section 2 we review some regularity properties of scalar conservation laws with convex flux, and quantitative compactness estimates. The examples presented in Section 3 show that solutions to Burgers' equation with an integrable source, in spite of their compactness properties, can exhibit a rather wild behavior. In Section 4 we still consider solutions to a scalar balance law with strictly convex flux and a source $g \in \mathbf{L}^1$. Toward an alternative compactness estimate, we consider the number $N(t)$ of times that the solution profile $u(t,\cdot)$ crosses up or down a given interval $[a,b] \in \mathbb{R}$. Two conjectures are proposed, bounding this number of crossings. Section 5 is concerned with solutions to the 2×2 strictly hyperbolic, genuinely nonlinear system of conservation laws (1.1). We observe that, working in Riemann coordinates, it is possible to define an auxiliary flow of piecewise Lipschitz functions where each component satisfies one-sided Lipschitz decay estimates. In turn, the entropic solutions to (1.1) can be approximated by periodically adding to this flow a source term g, globally bounded in $\mathbf{L}^1$. This leads to the conjecture that $\mathbf{L}^\infty$ solutions to (1.1) may share the same regularity properties as solutions to scalar balance laws with $\mathbf{L}^1$ source.

2 The scalar balance law

In this section we consider a scalar conservation law

$$u_t + f(u)_x \;=\; 0, \tag{2.1}$$

where f is a smooth flux. It is well known [18, 33] that in this case there exists a semigroup $\{S_t\,;\; t \geq 0\}$ which is contractive in $\mathbf{L}^1(\mathbb{R})$ and such that, for every initial datum

$$u(0,\cdot) \;=\; \bar u \in \mathbf{L}^1(\mathbb{R}), \tag{2.2}$$

the trajectory $t \mapsto u(t) = S_t \bar u$ is the unique entropy weak solution of the Cauchy problem.

2.1 A family of positively invariant domains.

Let A be a (possibly multivalued) nonlinear operator generating a contractive semigroup $\{\mathcal{S}_t\,;\; t \geq 0\}$ on a Banach space X. As in [20], this means that each trajectory $t \mapsto u(t) = S_t \bar u$ is the limit of a convergent sequence of Backward Euler approximations for the abstract Cauchy problem

$$\frac{d}{dt} u(t) \;=\; Au(t), \qquad\qquad u(0) = \bar u. \tag{2.3}$$

In this setting, the paper [19] introduced a definition of "generalized domain" $\mathcal{D}$ for the generator A, namely

$$\mathcal{D} \;\doteq\; \left\{ \bar u \in X\,;\quad \chi(\bar u) \;\doteq\; \sup_{0<t<1} \frac{\|S_t \bar u - \bar u\|}{t} \;<\; +\infty \right\}. \tag{2.4}$$

This consists of all initial data $\bar{u}$ for which the trajectory $t \mapsto S_t\bar{u}$ is globally Lipschitz continuous. Notice that, for the scalar conservation law (2.1), we have

$$\mathbf{L}^1 \cap BV \subseteq \mathcal{D}.$$

Furthermore, it was observed in [3] that a particular class of semigroup generators had regularizing properties, as in the case of linear analytic semigroups.

Motivated by the theory of fractional powers of sectorial operators [29, 36], together with (2.4) for $0 < \alpha < 1$ we define the intermediate domains

$$\mathcal{D}_\alpha \doteq \Big\{\bar{u} \in X\,; \quad \sup_{0<t<1} t^{-\alpha}\big\|S_t\bar{u} - \bar{u}\| < +\infty\Big\}. \tag{2.5}$$

These contain all the initial data whose trajectories are Hölder continuous with exponent α. Recalling the definition of $\chi(\bar{u})$ at (2.4), one can also consider the domains

$$\widetilde{\mathcal{D}}_\alpha \doteq \Big\{\bar{u} \in X\,; \quad \sup_{0<t<1} t^{1-\alpha}\cdot\chi(S_t\bar{u}) < +\infty\Big\}. \tag{2.6}$$

It is easy to check that $\widetilde{\mathcal{D}}_\alpha \subseteq \mathcal{D}_\alpha$, for any $0 < \alpha < 1$. Indeed, if $\bar{u} \in \widetilde{\mathcal{D}}_\alpha$, there exists a constant C such that

$$\chi(S_t\bar{u}) \leq Ct^{\alpha-1} \qquad \text{for all } t > 0.$$

In addition, for every $t, s > 0$ there holds

$$\|S_{t+s}\bar{u} - S_t\bar{u}\| \leq s\cdot\chi(S_t\bar{u}) \leq s\cdot Ct^{\alpha-1}. \tag{2.7}$$

Choosing $t_k = 2^{-k}t$, $k = 0, 1, 2\ldots$, and applying (2.7) with $s = t = 2^{-k}t$, we thus obtain

$$\|S_t\bar{u} - \bar{u}\| \leq \sum_{k\geq 0}\|S_{t_{k-1}}\bar{u} - S_{t_k}\bar{u}\| \leq \sum_{k\geq 0}(2^{-k}t)\cdot C(2^{-k}t)^{\alpha-1} = t^\alpha\cdot C\big(1 - 2^{-\alpha}\big)^{-1}.$$

In connection with the semigroup generated by a conservation law (2.1), we expect that the definitions (2.5) or (2.6) will identify some useful, positively invariant subdomains.

In the following, we shall assume that the flux function f is strictly convex, so that

$$f''(u) \geq c > 0 \qquad\qquad \text{for all } u \in \mathbb{R}. \tag{2.8}$$

In this section, for convenience we consider solutions of (2.1) or (1.2) within the space of periodic functions, so that $u(x+1) = u(x)$ for all x. This comes with the norm

$$\|u\|_{\mathbf{L}^1_{per}} \doteq \int_0^1 \big|u(x)\big|\,dx. \tag{2.9}$$

Of particular interest is to understand the range of solutions to the balance law (1.2), where the spatially periodic source term g satisfies

$$\big\|g(t,\cdot)\big\|_{\mathbf{L}^1_{per}} \leq C \qquad\qquad \text{for all } t \geq 0. \tag{2.10}$$

We claim that for $\tau > 0$ the solution $u(\tau,\cdot)$ to (1.2) lies in an intermediate domain of the form (2.6), with $\alpha = 1/2$.

Proposition 2.1. *Let the flux function f satisfy (2.8). Consider a spatially periodic solution u of (1.2), where g satisfies (2.10). Then for every $\tau > 0$ one has $u(\tau) \in \mathcal{D}_{1/2}$.*

Proof. Let $t \mapsto u(t)$ be any solution to (1.2), and fix $\tau > 0$. Using the decay of the total variation of solutions to the semigroup generated by (2.1), we obtain

$$\begin{aligned}\big\|S_\varepsilon u(\tau) - u(\tau)\big\|_{\mathbf{L}^1_{per}} &\leq \big\|S_\varepsilon u(\tau) - S_\varepsilon S_\delta u(\tau-\delta)\big\|_{\mathbf{L}^1_{per}} + \big\|S_\varepsilon S_\delta u(\tau-\delta) - S_\delta u(\tau-\delta))\big\|_{\mathbf{L}^1_{per}} \\ &\quad + \big\|S_\delta u(\tau-\delta) - u(\tau)\big\|_{\mathbf{L}^1_{per}} \\ &\leq C\delta + \varepsilon\cdot\chi\big(S_\delta u(\tau-\delta)\big) + C\,\delta.\end{aligned} \tag{2.11}$$

Next, by (2.10) it follows

$$\big\|u(\tau-\delta)\big\|_{\mathbf{L}^1_{per}} \leq \|\bar u\|_{\mathbf{L}^1_{per}} + C\tau. \tag{2.12}$$

Moreover, if $v(x) = \big(S_\delta u(\tau-\delta)\big)(x)$, then v satisfies the one-sided Lipschitz estimate

$$f'\big(v(x)\big) - f'\big(v(y)\big) \leq \frac{x-y}{\delta} \qquad \text{for all } x > y. \tag{2.13}$$

Combining (2.12) with (2.13) we conclude

$$\chi(v) \leq 2\sup_{x\in[0,1]}\big|f'(v(x))\big| \leq 2\sup\Big\{\big|f'(\omega)\big|\,;\;\; |\omega| \leq \|\bar u\|_{\mathbf{L}^1_{per}} + C\tau\Big\} + \frac{2}{\delta} \leq \frac{C_1}{\delta}, \tag{2.14}$$

for some constant C_1 and all $\delta \in]0,1]$. Inserting (2.14) into (2.11) and choosing $\delta = \varepsilon^{1/2}$ one obtains the desired estimate:

$$\big\|S_\varepsilon u(\tau) - u(\tau)\big\|_{\mathbf{L}^1_{per}} \leq C\varepsilon^{1/2} + \varepsilon\cdot\frac{C_1}{\varepsilon^{1/2}} + C\,\varepsilon^{1/2}.$$

□

2.2 Quantitative compactness estimates.

Consider again the balance law (1.2), in the spatially periodic case.

If the flux function f is strictly convex, the semigroup S generated by the conservation law without source is compact. More precisely, for every $\tau > 0$ and $M > 0$, the set

$$K_\tau \doteq \Big\{S_\tau \bar u\,;\;\; \|\bar u\|_{\mathbf{L}^1_{per}} \leq M\Big\}$$

is compact. Indeed, by (2.8), the Oleinik's one-sided Lipschitz conditions yield

$$(S_\tau\bar u)(x) - (S_\tau\bar u)(y) \leq \frac{x-y}{c\tau} \qquad \text{for all } x < y,$$

and hence, over the interval $x \in [0,1]$,

$$\text{Tot.Var.}(S_\tau\bar u) \leq \frac{2}{c\tau} \qquad \text{for all } \bar u.$$

Since, by conservation $\|S_\tau \bar{u}\|_{\mathbf{L}^1_{per}} \leq M$, this implies that, for any $\varepsilon > 0$ the set of functions $K_\tau \subset \mathbf{L}^1$ can be covered by a finite number of balls in $\mathbf{L}^1$ with radius ε. See [1, 25] for more precise quantitative estimates on the number of balls needed for this covering.

Here we observe that, for solutions to balance laws in $\mathbf{L}^1(\mathbb{R})$, a relaxed version of the one-sided Lipschitz condition remains valid, which is equally useful to achieve compactness.

Proposition 2.2. *Let $u = u(t,x)$ be a solution to the balance law (1.2), assuming that*

$$f''(u) \geq c > 0, \qquad \big\|g(t,\cdot)\big\|_{\mathbf{L}^1(\mathbb{R})} \leq C, \tag{2.15}$$

for all u, t. Then, for every $T > 0$ and $\varepsilon > 0$, there exists a subset $V_\varepsilon \subset \mathbb{R}$, with

$$meas(V_\varepsilon) \leq C\,\varepsilon^{1/2}, \tag{2.16}$$

such that

$$u(T,y) - u(T,x) \leq 2\varepsilon^{1/2} + \frac{y-x}{c\,\varepsilon} \qquad \text{for all } x, y \notin V_\varepsilon\,, \quad x < y\,. \tag{2.17}$$

Proof. Given $T > \varepsilon > 0$, let v be the solution to the conservation law without source

$$v_t + f(v)_x = 0, \qquad v(T-\varepsilon, x) = u(T-\varepsilon, x). \tag{2.18}$$

The second inequality in (2.15) implies

$$\|v(T) - u(T)\|_{\mathbf{L}^1} \leq C\varepsilon. \tag{2.19}$$

Calling

$$V_\varepsilon = \Big\{x \in \mathbb{R}\,; \ \ \big|v(T,x) - u(T,x)\big| > \varepsilon^{1/2}\Big\},$$

by (2.19) it follows (2.16).

Next, for $x, y \notin V_\varepsilon$, $x < y$, by Oleinik's inequality and the triangle inequality we conclude

$$u(T,y) - u(T,x) \leq 2\varepsilon + v(T,y) - v(T,x) \leq 2\varepsilon^{1/2} + \frac{y-x}{c\,\varepsilon}\,.$$

□

3 Examples of solutions to scalar balance laws

Throughout this section we consider Burgers' equation with an integrable source term:

$$u_t + \left(\frac{u^2}{2}\right)_x = g(t,x), \qquad u(0,x) = \bar{u}(x) \in \mathbf{L}^1(\mathbb{R}). \tag{3.1}$$

As remarked earlier, relying on Oleinik's inequalities one obtains good compactness estimates on the set of all solutions. Yet, the examples collected in this section show that these solutions can be quite wild.

Example 3.1. We start with an elementary example showing that the total variation of a solution to (3.1) can be infinite for all times $t \geq 0$. Consider the constant in time function

$$u(t,x) \;=\; \begin{cases} x \sin \frac{1}{x} & \text{if} \quad |x| \leq \frac{1}{\pi}\,, \\ 0 & \text{otherwise.} \end{cases}$$

This is a stationary solution, with unbounded variation, of

$$u_t + uu_x \;=\; g(x) \;=\; \begin{cases} x \sin^2 \frac{1}{x} - \sin \frac{1}{x} \cos \frac{1}{x} & \text{if} \quad |x| \leq \frac{1}{\pi}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Notice that here the source term $g(\cdot)$ has bounded $\mathbf{L}^1$ norm. We remark that (3.2) can be equivalently written as a conservation law with a Lipschitz continuous flux depending also on the space variable x, namely

$$u_t + \left(\frac{u^2}{2} - G(x)\right)_x \;=\; 0, \qquad\qquad G(x) \;\doteq\; \int_0^x g(y)\,dy.$$

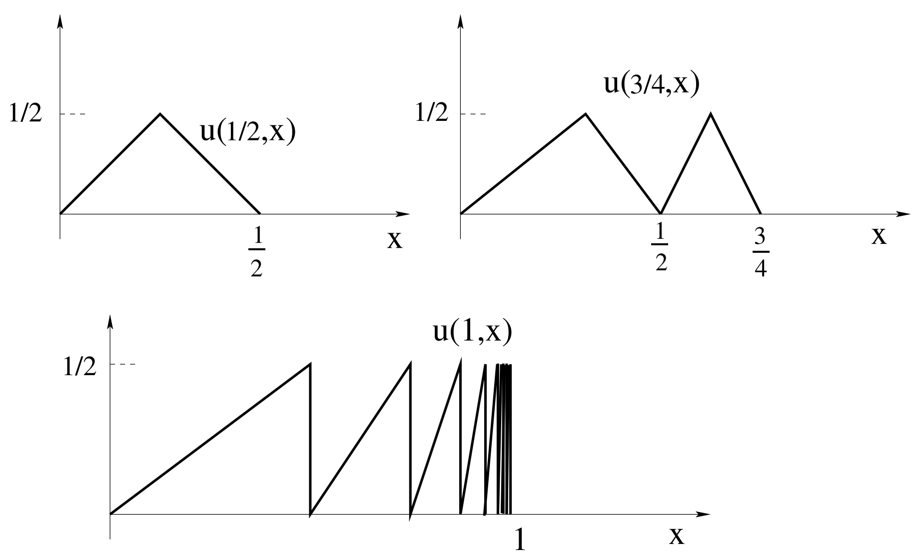

Figure 1: The solution to (3.1) constructed in Example 3.2, which has unbounded oscillation at time $t = 1$.

Example 3.2. Using a bounded source g, we can also construct a solution with zero initial data and such that, at time $T = 1$, it oscillates infinitely many times between 0 and 1/2 (see Fig. 1).

Choose a source $g = g(t,x)$ such that, at time $t_1 = 1/2$, the solution to (3.1) is the tent function

$$u\Big(\frac{1}{2}, x\Big) \;=\; \begin{cases} 2x & \text{if} \quad x \in \left[0, \frac{1}{4}\right], \\ 1 - 2x & \text{if} \quad x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Notice that, if no source is applied for $t > \frac{1}{2}$, this solution of Burgers' equation with initial data (3.3) remains continuous up to time $t = 1$.

Similarly, during the time interval

$$I_k = [t_{k-1}, t_k], \qquad\qquad t_k = 1 - 2^{-k},$$

we use the source g to construct an additional spike on the interval $x \in I_k$. Namely

$$u(t_k, x) = \min\Big\{2(x - t_{k-1}),\ 2(t_k - x)\Big\} \qquad\qquad x \in [t_{k-1}, t_k].$$

Notice that, if no source is applied for $t \in [t_k, 1]$, this solution remains continuous up to time $t = 1$.

Performing the same construction for all $k \geq 1$, at time $t = 1$, the solution satisfies

$$u(1, t_k-) = 1/2, \qquad\qquad u(1, t_k+) = 0, \qquad\qquad \text{for all } k \geq 1.$$

Notice, however, that the number of oscillations between 0 and 1/2 in infinite only at the particular time $t = 1$.

Example 3.3. Given $\varepsilon > 0$, there exists a positive source function $g \in \mathbf{L}^1\big(\mathbb{R}_+ \times \mathbb{R}\big)$ with $\big\|g\big\|_{\mathbf{L}^1} \leq \varepsilon$, such that the solution to (3.1) with zero initial data satisfies the following property. For every point (τ, y) with rational coordinates and with $\tau > 0$, one has

$$\lim_{x \to y-} u(\tau, x) = +\infty. \tag{3.4}$$

The construction will be given in three steps.

1. Following [15], we first construct a function g such that the solution of (3.1) with zero initial data blows up at the point $(\tau, y) = (1, 1)$. Define the source function

$$g(t, x) = \begin{cases} \dfrac{1}{1-t} & \text{if } \ x \in [a(t), b(t)] \ \text{ and } \ 0 < t < 1, \\ 0 & \text{if } \ x \notin [a(t), b(t)] \ \text{ or if } \ t \geq 1, \end{cases} \tag{3.5}$$

where, for $0 < t < 1$,

$$a(t) \doteq \int_0^t |\ln(1-s)|ds = t + (1-t)\ln(1-t)\,, \qquad\qquad b(t) \doteq 1 + (1-t)\ln(1-t).$$

Since $b(t) - a(t) = 1 - t$, it is clear that $\|g(t, \cdot)\|_{\mathbf{L}^1} = 1$ for $t < 1$. For $0 \leq t < 1$, the solution of (3.1), shown in Fig. 2, left, satisfies

$$u(t, x) = \begin{cases} |\ln(1-t)| & \text{if } \ x \in [a(t), b(t)], \\ \dfrac{1-x}{1-t}, & \text{if } \ x \in [b(t), 1], \\ 0 & \text{if } \ x \notin [0, 1]. \end{cases}$$

Note that, for $t \in [0, 1[$, we have

$$u_x(t, x) \begin{cases} \geq 0 & \text{if } \ 0 < x < a(t), \\ = 0 & \text{if } \ a(t) < x < b(t), \\ \geq \ -\frac{1}{1-t} & \text{if } \ b(t) < x < 1, \end{cases}$$

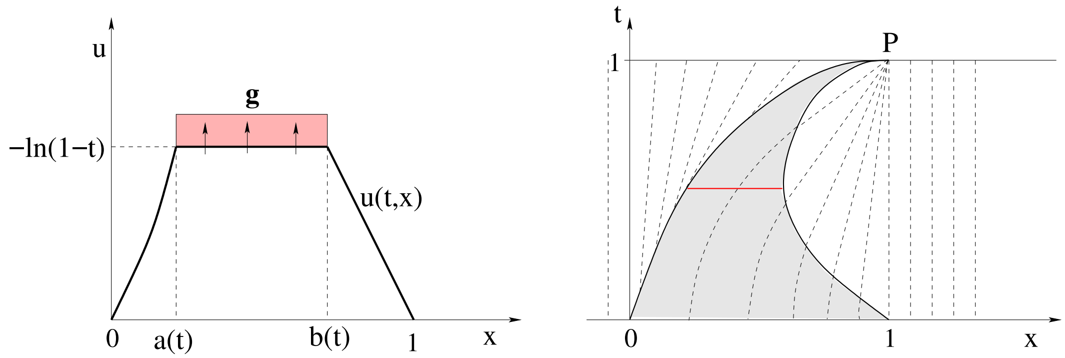

Figure 2: Constructing a solution of Burgers' equation with source, that blows up in finite time. Left: the profile of $u(t,\cdot)$ at some time $0<t<1$. Right: sketch of the characteristics in the t-x plane. Here $P=(1,1)$ is the blow up point.

hence no shock is formed for $t<1$. The $\mathbf{L}^\infty$ norm of this solution blows up as $t\to 1-$. Moreover, at time $t=1$ one has

$$\|u(1,\cdot)\|_{\mathbf{L}^1} \;=\; 1, \qquad\qquad \lim_{x\to 1-} u(1,x)=+\infty. \tag{3.6}$$

2. Next, consider any point $(\tau,y)\in\,]0,T]\times\mathbb{R}$ and any $n\geq 1$. We construct a source g_n with $\|g_n\|_{\mathbf{L}^1}\leq 2^{-n}\varepsilon$ and such that the corresponding solution to (3.1) blows up at the point (τ,y). Choose $\delta=\min\{\tau,2^{-n}\varepsilon\}$. Then consider the rescaled function

$$u_n(t,x) \;=\; \left\{\begin{array}{ccc} 0 & \text{if} & t\notin\,]\tau-\delta,\tau[\,, \\ u\left(\frac{t-(\tau-\delta)}{\delta},\,\frac{x-(y-\delta)}{\delta}\right) & \text{if} & t\in\,]\tau-\delta,\tau[\,. \end{array}\right. \tag{3.7}$$

Notice that we are shifting the blow up point $P=(1,1)$ of u to the blow up point $P_n=(\tau,y)$ of u_n. The function u_n satisfies the balance law

$$u_t+\left(\frac{u^2}{2}\right)_x \;=\; g_n\,,$$

where

$$g_n(t,x) \;\doteq\; \left\{\begin{array}{ccc} 0 & \text{if} & t\notin\,]\tau-\delta,\tau[\,, \\ \frac{1}{\delta}g\left(\frac{t-(\tau-\delta)}{\delta},\,\frac{x-(y-\delta)}{\delta}\right) & \text{if} & t\in\,]\tau-\delta,\tau[\,. \end{array}\right. \tag{3.8}$$

This yields

$$\|g_n\|_{\mathbf{L}^1} \;=\; \delta\cdot\|g\|_{\mathbf{L}^1} \;=\; \delta.$$

3. We now arrange all rational points inside $]0,T]\times\mathbb{R}$ into a sequence $P_n=(t_n,x_n)$. For each $n\geq 1$, consider the source function g_n defined as in (3.8), with (τ,y) replaced by (t_n,x_n). We the define the source

$$G(t,x) \;\doteq\; \sum_{n\geq 1} g_n(t,x).$$

This implies

$$\|G\|_{\mathbf{L}^1} \;=\; \sum_{n\geq 1}\|g_n\|_{\mathbf{L}^1} \;\leq\; \sum_{n\geq 1}2^{-n}\varepsilon \;\leq\; \varepsilon.$$

Calling $U = U(t,x)$ the solution to

$$U_t + \left(\frac{U^2}{2}\right)_x = G(t,x), \qquad U(0,x) = 0,$$

since $g_n \leq G$ for every n, by a comparison argument we conclude

$$u_n(t,x) \leq U(t,x)$$

for every t, x, n. In particular (3.4) holds at every rational point (τ, y). □

4 Regularity of solutions to scalar balance laws

As shown by the previous examples, for a source term g satisfying only an integral bound, solutions to the balance law (1.2) can be quite wild. Yet, if the flux function f is strictly convex, the oscillations produced by the source term do not prevent compactness estimates. In particular, any weakly convergent sequence of solutions $u_n \rightharpoonup u$ is also strongly convergent.

One wonders what kind of uniform regularity properties can be proved for these solutions. Proposition 2.2 provides a simple result in this direction. Comparing the solutions u of the balance law (1.2) with the solution v of the homogeneous problem (2.18), for any given $\varepsilon > 0$ one can change the profile $u(t,\cdot)$ by an amount $\mathcal{O}(1) \cdot \varepsilon$ in the $\mathbf{L}^1$ distance, and obtain a function $v \in \mathbf{L}^1(\mathbb{R})$ that satisfies Oleinik's one-sided Lipschitz estimates

$$v(y) - v(x) \leq \frac{y-x}{\varepsilon} \qquad \text{for all } x < y.$$

An alternative, more direct way to measure the regularity of these solutions is to quantify the amount of oscillations. More precisely, consider any interval $[a,b]$, and denote by $N = N_{[a,b]}(t)$ the number of times that the function $x \mapsto u(t,x)$ crosses the interval $[a,b]$. That means: there exist $x_1 < x_2 < \cdots < x_{2N}$ such that

$$\begin{cases} u(t,x_k) \leq a & \text{for } k \text{ odd}, \\ u(t,x_k) \geq b & \text{for } k \text{ even}. \end{cases}$$

As a possible bound on the number of these oscillations, the following comes to mind:

Conjecture 4.1. *Assume that the flux f is strictly convex, so that (2.8) holds.*
Then there exists a constant C such that, for any solution $u = u(t,x)$ to (1.2), with initial data $\bar{u} \in \mathbf{L}^1(\mathbb{R})$ and integrable source $g \in \mathbf{L}^1(\mathbb{R}_+ \times \mathbb{R})$, one has

$$\int_0^{+\infty} N_{[a,b]}(t)\, dt \leq C \cdot \frac{\|\bar{u}\|_{\mathbf{L}^1} + \|g\|_{\mathbf{L}^1}}{(b-a)^2}. \tag{4.1}$$

A few remarks are in order.

(i) In the special case $g = 0$, Oleinik's estimate would yield $N_{[a,b]}(t) \leq \mathcal{O}(1) \cdot t^{-1}$, which is not useful to achieve (4.1).

(ii) Without loss of generality, one can assume $\bar{u} = 0$.

(iii) For simplicity, one can consider Burgers' equation (3.1), with zero initial data. In this case, by a rescaling of coordinates, it suffices to prove the inequality for $a = -1, b = 1$. The bound (4.1) thus takes the simpler form

$$\int_0^{+\infty} N_{[-1,1]}(t)\, dt \;\leq\; C\, \|g\|_{\mathbf{L}^1}\,. \tag{4.2}$$

A simpler estimate, apparently related to the previous one, is:

Conjecture 4.2. *Let $u = u(t,x)$ be the solution to Burgers' equation (3.1) with zero initial data and an integrable source term g. Then*

$$meas\left(\left\{t > 0\,;\;\; \operatorname*{ess-sup}_{x\in\mathbb{R}} u(t,x) \geq 1\right\}\right) \;\leq\; C\, \|g\|_{\mathbf{L}^1}\,, \tag{4.3}$$

for some constant C independent of g.

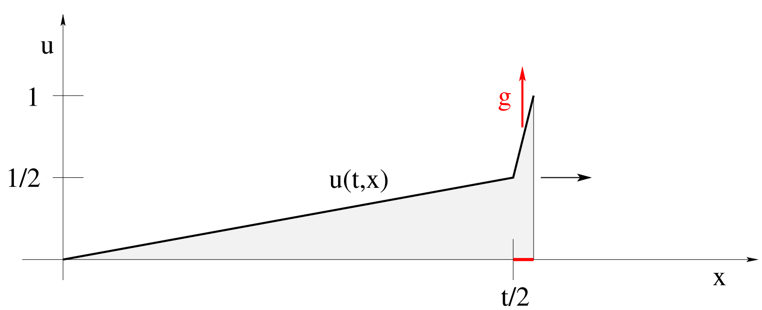

Figure 3: A source of size $\big\|g(t,\cdot)\big\|_{\mathbf{L}^1} \leq \frac{1}{8}$, located behind the shock, suffices to maintain the supremum $\sup_{x\in\mathbb{R}} u(t,x) = 1$ for all times $t > 0$.

At the present time, it is unclear whether any of the above conjectures might be true. To get some intuition, the following examples may be useful.

Example 4.3. As shown in Fig. 3, consider the function

$$u(t,x) \;=\; \begin{cases} 0 & \text{if} \quad x \notin \left[0,\ \frac{t}{2}+\varepsilon\right], \\[2mm] \dfrac{x}{t} & \text{if} \quad x \in \left[0,\ \frac{t}{2}\right], \\[2mm] \dfrac{1}{2} + \dfrac{x - t/2}{2\varepsilon} & \text{if} \quad x \in \left[\frac{t}{2},\ \frac{t}{2}+\varepsilon\right]. \end{cases} \tag{4.4}$$

This is a solution to the balance law (3.1), with

$$g(t,x) \;=\; u_t + uu_x \;=\; \begin{cases} 0 & \text{if} \quad x \notin \left[\frac{t}{2},\ \frac{t}{2}+\varepsilon\right], \\[2mm] \dfrac{x - t/2}{2\varepsilon}\cdot\dfrac{1}{2\varepsilon} & \text{if} \quad x \in \left[\frac{t}{2},\ \frac{t}{2}+\varepsilon\right]. \end{cases}$$

Notice that here $\big\|g(t,\cdot)\big\|_{\mathbf{L}^1} = 1/8$ for every $t > 0$. Therefore, a source of strength $\big\|g(t,\cdot)\big\|_{\mathbf{L}^1} \leq \frac{1}{8}$ suffices to sustain one oscillation across the interval $[0,1]$. This indicates that the constant C in (4.3) cannot be smaller than 8.

Example 4.4. To appreciate the subtleties involved in a proof of Conjecture 4.2 we observe that, if the flux $f(u) = u^2/2$ is replaced by a piecewise affine flux as in [21], then the estimate (4.3) cannot hold. To construct a counterexample, let us partition the interval $[0,1]$ into n equal subintervals, inserting the points $s_k = k/n$, $k = 0, 1, \ldots, n$. Call f_n the piecewise affine flux function which coincides with f at every point s_k, and let

$$\lambda_k \doteq \frac{s_k + s_{k-1}}{2} = \frac{2k-1}{2n}$$

be the speed of a jump connecting the states s_{k-1} and s_k.

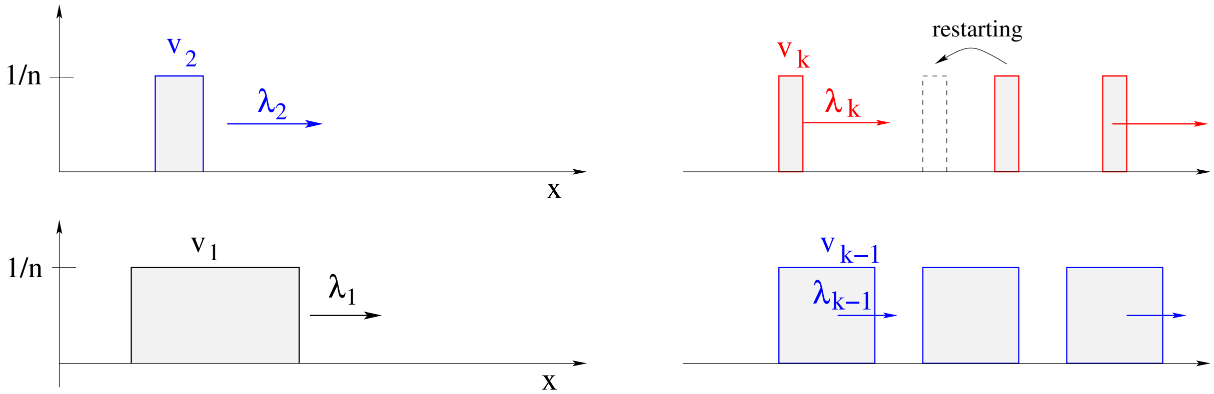

Figure 4: The functions v_k constructed in Example 4.4. The support of v_k is an interval that shifts in time with speed $\lambda_k > \lambda_{k-1}$. Therefore, at certain times τ_k, $k = 1, \ldots, N_k$, the support of $v_k(t, \cdot)$ will touch the boundary of the support of v_{k-1}. When this happens, the function v_k must be restarted.

Let $\varepsilon > 0$ be given. We shall construct a solution to

$$u_t + f_n(u)_x = g(t,x), \qquad u(0,x) = 0, \tag{4.5}$$

with

$$\|g\|_{\mathbf{L}^1} < \varepsilon, \qquad \text{meas}\left(\left\{t \in [0,1]\,;\ \operatorname*{ess-sup}_{x\in\mathbb{R}} u(t,x) = 1\right\}\right) > 1 - \varepsilon. \tag{4.6}$$

1. As a first step, we construct a piecewise constant function $v = v(t,x)$ taking values inside the discrete set $\{k/n\,;\ k = 0, 1, \ldots, n\}$, such that, for a suitable partition $0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$, there holds:

(i) Restricted to each time interval $I_\ell = [t_{\ell-1}, t_\ell[$ the function v provides a solution to the conservation law

$$v_t + f_n(v)_x = 0.$$

(ii) The changes in the function $v(t,\cdot)$ at the restarting times t_ℓ satisfy

$$\sum_{\ell=1}^{N} \big\|v(t_\ell, \cdot) - v(t_\ell -, \cdot)\big\|_{\mathbf{L}^1} < \varepsilon. \tag{4.7}$$

The solution v is defined as a sum:

$$v(t,x) = \sum_{k=1}^{n} v_k(t,x), \tag{4.8}$$

where the functions $v_k : [0,1]\times\mathbb{R}\mapsto\{0, n^{-1}\}$ satisfy

$$0 \;\leq\; v_n(t,x) \;\leq\; v_{n-1}(t,x) \;\leq\; \cdots \;\leq\; v_2(t,x) \;\leq\; v_1(t,x) \;\leq\; \frac{1}{n}\,. \tag{4.9}$$

Denoting by $\chi_{_J}$ the characteristic function of the set $J\subset\mathbb{R}$, the functions v_k are defined inductively as follows.

(i) The function v_1 is a step function traveling with speed λ_1, namely

$$v_1(t,x) \;=\; \frac{1}{n}\chi_{[\lambda_1 t,\ \lambda_1 t+\varepsilon_1]}(x),$$

for some $\varepsilon_1<\varepsilon$.

(ii) The function v_2 has the form

$$v_2(t,x) \;=\; \frac{1}{n}\chi_{[\alpha_{2j}+\lambda_2 t,\ \alpha_{2j}+\lambda_2 t+\varepsilon_2]}(x), \qquad t\in I_{2j}\,,\quad j=1,2,\ldots,N_2\,,$$

for some $\varepsilon_2 << \varepsilon_1$. Here the intervals I_{2j} and the constants α_{2j} are chosen in order to satisfy the inequality $v_2(t,x)\leq v_1(t,x)$ for all t,x.

(iii) By induction, assume that v_{k-1} has been constructed. We then choose $\varepsilon_k << \varepsilon_{k-1}$ and let v_k be a function of the form

$$v_k(t,x) \;=\; \frac{1}{n}\chi_{[\alpha_{kj}+\lambda_k t,\ \alpha_{kj}+\lambda_1 t+\varepsilon_k]}(x), \qquad t\in I_{kj}\,,\quad j=1,2,\ldots,N_k\,,$$

Here the intervals I_{kj} and the constants α_{kj} are chosen in order to satisfy the inequality $v_k(t,x)\leq v_{k-1}(t,x)$ for all t,x.

We now estimate the total amount of source needed to achieve the above function $v=\sum_k v_k$.

- The construction of v_1 requires a source of total size $\frac{1}{n}\varepsilon_1$. We choose $\varepsilon_1<\varepsilon$.
- The construction of v_2 requires a source of size $\frac{1}{n}\varepsilon_2\cdot N_2$. Here N_2 depends only on ε_0. We choose $\varepsilon_2<\frac{\varepsilon}{N_2}$
- In general, the construction of v_k requires a source of size $\frac{1}{n}\varepsilon_k\cdot N_k$. Here N_k depends only on ε_{k-1} and N_{k-1}. We choose $\varepsilon_k<\frac{\varepsilon}{N_k}$.

The total amount of source required is estimated by

$$\frac{\varepsilon_1}{n}+\frac{\varepsilon_2 N_2}{n}+\cdots+\frac{\varepsilon_n N_n}{n} \;<\; \frac{\varepsilon}{n}+\frac{\varepsilon}{n}+\cdots+\frac{\varepsilon}{n} \;=\; \varepsilon.$$

2. In view of (4.7), the function v provides a solution to (4.5) where the source term g is replaced by a measure μ of total mass $|\mu|\big([0,1]\times\mathbb{R}\big)<\varepsilon$, concentrated at the times t_ℓ. By approximating μ with an $\mathbf{L}^1$ function g having the same global bound, we obtain a solution u of (4.5), for which (4.6) holds. This shows that Conjecture 4.2 cannot hold for a piecewise affine flux.

We remark that, in the above example, the sets where $u(t,x)=1$ are extremely small. In fact, as $n\to\infty$, even the sets where $u(t,x)\geq 1/2$ have measure which approaches zero.

5 Decay of solutions to a diagonal hyperbolic system

Our ultimate goal is to gain some insight on the regularity of $\mathbf{L}^1$ solutions to a 2×2 strictly hyperbolic system of conservation laws (1.1), without restrictions on the total variation. Call $\lambda_1(u), \lambda_2(u)$ the characteristic speeds, i.e., the eigenvalues of the Jacobian matrix $Df(u)$. Working with a set of Riemann coordinates $w = (w_1, w_2)$, smooth solutions to (1.1) can be obtained by solving the hyperbolic system in (non-conservative) diagonal form

$$\begin{cases} w_{1,t} + \lambda_1(w_1, w_2) w_{1,x} &= 0, \\ w_{2,t} + \lambda_2(w_1, w_2) w_{2,x} &= 0. \end{cases} \tag{5.1}$$

We consider solutions to (5.1) on a domain of bounded $\mathbf{L}^1$ functions, namely

$$\mathcal{D} \doteq \Big\{ w \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}^2)\,; \quad \big(w_1(x), w_2(x)\big) \in [a_1, b_1] \times [a_2, b_2] \quad \text{for all } x \in \mathbb{R} \Big\}. \tag{5.2}$$

Throughout the following, we shall assume

(A1) *The characteristic speeds λ_1, λ_2 are $\mathcal{C}^2$ in an open domain $\Omega \supset [a_1, b_1] \times [a_2, b_2]$. For every $(w_1, w_2) \in \Omega$ one has*

$$\lambda_1(w_1, w_2) \;\leq\; -\delta_0 \;<\; 0 \;<\; \delta_0 \;\leq\; \lambda_2(w_1, w_2). \tag{5.3}$$

In addition, genuine nonlinearity holds:

$$\frac{\partial}{\partial w_1} \lambda_1(w_1, w_2) \;\geq\; \kappa \;>\; 0, \qquad\qquad \frac{\partial}{\partial w_2} \lambda_2(w_1, w_2) \;\geq\; \kappa \;>\; 0. \tag{5.4}$$

(A2) *As (w_1, w_2) range in the domain Ω, the other two partial derivatives $\frac{\partial}{\partial w_2}\lambda_1$ and $\frac{\partial}{\partial w_1}\lambda_2$ have a constant sign.*

It will be convenient to work within the set of functions (see Fig. 5)

$$\mathcal{F} \;\doteq\; \Big\{ u \in \mathbf{L}^1(\mathbb{R})\,; \;\; u \text{ is piecewise Lipschitz continuous with finitely many downward jumps,} \\ u_x(x) \geq 0 \text{ for a.e. } x \in \mathbb{R} \Big\}. \tag{5.5}$$

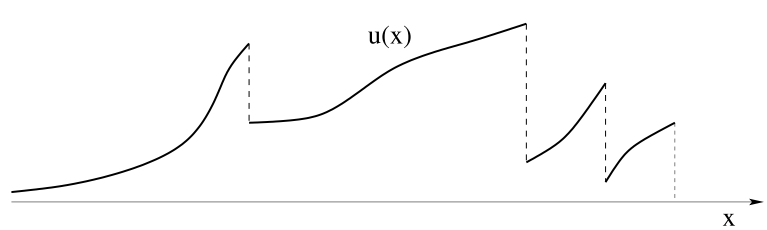

Figure 5: A function u in the class $\mathcal{F}$, as defined at (5.5).

To introduce a concept of "solution" for the non-conservative system (5.1), in the case of functions $w = (w_1, w_2)$ with both components in $\mathcal{F}$, one needs to assign the speed of downward jumps. This can be defined in terms of a non-conservative product [16, 23, 24]. For example, one could require these speeds to be the average values:

$$\lambda_1(w_1^-, w_1^+, w_2) \doteq \frac{\displaystyle\int_{w_1^+}^{w_1^-} \lambda_1(s, w_2)\, ds,}{w_1^- - w_1^+}, \qquad \lambda_2(w_1, w_2^-, w_2^+) \doteq \frac{\displaystyle\int_{w_2^+}^{w_2^-} \lambda_2(w_1, s)\, ds}{w_2^- - w_2^+}. \tag{5.6}$$

For our purpose, however, it will be convenient to directly introduce two additional functions, prescribing the speed of the jumps:

$$\Lambda_1(w_1^-, w_1^+, w_2), \qquad \Lambda_2(w_2^-, w_2^+, w_1). \tag{5.7}$$

We shall assume that Λ_1, Λ_2 depend smoothly on all variables. Moreover, for $w_i^+ < w_i^-$, these speeds should satisfy

$$\begin{aligned} \lambda_1(w_1^+, w_2) \;&\leq\; \Lambda_1(w_1^-, w_1^+, w_2) \;\leq\; \lambda_1(w_1^-, w_2), \\ \lambda_2(w_1, w_2^+) \;&\leq\; \Lambda_2(w_1, w_2^-, w_2^+) \;\leq\; \lambda_2(w_1^-, w_2). \end{aligned} \tag{5.8}$$

$$\begin{aligned} \Big|\Lambda_1(w_1^-, w_1^+, w_2) - \lambda_1(w_1^-, w_1^+, w_2)\Big| \;&\leq\; \kappa|w_1^+ - w_1^-|^2, \\ \Big|\Lambda_2(w_2^-, w_2^+, w_1) - \lambda_2(w_2^-, w_2^+, w_1)\Big| \;&\leq\; \kappa|w_2^+ - w_2^-|^2. \end{aligned} \tag{5.9}$$

Definition 5.1. *Let the jumps speeds Λ_i in (5.7) be given. A piecewise Lipschitz function $w = w(t,x)$, with $w_i(t,\cdot) \in \mathcal{F}$ for every $t \in [0,T]$, $i = 1,2$, is called a* **generalized solution** *to the hyperbolic system (5.1) if the following holds. Consider the limits*

$$w_i^{\pm}(t,x) = \lim_{y \to x\pm} w_i(t,y)$$

which are well defined because $w_i \in \mathcal{F}$. Then the domain $]0,T[\times\mathbb{R}$ can be decomposed as

$$]0,T[\times\mathbb{R} \;=\; V \cup \Big(\bigcup_j \gamma_{1j}\Big) \cup \Big(\bigcup_j \gamma_{2j}\Big) \cup J, \tag{5.10}$$

where

(i) V is an open set where w is continuous. The equations (5.1) are satisfied a.e. on this set.

(ii) Each $\gamma_{1j} :]t_j^-, t_j^+[\mapsto \mathbb{R}$ is a Lipschitz curve where a downward 1-jump occurs. Namely, $w_1^+ < w_1^-$, $w_2^+ = w_2^-$. The speed of this curve is $\dot\gamma_{ij}(t) = \Lambda_1(w_1^-, w_1^+, w_2)$.

Similarly, each $\gamma_{2j} :]\tau_j^-, \tau_j^+[\mapsto \mathbb{R}$ is a Lipschitz curve where a downward 2-jump occurs. Namely, $w_1^+ = w_1^-$, $w_2^+ < w_2^-$. The speed of this curve is $\dot\gamma_{2j}(t) = \Lambda_2(w_1, w_2^-, w_2^+)$.

(iii) The set J consist of finitely many points, where two or more jumps interact.

Given initial data

$$w_i(0,x) \;=\; \overline{w}_i(x) \;\in\; [a_i, b_i], \qquad i = 1,2, \quad x \in \mathbb{R}, \tag{5.11}$$

in the class of piecewise Lipschitz functions $\mathcal{F}$, generalized solutions to (5.1) are easily constructed.

Proposition 5.1. *Let the system (5.1) satisfy* **(A1)**, *and consider initial data (5.11) with* $\overline{w}_1, \overline{w}_2 \in \mathcal{F}$. *Then the Cauchy problem has a unique generalized solution, with components* $w_i(t,\cdot) \in \mathcal{F}$ *for all* $t > 0$.

Proof. 1. The construction of local solutions within the class of piecewise Lipschitz functions with downward jumps is a straightforward task. It can be accomplished by solving the system (1.3) in the regions where the functions w_1, w_2 are Lipschitz, then locating the positions of the finitely many downward jumps, using the ODE determined by the speeds Λ_1, Λ_2. In view of (5.11), is clear that the components satisfy $w_i(t,x) \in [a_i, b_i]$ for all t, x.

2. We now check that the components of the solution remain in $\mathcal{F}$. To show that $w_{1,x}(t,x) \geq 0$ for all t, x, we differentiate the first equation in (5.1) and obtain

$$w_{1,xt} + \lambda_1(w_1, w_2) w_{1,xx} \;=\; -\lambda_{1,w_1} w_{1,x}^2 - \lambda_{1,w_2} w_{1,x} w_{2,x}\,. \tag{5.12}$$

Along a characteristic $t \mapsto x(t)$ with $\dot{x} = \lambda_1\big(w(t,x)\big)$, this implies

$$\frac{d}{dt} w_{1,x}(t, x(t)) \;\geq\; -C\, w_{1,x}\big(t, x(t)\big), \tag{5.13}$$

for some constant C. At a time τ when this characteristic crosses a 2-jump with speed Λ_2, the gradients $w_{1,x}^{\pm} = w_{1,x}\big(\tau\pm, x(\tau\pm)\big)$ before and after the crossing are related by

$$\frac{w_{1,x}^+}{w_{1,x}^-} \;=\; \frac{\Lambda_2 - \lambda_1^-}{\Lambda_2 - \lambda_1^+}\,. \tag{5.14}$$

Here $\lambda_1^{\pm}$ denote the 1-characteristic speed before and after the crossing. Combining (5.12) with (5.14), we conclude that $w_{1,x}(t, x(t)) \geq 0$ at all times $t \geq 0$. As a consequence, no new jumps ever develop, and the components of the solution remain in $\mathcal{F}$.

3. Finally, we observe that two jumps of opposite families simply cross each other without changing strength. Two jumps of the same family join together in a single jump. As a consequence, the total number of jumps can only decrease, and the total number of interactions between jumps is finite. The solution can thus be constructed globally in time, in a finite number of steps. □

5.1 Decay of positive gradients.

In this subsection, we wish to prove that the positive gradients of the components: $w_{1,x}, w_{2,x}$ satisfy an Oleinik-type decay estimate, provided that the jump speeds Λ_i at (5.7) are suitably chosen.

Theorem 5.2. *Let the characteristic speeds satisfy the assumptions* **(A1)-(A2)**. *Then it is possible to choose jump speeds* Λ_1, Λ_2 *as in (5.8)-(5.9), such that, for some constant* $C > 0$, *the following holds. For every piecewise Lipschitz solution* $w = (w_1, w_2)$ *of (5.1) with components* $w_1, w_2 \in \mathcal{F}$, *one has the decay estimates*

$$\frac{w_i(t,x_2) - w_i(t,x_1)}{x_2 - x_1} \;\leq\; \frac{C}{t} \qquad \text{for all } t > 0,\ x_1 < x_2\,,\ i = 1,2\,. \tag{5.15}$$

Proof. 1. As a first step, consider any Lipschitz solution of (5.1), without jumps. Differentiating the second equation w.r.t. x, we obtain

$$w_{2,xt} + \lambda_2(w_1, w_2)w_{2,xx} \;=\; -\lambda_{2,w_1} w_{1,x} w_{2,x} - \lambda_{2,w_2} w_{2,x}^2\,. \tag{5.16}$$

In particular, if $t \mapsto x(t)$ is a 2-characteristic (see Fig. 6, left), so that

$$\dot{x} \;=\; \lambda_2\big(w_1(t,x), w_2(t,x)\big), \tag{5.17}$$

we find

$$\frac{d}{dt} w_{2,x}\big(t, x(t)\big) \;=\; -\lambda_{2,w_1} w_{1,x} w_{2,x} - \lambda_{2,w_2} w_{2,x}^2\,. \tag{5.18}$$

Observing that

$$w_2\big(t, x(t)\big) \;=\; \overline{w}_2 \tag{5.19}$$

is a constant, while

$$\frac{d}{dt} w_1\big(t, x(t)\big) \;=\; \big[\lambda_2(w_1, w_2) - \lambda_1(w_1, w_2)\big]\, w_{1,x}\,, \tag{5.20}$$

from (5.18) one obtains

$$\frac{d}{dt} w_{2,x}\big(t, x(t)\big) \;=\; -\frac{\lambda_{2,w_1}}{\lambda_2 - \lambda_1} \cdot \left(\frac{d}{dt} w_1\big(t, x(t)\big)\right) w_{2,x} - \lambda_{2,w_2} w_{2,x}^2\,. \tag{5.21}$$

Setting $z(t) \doteq w_{2,x}\big(t, x(t)\big)$, we thus obtain the ODE

$$\dot{z}(t) \;=\; -\frac{\lambda_{2,w_1}}{\lambda_2 - \lambda_1} \cdot \left(\frac{d}{dt} w_1\big(t, x(t)\big)\right) z(t) - \lambda_{2,w_2} z^2(t)\,. \tag{5.22}$$

To integrate (5.22), we introduce the function

$$\Phi(w_1, w_2) \;\doteq\; -\int_0^{w_1} \frac{\lambda_{2,w_1}(s, w_2)}{\lambda_2(s, w_2) - \lambda_1(s, w_2)}\, ds. \tag{5.23}$$

Since $w_2\big(t, x(t)\big) = \overline{w}_2$ is constant in time, we can write (5.22) in the form

$$\begin{aligned} \dot{z}(t) \;&=\; \frac{d}{dt}\Phi\big(w_1(t,x(t)), \overline{w}_2\big)\, z(t) - \lambda_{2,w_2}\big(w_1(t,x(t)), \overline{w}_2\big)\, z^2(t) && (5.24)\\ &\leq\; \frac{d}{dt}\Phi\big(w_1(t,x(t)), \overline{w}_2\big)\, z(t) - \kappa\, z^2(t). && (5.25) \end{aligned}$$

2. Assume that $z(0) > 0$ and set $\phi(t) \doteq \Phi\big(w_1(t,x(t)), \overline{w}_2\big)$. From (5.25) it follows

$$\dot{z} \;\leq\; \dot{\phi}\, z - \kappa\, z^2, \qquad\qquad \frac{d}{dt}\left(\frac{1}{z}\, e^{\phi}\right) \;\geq\; \kappa\, e^{\phi},$$

$$\frac{e^{\phi(t)}}{z(t)} \;\geq\; \frac{e^{\phi(0)}}{z(0)} + \kappa \int_0^t e^{\phi(\tau)} d\tau \;\geq\; \kappa \int_0^t e^{\phi(\tau)} d\tau.$$

Therefore

$$z(t) \;\leq\; \frac{e^{\phi(t)}}{\kappa \int_0^t e^{\phi(\tau)} d\tau}\,. \tag{5.26}$$

Let Φ^- and Φ^+ be respectively a lower and an upper bound for the function Φ defined at (5.23). In particular, $\Phi\big(w_1(t,x(t)),\overline{w}_2\big) \in [\Phi^-,\Phi^+]$. By (5.26) it now follows

$$z(t) \;\leq\; \frac{e^{\Phi^+-\Phi^-}}{\kappa\, t} \qquad\qquad \text{for all } t>0. \tag{5.27}$$

3. Next, assume that w_1 is only piecewise Lipschitz, with downward jumps. We compute the change in $w_{2,x}(t,x(t))$ along a characteristic in two cases:

(i) The 2-characteristic crosses a single 1-shock, with left and right states $(w_1^-,\overline{w}_2)$, $(w_1^+,\overline{w}_2)$, as in Fig. 6, center. If this shock travels with speed Λ_1, the gradients before and after the interaction are computed by

$$\frac{w_{2,x}^+}{w_{2,x}^-} \;=\; \frac{\lambda_2(w_1^-,\overline{w}_2)-\Lambda_1}{\lambda_2(w_1^+,\overline{w}_2)-\Lambda_1}\,. \tag{5.28}$$

(ii) The 2-characteristic crosses a family of 1-compressions, joining the same left and right states, as shown in Fig. 6, right. In this case, according to (5.21), the gradients $w_{2,x}^-$, $w_{2,x}^+$ before and after the crossing are related by the ODE

$$\frac{dz}{ds} \;=\; -\frac{\lambda_{2,w_1}(s,\overline{w}_2)}{\lambda_2(s,\overline{w}_2)-\lambda_1(s,\overline{w}_2)}\,z(s), \qquad z(w_1^-)=w_{2,x}^-, \qquad z(w_1^+)=w_{2,x}^+\,. \tag{5.29}$$

To compare the two above expressions, consider the middle point

$$\widehat{w}_1 \;\doteq\; \frac{w_1^-+w_1^+}{2}\,,$$

and assume that the shock speed is precisely the characteristic speed at this middle point:

$$\Lambda_1 \;\doteq\; \lambda_1(\widehat{w}_1,\overline{w}_2) \;=\; \frac{1}{w_1^--w_1^+}\int_{w_1^+}^{w_1^-}\lambda_1(s,\overline{w}_2)\,ds+\mathcal{O}(1)\cdot(w_1^--w_1^+)^2. \tag{5.30}$$

Since the map $w_{2,x}^-\mapsto w_{2,x}^+$ is linear, without loss of generality, we can assume $w_{2,x}^-=1$. We wish to compute the difference between the two values for $w_{2,x}^+$ determined by (5.28) and (5.29), respectively.

Integrating (5.29) one obtains

$$\begin{aligned}\ln z(s)\Big|_{w_1^-}^{w_1^+} \;&=\; \ln w_{2,x}^+ \;=\; \int_{w_1^+}^{w_1^-}\frac{\lambda_{2,w_1}(s,\overline{w}_2)}{\lambda_2(s,\overline{w}_2)-\lambda_1(s,\overline{w}_2)}\,ds\\ &=\; \frac{\lambda_{2,w_1}(\widehat{w}_1,\overline{w}_2)}{\lambda_2(\widehat{w}_1,\overline{w}_2)-\lambda_1(\widehat{w}_1,\overline{w}_2)}\cdot(w_1^--w_1^+)+\mathcal{O}(1)\cdot(w_1^--w_1^+)^3.\end{aligned} \tag{5.31}$$

Notice that the last equality is trivially true because the integrand is a smooth function. On the other hand, from (5.28) it follows

$$\begin{aligned}\ln w_{2,x}^+ \;&=\; \ln\big(\lambda_2(w_1^-,\overline{w}_2)-\Lambda_1\big)-\ln\big(\lambda_2(w_1^+,\overline{w}_2)-\Lambda_1\big)\\ &=\; \int_{w_1^+}^{w_1^-}\frac{\lambda_{2,w_1}(s,\overline{w}_2)}{\lambda_2(s,\overline{w}_2)-\Lambda_1}\,ds\\ &=\; \frac{\lambda_{2,w_1}(\widehat{w}_1,\overline{w}_2)}{\lambda_2(\widehat{w}_1,\overline{w}_2)-\lambda_1(\widehat{w}_1,\overline{w}_2)}\cdot(w_1^--w_1^+)+\mathcal{O}(1)\cdot(w_1^--w_1^+)^3.\end{aligned} \tag{5.32}$$

Comparing the two expressions for $w^+_{2,x}$ in (5.31) and (5.32) we see that they only differ for an infinitesimal of order $\mathcal{O}(1)\cdot(w_1^- - w_1^+)^3$. Hence, by changing the shock speed Λ_1 by an amount $\mathcal{O}(1)\cdot(w_1^- - w_1^+)^2$, we can render the value in (5.28) smaller than the one determined by (5.29). $\square$

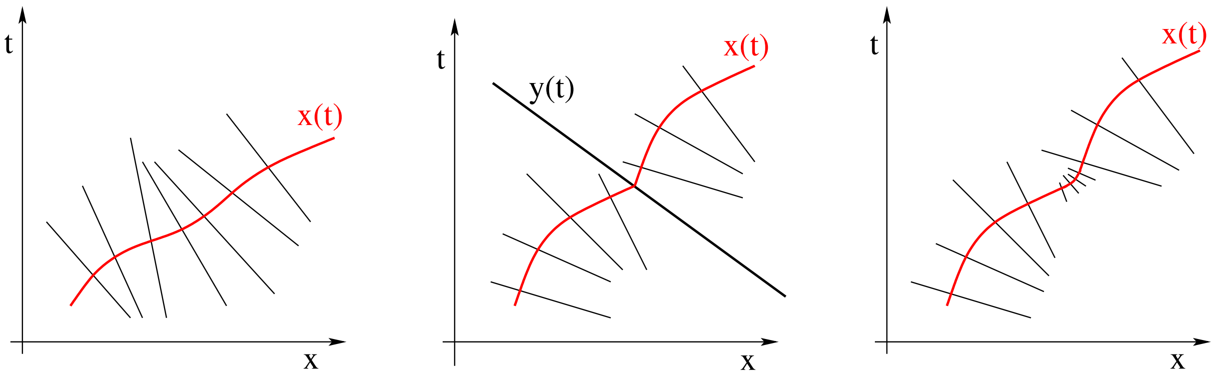

Figure 6: Left: a 2-characteristic $x(t)$, crossing a family of 1-characteristics. Center: a 2-characteristic $x(t)$, crossing a family of 1-rarefactions and a 1-shock. Right: an approximate configuration, where the 1-shock is replaced by 1-compressions. By slightly changing the speed Λ_1 assigned to the jump at $y(\cdot)$, the derivative $w_{2,x}\big(t, x(t)\big)$ will be smaller than in the case of smooth compression waves.

6 Approximate solutions to the system of conservation laws

By the previous analysis, one can construct a dense set of generalized solutions to the non-conservative system (5.1) which are piecewise Lipschitz with finitely many jumps. These have very similar properties as the solutions to a scalar conservation law with strictly convex flux.

If shock and rarefaction curves for (1.1) do not coincide, in the presence of jumps, these generalized solutions are not entropy weak solutions to the original 2×2 system of conservation laws (1.1). We remark, however, that the difference is of third order w.r.t. the size $\sigma = w_i^- - w_i^+$ of the jumps. More precisely (see Fig. 7), consider a jump in the first Riemann coordinate.

- Let $w^- = (w_1^-, w_2)$, $w^+ = (w_1^+, w_2)$, with $w_1^+ < w_1^-$ be the left and right states for a 1-jump in the Riemann coordinates. Let $u^- = u(w_1^-, w_2)$, $u^+ = u(w_1^+, w_2)$ be the corresponding values of the conserved variables. Let $u = u(t,x)$ be the exact solution of the Riemann problem for (1.1), with left and right states u^-, u^+. Going back to Riemann coordinates, this yields a function $w^{exact}(t,x)$.
- Next, call $w^{diag}(t,x)$ the solution to the diagonal, nonconservative system (5.1), consisting of a single jump traveling with speed Λ_1, namely

$$w^{diag}(t,x) \;=\; \begin{cases} (w_1^-, w_2) & \text{if} \quad x \,<\, t\,\Lambda_1(w_1^-, w_1^+, w_2),\\ (w_1^+, w_2) & \text{if} \quad x \,>\, t\,\Lambda_1(w_1^-, w_1^+, w_2). \end{cases} \tag{6.1}$$

Recalling that shock and rarefaction curves have a second order tangency [7, 22, 30], by the assumption (5.9) on the wave speed we conclude that the difference has size

$$\frac{1}{t}\int \big|w^{exact}(t,x) - w^{diag}(t,x)\big|\,dx \;=\; \mathcal{O}(1)\cdot|w_1^+ - w_1^-|^3. \tag{6.2}$$

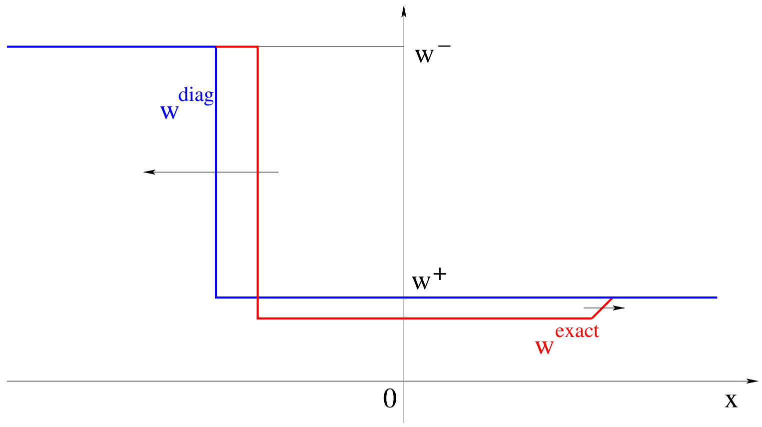

Figure 7: Two ways for solving a Riemann problem where the initial data contain a single jump in the coordinate w_1. The function w^{diag} consists of a single jump traveling with speed Λ_1 as in (6.1). The function w^{exact} is the exact solution to the conservation law (1.1), written in Riemann coordinates (w_1, w_2). For every $t > 0$, the $\mathbf{L}^1$ difference between the two solutions is $\mathcal{O}(1) \cdot |w^+ - w^-|^3\, t$.

This suggests a possible way to construct approximate solutions to the Cauchy problem for the 2×2 system of conservation laws

$$u_t + f(u)_x \;=\; 0, \qquad\qquad u(0,x) \;=\; \bar u(x). \tag{6.3}$$

Fix $\varepsilon > 0$, and define the times $t_k = k\varepsilon$, $k = 0, 1, 2, \ldots$

Choose an initial datum with $\|u_0 - \bar u\|_{\mathbf{L}^1} \le \varepsilon$ and such that the corresponding Riemann coordinates $w_{1,0}, w_{2,0}$ lie inside $\mathcal{F}$.

By induction on k, assume that $u_k = u(t_k, \cdot)$ has been constructed, in such a way that the corresponding Riemann coordinates satisfy $w_{1,k}, w_{2,k} \in \mathcal{F}$.

For $t \in [t_k, t_{k+1}[\,$, let $w(t,\cdot)$ be the generalized solution to the diagonal system (5.1), with initial data

$$w(t_k, x) \;=\; w_k(x).$$

By Proposition 5.1, this will be a piecewise Lipschitz function, with components $w_i(t,\cdot) \in \mathcal{F}$, $i = 1, 2$.

If this generalized solution contains jumps, then it will not be a solution to the original problem (6.3). We thus need to add a source to account for this difference. To fix ideas, for $\tau \in [t_k, t_{k+1}[\,$, let $x_\alpha(\tau)$, $\alpha \in \{1, \ldots, N\}$ be the locations of these jumps, and let $w_\alpha^-(\tau), w_\alpha^+(\tau) \in \mathbb{R}^2$ be the left and right values of the corresponding Riemann coordinates. As in (6.2), we consider the two different ways to solve the Riemann problem with data $\big(w_\alpha^-(\tau), w_\alpha^+(\tau)\big)$, and define the vector

$$\mathbf{v}_\alpha(\tau) \;\doteq\; \frac{1}{t}\int \big[w^{exact}(t,x) - w^{diag}(t,x)\big]\, dx \;\in\; \mathbb{R}^2. \tag{6.4}$$

Note that, by the self-similarity of the solutions to the Riemann problem, the right hand side does not depend on t. In turn, this yields a vector measure μ, concentrating a mass $\mathbf{v}_\alpha$ at each point x_α. More precisely, for every continuous function $\varphi : [t_k, t_{k+1}] \times \mathbb{R} \mapsto \mathbb{R}$,

$$\int \varphi\, d\mu \;=\; \sum_\alpha \int \varphi\big(\tau, x_\alpha(\tau)\big)\mathbf{v}_\alpha(\tau)\, d\tau. \tag{6.5}$$

To compensate for this error, at the terminal time t_{k+1} we perform a restarting procedure, and define

$$w(t_{k+1}, x) \doteq w(t_{k+1}-, x) + g_k(x), \tag{6.6}$$

where $g_k : \mathbb{R} \mapsto \mathbb{R}^2$ is a piecewise Lipschitz function, with components in $\mathcal{F}$, which approximates the integral of the measure μ over the interval $[t_k, t_{k+1}[$. For example, we could require

$$\left| \int g_k(x)\phi(x)\, dx - \sum_\alpha \int_{t_k}^{t_{k+1}} \phi\big(x_\alpha(\tau)\big)\mathbf{v}_\alpha(\tau)\, d\tau \right| < \varepsilon \tag{6.7}$$

for every Lipschitz continuous test function ϕ with Lipschitz constant $Lip(\phi) \le \varepsilon^{-1}$. Assuming that $w(t_{k+1}, \cdot)$ remains in the domain $\mathcal{D}$ at (5.2) where the Riemann coordinates are defined (see [31] for a general result on positive domain invariance) the induction can then be continued.

In Riemann coordinates, we thus construct a solution to (5.1) with sources added at the discrete set of times $t_1, t_2, \ldots$ Since the sum of the cubes of the shock strengths can be controlled by the decrease of a strictly convex entropy, the total strength of the sources is uniformly bounded:

$$\sum_k \|\tilde{g}_k\|_{\mathbf{L}^1} \le C.$$

In view of the strong regularizing properties (5.15) of the homogeneous system (5.1), this leads us to conjecture that all these approximate solutions will enjoy the same regularity properties discussed in the previous sections for scalar balance laws with a bounded source.

A proof of this fact, however, is far from straightforward. The main difficulty stems from the fact that the system (5.1) is not conservative and does not generate a contractive semigroup. On the positive side, we observe that the measure μ at (6.5), accounting for entropy dissipation, is absolutely continuous w.r.t. 1-dimensional Hausdorff measure. Using the strict hyperbolicity assumption (5.3), one can show that all source functions g_k are bounded in $\mathbf{L}^\infty$.

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