MA 401, Applied Differential Equations, Fall 2021

Tien Khai Nguyen, Department of Mathematics, NCSU

1 Introduction

1.1 Classification of Differential Equations

Definition 1.1 *A differential equation is an equation which contains derivatives of the unknown (Usually it is a mathematical model of some physical phenomenon).*

Example 1.
a) Model of population of ecology:

$$\dot{u}(t) \;=\; ru(t)\left(1-\frac{u(t)}{K}\right) \qquad (ODE)$$

where

- r, K are given constants;
- t is time variable and u is an unknown function of t.

b) Model of traffic flow on a single road

$$u_t(x,t) + f(u(x,t))_x \;=\; 0 \qquad (PDE)$$

where

- t is time variable and x is state variable;
- f is a given flux;
- u is a unknown function of t and x.

Notations:

- $\dot{u}(t) \;=\; \frac{du}{dt}$: ordinary derivative.
- $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $u_{tx} = \frac{\partial^2 u}{\partial t \partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$: partial derivatives.

There are two classes of differential equations:

- Ordinary differential equations (ODEs).
- Partial differential equations (PDEs).

1.2 A review on ordinary differential equations

Definition 1.2 *A ordinary differential equation is an equation with ordinary derivative of the unknown u that depends only on one variable.*

First order differential equations. Consider the ordinary differential equaition

$$u'(t) \;=\; f(t, u(t))$$

where f is a given function and u is an unknown of t.

Goal: Solve the above ODE.

1.2.1 Linear equations: Method of integrating factors

The function $f(t, u)$ is linear function in u, we can write

$$f(t, u) \;=\; -\,p(t) \cdot u + q(t)$$

where p, q are given functions of t.

We will study the equation

$$u'(t) + p(t)u(t) \;=\; q(t)\,. \tag{1.1}$$

Method of integrating factors.

Step 1: Compute the integrating factor

$$\mu(t) \;=\; \exp\left(\int p(t)\;dt\right).$$

Step 2: The general solution is

$$u(t) \;=\; \frac{1}{\mu(t)} \cdot \left[\int \mu(t)q(t)\;dt + C\right].$$

Example 2. Solving the following initial value problems

a) $u'(t) + u(t) \;=\; e^{2t}, \qquad u(0) \;=\; 1.$

b) $tu'(t) - u(t) \;=\; t^2e^{-t}$ for all $t \geq 1, \qquad u(1) = 1 - e^{-1}.$

Answer. (a) We have

$$p(t) \;=\; 1, \qquad q(t) \;=\; e^{2t}\,.$$

The integrating factor

$$\mu(t) \;=\; \exp\left(\int p(t)\;dt\right) \;=\; \exp\left(\int 1\;dt\right) \;=\; e^t\,.$$

The general solution

$$u(t) = \frac{1}{\mu(t)} \cdot \left[\int \mu(t) q(t)\ dt + C \right]$$
$$= \frac{1}{e^t} \cdot \left[\int e^{3t}\ dt + C \right] = \frac{1}{3} \cdot e^{2t} + C \cdot e^{-t}.$$

The initial condition implies that

$$1 = u(0) = \frac{1}{3} + C \qquad \Longrightarrow \qquad C = \frac{2}{3}.$$

The solution

$$u(t) = \frac{1}{3} \cdot e^{2t} + \frac{2}{3} \cdot e^t.$$

(b). Rewrite the equation

$$u'(t) - \frac{1}{t} \cdot u(t) = te^{-t}.$$

We have

$$p(t) = -\frac{1}{t} \qquad \text{and} \qquad q(t) = te^{-t}.$$

The integrating factor

$$\mu(t) = \exp\left(\int -\frac{1}{t}\ dt \right) = e^{-\ln(t)} = \frac{1}{t}.$$

The general solution

$$u(t) = \frac{1}{\mu(t)} \cdot \left[\int \mu(t) q(t)\ dt + C \right]$$
$$= t \cdot \left[\int e^{-t}\ dt + C \right] = -te^{-t} + Ct.$$

The initial condition implies that

$$e^{-1} + 1 = u(1) = e^{-1} + C \qquad \Longrightarrow \qquad C = 1.$$

The solution

$$u(t) = -te^{-t} + t.$$

1.2.2 Separable equations

Assume that $f(t,u)$ can be separated

$$f(t,u) = \frac{M(t)}{N(u)}.$$

We will study the equation

$$\frac{du}{dt} = f(t,u) = \frac{M(t)}{N(u)}. \tag{1.2}$$

Equivalently,

$$N(u)du \;=\; M(t)dt \qquad \Longrightarrow \qquad \int N(u)\; du \;=\; \int M(t)\; dt$$

and it yields an *implicit* formula for the solution u

Example 3. Consider the equation

$$u'(t) \;=\; \frac{\cos t}{1-u^2}, \qquad\qquad u(\pi/2) \;=\; 3\,.$$

We can separate the variables

$$(1-u^2)du \;=\; \cos t\; dt \qquad \Longrightarrow \qquad \int (1-u^2)du \;=\; \int \cos t\; dt\,.$$

This yields

$$u - \frac{1}{3}u^3 \;=\; \sin(t) + C\,.$$

Since $u(\pi/2) = 3$, we have

$$3 - \frac{1}{3}\cdot 3^3 \;=\; 1 + C \qquad \Longrightarrow \qquad C \;=\; -7\,.$$

The solution u is given implicitly as

$$u - \frac{1}{3}u^3 \;=\; \sin(t) + 7\,.$$

1.2.3 Second Order Linear Equations

The general form of these equations is

$$a_2(t)u''(t) + a_t(t)u'(t) + a_0u(t) \;=\; b(t)\,.$$

where a_0, a_1, a_2 and b are given functions and u is an unknown of t.

If $b(t) \equiv 0$, we call it *homogeneous.* Otherwise, it is called *non-homogeneous.*

1.2.4 Homogeneous equations with constant coefficients

The linear equation

$$au'' + bu' + cu \;=\; 0 \tag{1.3}$$

where a, b, c are given constants.

The principle of superposition. *If u_1 and u_2 are solutions of (1.3), then $u = c_1u_1 + c_2u_2$ is also a solution of (1.3) for arbitrary constants c_1, c_2.*

How to find u_1 and u_2?

The characteristic equation of (1.3)

$$ar^2 + br + c \;=\; 0\,. \tag{1.4}$$

Denote by

$$D = b^2 - 4ac\,.$$

Three cases can occur:

- If $D > 0$ then (1.4) has two real roots

$$r_1 = \frac{-b+\sqrt{D}}{2a}, \qquad r_2 = \frac{-b-\sqrt{D}}{2a}$$

  Two particular solutions

$$u_1(t) = e^{r_1 t}, \qquad u_2(t) = e^{r_2 t}\,.$$

  The general solution of (1.3) is

$$u(t) = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}\,.$$

- If $D = 0$ then (1.4) has a repeated root

$$r_1 = r_2 = \bar{r} = \frac{-b}{2a}\,.$$

  Two particular solutions

$$u_1(t) = e^{\bar{r}t}, \qquad u_2(t) = te^{\bar{r}t}\,.$$

  The general solution of (1.3) is

$$u(t) = c_1 \cdot e^{\bar{r}t} + c_2 \cdot te^{\bar{r}t}\,.$$

- If $D < 0$ then (1.4) has two complex conjugate roots

$$r_1 = \alpha + i\beta, \qquad r_2 = \alpha - i\beta$$

  where

$$\alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{|D|}}{2a}\,.$$

  Two particular solutions

$$u_1(t) = e^{\alpha t}\cdot\cos(\beta t), \qquad u_2(t) = e^{\alpha t}\cdot\sin(\beta t)\,.$$

  The general solution of (1.3) is

$$u(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)\,.$$

Example 4. Solve the second order linear ODE

$$u'' + 3u' + 2u = 0 \quad \text{with} \quad u(0) = 1, u'(0) = 2\,.$$

Answer. The characteristic equation

$$r^2 + 3r + 2 \ = \ 0 \, .$$

Since $D = 3^2 - 4 \cdot 2 \cdot 1 = 1 > 0$, we have

$$r_1 \ = \ -1, \qquad\qquad r_2 \ = \ -2 \, .$$

The general solution

$$u(t) \ = \ c_1 e^{-t} + c_2 e^{-2t} \, .$$

Initial conditions imply that

$$1 \ = \ u(0) \ = \ c_1 + c_2$$

and

$$2 \ = \ u'(0) \ = \ -c_1 - 2c_2 \, .$$

Solving the system of algebra equations, we obtain

$$c_1 \ = \ 4, \qquad\qquad c_2 \ = \ -3 \, .$$

The solution

$$u(t) \ = \ 4 \cdot e^{-t} - 2e^{-2t} \, .$$

□

1.2.5 Cauchy-Euler equations

Consider the second order equation of the form

$$ax^2 u'' + bxu' + cu \ = \ 0 \, .$$

Try to look for particular solutions of the form $u(x) \ = \ x^r$. This yields the characteristic equation

$$ar(r-1) + br + c \ = \ 0 \, .$$

This quadratic equation has two roots r_1, r_2. Three cases may occur:

- If r_1 and r_2 are two distinct real roots, then the general solution

$$u(x) \ = \ c_1 x^{r_1} + c_2 x^{r_2} \, .$$

- If $r_1 = r_2 = \bar{r}$, then the general solution

$$u(x) \ = \ c_1 x^{\bar{r}} + c_2 x^{\bar{r}} \ln x$$

- If r_1 and r_2 are two complex conjugate roots, i.e.,

$$r_1 \ = \ \alpha + i\beta, \qquad\qquad r_2 \ = \ \alpha - i\beta \, .$$

  then the general solution

$$u(x) \ = \ c_1 x^{\alpha} \sin(\beta \ln x) + c_2 x^{\alpha} \cos(\beta \ln x) \, .$$

1.3 Partial Differential Equations.

Definition 1.3 *A partial differential equation is an equation with partial derivatives of the unknown u that depends on several variables.*

Some basic concepts related to differential equations:

- Order of PDEs: the highest order of derivatives.
- Linear PDEs: the term with u and its derivatives are in a linear form.
- Nonlinear PDEs: the term with u and its derivatives are in a nonlinear form.

Example 1. Let u be a function of two variables t, x. Identify the order and linearity of the following equations.

(a). $u_t + 2u_x \;=\; 0$

(b). $u_{tt} \;=\; c^2 \cdot u_{xx}$ (Wave equation)

(c). $u_{xx} + u_{yy} \;=\; 0$ (Laplace equation)

(d). $u_t \;=\; u_{xx} + u_{yy}$ (2D heat equation)

(e). $u_t + \left(\frac{u^2}{2}\right)_x \;=\; 0$ (Burger's equation)

(f). $u_{xx} + u_{yy} \;=\; f(x,y)$ (Poisson equation)

(g). $u_{tt} - 4u_{xt} + u_{xx} + x^3 u + t u_x \;=\; 0\,.$

Definition 1.4 *The function u is a solution if it satisfies the equation and any boundary or initial conditions.*

Example 2. (a) Given any smooth functions F, the function

$$u(x,t) \;\doteq\; F(2t-x) \qquad \text{for all } (t,x) \in (0,\infty) \times \mathbb{R}$$

is a solution of the equation in (a) of example 1.

Proof. Using the change rule, one computes that

$$u_x(x,t) \;=\; \frac{d}{dx}F(2t-x) \;=\; -F'(2t-x) \qquad \text{and} \qquad u_t(x,t) \;=\; \frac{d}{dt}F(2t-x) \;=\; 2F'(2t-x)$$

This implies that

$$u_t + 2u_x \;=\; 2F'(2t-x) - 2F'(2t-x) \;=\; 0\,.$$

□

(b) Showing that the function

$$u(x,t) \;=\; e^{f(t)} \cdot g(x)$$

solves the equation

$$u \cdot u_{tx} \;=\; u_t \cdot u_x$$

Proof. Using the change rule, one computes

$$u_t \;=\; \frac{d}{dt}e^{f(t)} \cdot g(x) \;=\; f'(t)e^{f(t)}g(x), \qquad u_x \;=\; e^{f(t)} \cdot \frac{d}{dx}g(x) \;=\; e^{f(t)}g'(x)$$

and

$$u_{tx} \;=\; \frac{d}{dt}e^{f(t)} \cdot \frac{d}{dx}g(x) \;=\; f'(t)g'(x)e^{f(t)}\,.$$

Therefore

$$u \cdot u_{tx} \;=\; e^{f(t)}g(x) \cdot f'(t)g'(x)e^{f(t)} \;=\; f'(t)e^{f(t)}g(x) \cdot e^{f(t)}g'(x) \;=\; u_t \cdot u_x\,.$$

□

Definition 1.5 *Let L be a differential operator. We say that*

(H) The equation $L(u) = 0$ is homogeneous.

(NH) The quation $L(u) = f$ is non-homogeneous for all $f \neq 0$.

The principle of superposition. Assume that L is a linear differential operator, i.e.,

$$L(u+v) \;=\; L(u) + L(v) \qquad \text{and} \qquad L(\lambda \cdot u) \;=\; \lambda \cdot L(u)\,.$$

Then the followings hold:

(i) If u_1 and u_2 are solutions of the homogeneous equation

$$L(u) \;=\; 0$$

then $u = \lambda_1 \cdot u_1 + \lambda_2 \cdot u_2$ is also a solution for any $\lambda_1, \lambda_2 \in \mathbb{R}$.

(ii) If u_1 is a solution of the homogeneous equation $L(u) = 0$, and u_2 is a solution of the non-homogeneous equation $L(u) = f$, then $u = u_1 + u_2$ is a solution of $L(u) = f$.

Classification of PDEs. Consider the second order PDEs

$$Au_{xx} + Bu_{xt} + Cu_{tt} + F(x,t,u,u_x,u_t) \;=\; 0 \tag{1.5}$$

where A, B, C are given constants, F is a given function, and u is an unknown.

Denote by

$$\Delta \;=\; B^2 - 4AC\,.$$

There are three cases:

- If $\Delta > 0$ then (3.6) is hyperbolic;
- If $\Delta < 0$ then (3.6) is elliptic;
- If $\Delta = 0$ then (3.6) is parabolic.

2 Scalar Conservation Laws

General form

$$u_t + \frac{d}{dx}\Phi(t,x,u) \ = \ g$$

where

- u is the density which depends on the time variable $t \geq 0$ and the state variable $x \in \mathbb{R}$;
- Φ is a given flux;
- g is a given source term (external force).

Example 1. (Traffic flow) On a single road, let's denote by

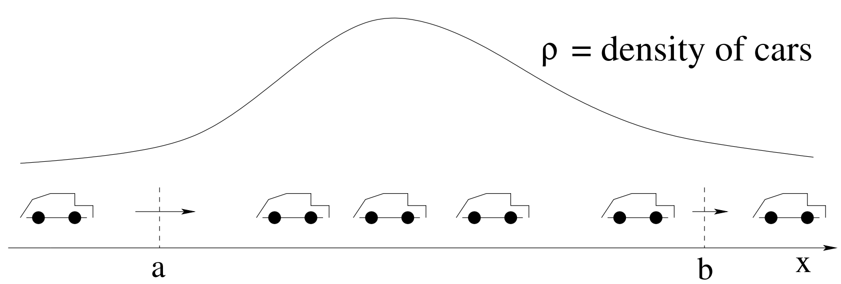
- $u(x,t)$ is the traffic density at the location x at time t.
- v is the velocity of cars which depends on the traffic density.
- The flux

$$f(u) \ \doteq \ u \cdot v(u)$$

  describes the total number of cars crossing the location x at time t.

Giving two locations a and b on the road, the integral

$$\int_a^b u(x,t)\ dx \ = \ \text{total number of cars in } [a,b] \text{ at time } \ t\,.$$



We compute

$$\begin{aligned}\frac{d}{dt}\int_a^b u(x,t)\ dx &= f(u(a,t)) - f(u(b,t)) \\ &= -\int_a^b \frac{d}{dx}\ f(u(x,t))\ dx\,.\end{aligned}$$

This implies that

$$\int_a^b u_t(x,t) + f(u(x,t))_x\ dx \ = \ 0 \qquad\qquad \text{for all } a < b\,.$$

A PDE for traffic flow

$$u_t(x,t) + f(u(x,t))_x \ = \ 0\,. \tag{2.1}$$

GOAL: describe the traffic density at time t.

2.1 Linear advection equations

In this subsection, we will study linear advection equations of form

$$u_t(x,t) + c(x,t) \cdot u_x(x,t)) \;=\; g(x,t,u)$$

where

- t is the time variable and x is the state variable;
- g is a given source term;
- c is a given speed of t and x

Goal: Find the density u at the location x and the time t.

2.1.1 Homogeneous linear advection equations with constant speed

Consider the Cauchy problem

$$\begin{cases} u_t(x,t) + c \cdot u_x(x,t) \;=\; 0\,, \\ u(x,0) \;=\; u_0(x) \end{cases} \tag{2.2}$$

where

- c is a given constant speed;
- the function $u_0(x)$ is the initial data.

Observe that

$$\frac{d}{dt}\, u(x_0 + ct, t) \;=\; c \cdot u_x(x_0 + ct, t) + u_t(x_0 + ct, t) \;=\; 0\,.$$

Hence, u is constant along every line $(x_0 + ct, t)$. In particular, one has

$$u(t, x_0 + ct) \;=\; u(x_0, 0) \;=\; u_0(x_0)\,.$$

Set $x = x_0 + ct$, we have $x_0 = x - ct$. The solution is

$$u(x,t) \;=\; u_0(x - ct)\,.$$

Remark. The general solution of (2.2) has form

$$u(x,t) \;=\; F(x - ct)$$

for smooth function F.

Example 1. Consider the Cauchy problem

$$\begin{cases} u_t(x,t) + 2 \cdot u_x(x,t) \;=\; 0\,, \\ u(x,0) \;=\; \dfrac{1}{1+x^2}\,. \end{cases}$$

Find $u(x,1)$.

Answer. $c = 2$ and $u_0(x) = \dfrac{1}{1+x^2}$. Thus, the solution

$$u(x,t) \;=\; \frac{1}{1+(x-2t)^2}\,.$$

In particular, we have

$$u(x,1) \;=\; \frac{1}{1+(x-2)^2} \;=\; \frac{1}{x^2-4x+5}\,.$$

$\square$

Example 2. Solve the initial value problem (IVP)

$$\begin{cases} u_t(x,t) - 3\cdot u_x(x,t) \;=\; 0\,, \\ u(x,0) \;=\; \begin{cases} -1 & \text{if} \quad x>0 \\ 1 & \text{if} \quad x\le 0\,. \end{cases} \end{cases}$$

Answer. We have

$$c \;=\; -3 \qquad \text{and} \qquad u_0(x) \;=\; \begin{cases} -1 & \text{if} \quad x>0 \\ 1 & \text{if} \quad x\le 0\,. \end{cases}$$

Thus, the solution is

$$u(x,t) \;=\; u_0(x+3t) \;=\; \begin{cases} -1 & \text{if} \quad x>-3t \\ 1 & \text{if} \quad x\le -3t\,. \end{cases}$$

$\square$

Example 3. Find the solution of the following initial value problem

$$\begin{cases} u_t(x,t) - 2\cdot u_x(x,t) + 3u(x,t) \;=\; 0\,, \\ u(x,0) \;=\; xe^{-x^2}\,. \end{cases}$$

Answer. Set $v(x,0) = e^{3t}u(x,0)$. We have

$$v_x(x,0) \;=\; e^{3t}u(x,0) \qquad \text{and} \qquad v_t \;=\; e^{3t}\cdot[u_t(x,t)+3u(x,t)]\,.$$

Thus,

$$\begin{cases} v_t(x,t) - 2\cdot v_x(x,t) \;=\; 0\,, \\ v(x,0) \;=\; v_0(x) \;=\; xe^{-x^2}\,. \end{cases}$$

Solving the above equation, we get

$$v(x,t) \;=\; v_0(x+2t) \;=\; (x+2t)e^{-(x+2t)^2}\,.$$

Recalling that

$$u(x,t) \;=\; e^{-3t}\cdot v(x,t)\,,$$

the solution u is

$$u(x,t) \;=\; (x+2t)e^{-(x+2t)^2-3t}\,.$$

□

2.1.2 Non-homogeneous linear advection equations with constant speed

Consider the Cauchy problem

$$\begin{cases} u_t(x,t)+c\cdot u_x(x,t)+a(t)u(x,t) \;=\; g(x,t)\,, \\ u(0,x) \;=\; u_0(x) \end{cases} \tag{2.3}$$

where

- c is a given constant speed;
- the function $u_0(x)$ is the initial data.
- $a(t)$, $g(x,t)$ are given functions.

How to solve (2.3)?

Answer. It is divided into several steps:

Step 1: Introduce new functions

$$v(x,0) \;\doteq\; e^{\mu(t)}\cdot u(x,t) \qquad \text{and} \qquad k(x,t) \;\doteq\; e^{\mu(t)}g(x,t)$$

where μ is the integrating factor

$$\mu(t) \;=\; \int_0^t a(s)\;ds\,.$$

We compute that

$$v_x(x,t) \;=\; e^{\mu(t)}\cdot u(x,t), \qquad\qquad v_t(x,t) \;=\; e^{\mu(t)}\cdot[u_t(x,t)+a(t)u(x,t)]$$

and

$$u(x,0) \;=\; e^{\mu(0)}\cdot u(x,0) \;=\; u_0(x)\,.$$

Thus, v is the solution of

$$\begin{cases} v_t(x,t)+c\cdot v_x(x,t) \;=\; k(x,t)\,, \\ v(x,0) \;=\; u_0(x)\,. \end{cases}$$

Step 2: Set $V(x,t) \; = \; v(x+ct,t)$. We have

$$V_t \; = \; v_t + cv_x \; = \; k(x+ct,t)\,.$$

Solving the ordinary different equation in time t

$$V_t(x,t) \; = \; k(x+ct,t) \qquad \text{with} \qquad V(x,0) \; = \; u_0(x)$$

we obtain that

$$V(x,t) \; = \; u_0(x) + \int_0^t k(x+cs,s)\; ds\,.$$

Step 3: The general solution

$$\begin{aligned} u(x,t) \; &= \; e^{-\mu(t)} \cdot v(x,t) \\ &= \; e^{-\mu(t)} \cdot V(x-ct,t)\,. \end{aligned}$$

□

Example 1. a). Find the general solution

$$u_t - 2u_x + 2u \; = \; e^{-t}\,.$$

b) Assume that $u(x,0) = e^{-x}$. Compute $u(2,1)$.

Answer. *Step 1.* We have

$$c \; = \; -2, \qquad a \; = \; 2, \qquad g(t) \; = \; e^{-t}\,.$$

The function

$$\mu(t) \; = \; \int_0^t 2\; ds \; = \; 2t\,.$$

We set

$$v(x,t) \; = \; e^{2t} \cdot u(x,t) \qquad \text{and} \qquad k(t) \; = \; e^t\,.$$

Then, $v(x,t)$ solves the PDE

$$v_t - 2v_x \; = \; e^t.$$

Step 2. Set $V(x,t) \; = \; v(x-2t,t)$. We have

$$V_t(x,t) \; = \; e^t\,.$$

Thus,

$$V(x,t) \; = \; \int e^s\; ds \; = \; e^t + F(x)\,.$$

Step 3. The general solution

$$\begin{aligned} u(x,t) \; &= \; e^{-2t} v(x,t) \; = \; e^{-2t} V(x+2t,t) \\ &= \; e^{-2t} F(x+2t) + e^{-2t} \cdot e^t \; = \; = e^{-2t} F(x+2t) + e^{-t}. \end{aligned}$$

(b). The initial condition $u(x,0)=e^{-x}$ implies that

$$e^{-x} \;=\; F(x)+1 \qquad \Longrightarrow \qquad F(x) \;=\; e^{-x}-1\,.$$

Thus,

$$u(x,t) \;=\; e^{-2t}\cdot e^{-(x+2t)}+e^{-t}-e^{-2t} \;=\; e^{-x}+e^{-t}-e^{-2t}\,.$$

In particular,

$$u(2,1) \;=\; e^{-1}+e^{-2}-e^{-2} \;=\; e^{-1}\,.$$

□

Example 2. Find the solution of the Cauchy problem

$$\begin{cases} u_t(x,t)+3\cdot u_x(x,t)+2t\cdot u(x,t) \;=\; t\,, \\ u(x,0) \;=\; x+\dfrac{1}{2}\,. \end{cases}$$

Answer. *Step 1.* We have

$$c \;=\; -2, \qquad a(t) \;=\; 2t \qquad \text{and} \qquad g(t) \;=\; t\,.$$

The function

$$\mu(t) \;=\; \int_0^t 2s\;ds \;=\; t^2\,.$$

We set

$$V(x,t) \;=\; e^{t^2}\cdot u(x,t) \qquad \text{and} \qquad k(t) \;=\; te^{t^2}\,.$$

Then, v is the solution of the Cauchy problem

$$\begin{cases} v_t+3\cdot v_x \;=\; te^{t^2}\,, \\ v(x,0) \;=\; x+\dfrac{1}{2}\,. \end{cases}$$

Step 2. Set $V(t,x) \;=\; v(t,x+3t)$. We have

$$V_t(x,t) \;=\; te^{t^2} \qquad \text{and} \qquad V(x,0) \;=\; x+\frac{1}{2}\,.$$

Thus,

$$V(x,t) \;=\; x+\frac{1}{2}+\int_0^t se^{s^2}\;ds \;=\; x+\frac{e^{t^2}}{2}\,.$$

Step 3. The solution

$$\begin{aligned} u(x,t) \;&=\; e^{-t^2}v(x,t) \;=\; e^{-t^2}V(x-3t,t) \\ &=\; e^{-t^2}\cdot(x-3t)+\frac{1}{2}\,. \end{aligned}$$

□

Example 3. Solve the initial value problem

$$\begin{cases} u_t(x,t) + u_x(x,t) + 3u(x,t) &= xe^{-3t}\,, \\ u(x,0) &= x^2 - 1\,. \end{cases}$$

Answer. *Step 1.* We have

$$c = 1, \qquad a(t) = 3 \qquad \text{and} \qquad g(x,t) = xe^{-3t}\,.$$

The function

$$\mu(t) = \int_0^t 3\; ds = 3t\,.$$

We set

$$v(x,t) = e^{3t} \cdot u(x,t) \qquad \text{and} \qquad k(x,t) = x\,.$$

Then, v is the solution of the Cauchy problem

$$\begin{cases} v_t(x,t) + v_x(x,t) &= k(x,t)\,, \\ u(x,0) &= x^2 - 1\,. \end{cases}$$

Step 2. Set $V(x,t) = v(x+t,t)$. We have

$$V_t(x,t) = k(x+t,t) = x+t, \qquad V(x,0) = x^2 - 1\,.$$

Thus,

$$V(x,t) = x^2 - 1 + \int_0^t (x+s)\; ds = x^2 - 1 + xt + \frac{t^2}{2}\,.$$

Step 3. The solution

$$\begin{aligned} u(x,t) &= e^{-3t} v(x,t) = e^{-3t} V(x-t,t) \\ &= e^{-3t} \cdot \left[(x-t)^2 - 1 + (x-t)t + \frac{t^2}{2} \right] \\ &= e^{-3t} \cdot \left[x^2 - xt + \frac{t^2}{2} - 1 \right]\,. \end{aligned}$$

2.1.3 Homogeneous linear advection equations with nonconstant speed

Consider the Cauchy problem

$$\begin{cases} u_t(x,t) + c(x,t) \cdot u_x(x,t) &= 0\,, \\ u(x,0) &= u_0(x) \end{cases} \tag{2.4}$$

where the speed $c(x,0)$ is a given function of x and t.

Goal: Find the solution u.

• **The method of characteristics.** Let $x(t)$ be the solution of

$$\dot{x}(t) = c(x,t), \qquad x(0) = x_0 .$$

The curve $(x(t),t)$ is called a characteristic curve.

Observe that

$$\begin{aligned} \frac{d}{dt}\, u(x(t),t) &= u_t(x(t),t) + \dot{x}(t)\cdot u_x(x(t),t) \\ &= u_t(x(t),t) + c(x(t),t)\cdot u_x(x(t),t) = 0 . \end{aligned}$$

This implies that the function u is constant along the *characteristic curve* $(x(t),t)$. In particular, we have

$$u(x(t),t) = u(x(0),t) = u_0(x_0) .$$

Therefore, the solution u can be solved backward along characteristic curves.

• **How to solve the equation (2.4)?**

Step 1. Solve the ODE

$$\dot{x}(t) = c(x,t)$$

and get the general solution of form

$$\xi(x,t) = C .$$

Step 2. The general solution is

$$u(x,t) = F(\xi(x,t))$$

for some smooth function F.

Step 3. Find F by using the initial condition. □

Example 1. Find a general solution of the ODE

$$u_t(x,t) + 2tu_x(x,t) = 0 .$$

Answer. *Step 1.* Solve the ODE

$$\dot{x}(t) = 2t$$

we obtain that

$$x(t) = t^2 + C \qquad \Longrightarrow \qquad x - t^2 = C .$$

Thus,

$$\xi(x,t) = x - t^2 .$$

Step 2. The general solution

$$u(x,t) = F(\xi(x,t)) = F(x-t^2)$$

for some smooth function F. □

Example 2. Consider the first order linear PDE

$$u_t + t^2 u_x \;=\; 0\,.$$

(a) Find $u(x,t)$ if $u(x,0) = \sin x$.

(b) Find $u(x,t)$ if $u(x,1) = e^{-x^2}$.

Answer. Solve the ODE

$$\dot{x}(t) \;=\; t^2$$

we obtain that

$$x(t) \;=\; \frac{1}{3}\cdot t^3 + C \qquad \Longrightarrow \qquad x - \frac{t^3}{3} \;=\; C\,.$$

Thus,

$$\xi(x,t) \;=\; x - \frac{t^3}{3}$$

and the general solution

$$u(x,t) \;=\; F(\xi(x,t)) \;=\; F(x - t^3/3)$$

for some smooth function F.

(a). If $u(x,0) = \sin x$ then

$$F(x) \;=\; \sin x.$$

The solution

$$u(x,t) \;=\; \sin(x - t^3/3)\,.$$

(b). If $u(x,1) = e^{-x^2}$ then

$$F(x - 1/3) \;=\; e^{-x^2} \qquad \Longrightarrow \qquad F(x) \;=\; e^{-(x+1/3)^2}\,.$$

The solution

$$u(x,t) \;=\; F(x - t^3/3) \;=\; e^{-\left(x - \frac{t^3-1}{3}\right)^2}\,.$$

□

Example 3. Consider the initial value problem

$$\begin{cases} u_t(x,t) + txu_x(x,t)) \;=\; 0\,, \\ u(x,0) \;=\; e^{-x}\,. \end{cases}$$

Find $u(x,2)$.

Answer. Solve the ODE

$$\dot{x} \;=\; tx \qquad \Longrightarrow \qquad x \;=\; Ce^{\frac{t^2}{2}} \qquad \Longrightarrow \qquad x\cdot e^{-t^2/2} \;=\; C\,.$$

Thus,

$$\xi(x,t) \;=\; x\cdot e^{-t^2/2}\,.$$

The general solution

$$u(x,t) \;=\; F(\xi(x,t)) \;=\; F(xe^{-t^2/2})\,.$$

The initial data $u(x,0)=e^{-x}$ implies that

$$F(x) \;=\; e^{-x}\,.$$

Therefore, the solution

$$u(x,t) \;=\; F(xe^{-t^2/2}) \;=\; e^{-xe^{-t^2/2}}\,.$$

In particular,

$$u(x,2) \;=\; e^{-xe^{-2}}\,.$$

□

2.1.4 Nonhomogeneous linear advection equations with nonconstant speed

Consider the Cauchy problem

$$\begin{cases} u_t(x,t)+c(x,t)\cdot u_x(x,t) \;=\; g(x,t)\,, \\ u(x,0) \;=\; u_0(x) \end{cases} \tag{2.5}$$

where

- the speed $c(x,t)$ is a given function of x and t.
- g is a given source term of x and t.

Goal: Find the solution u.

As in the previous case, let $x(t)$ be the characteristic associated with (2.5), i.e.,

$$\dot{x}(t) \;=\; c(x,t), \qquad x(0) \;=\; x_0\,.$$

We compute that

$$\frac{d}{dt}\,u(x(t),t) \;=\; u_t(x(t),t)+c(x(t),t)\cdot u_x(x(t),t) \;=\; g(x(t),t)\,.$$

This implies that

$$u(x(t),t) \;=\; u_0(x_0)+\int_0^t g(x(s),s)\,ds\,.$$

Therefore, the solution u can be solved backward along characteristic curves.

How to solve 2.5?

It is divided into three steps.

Step 1: Solve the ODE

$$\dot{x}(t) \;=\; c(x,t)$$

and get the general solution of form

$$\xi(x,t) \;=\; C\,.$$

Step 2: Change of coordinate

$$u(x,t) \;=\; V(\xi(x,t),t)\,.$$

we then have

$$V_t(\xi,t) \;=\; f(x,t) \;=\; F(\xi,t)$$

where

$$f(x,t) \;=\; F(\xi(x,t),t)\,.$$

Step 3: Solve the ODE

$$V_t(\xi,t) \;=\; F(\xi,t)$$

to obtain V and then recover $u(x,t)$. □

Example 1. Solve the initial value problem

$$\begin{cases} u_t(x,t) + x u_x(x,t) &= e^t\,, \\ u(x,0) &= \sin x \end{cases}$$

Answer.
Step 1. Solve the ODE

$$\dot{x}(t) \;=\; x(t) \qquad \Longrightarrow \qquad x(t) \;=\; C\cdot e^t \qquad \Longrightarrow \qquad xe^{-t} \;=\; C\,.$$

Thus,

$$\xi(x,t) \;=\; x\cdot e^{-t}\,.$$

Step 2. Set $u(x,t) \;=\; V(\xi,t) \;=\; V(xe^{-t},t)$. We have

$$V_t(\xi,t) \;=\; e^t\,.$$

This implies that

$$V(\xi,t) \;=\; e^t + g(\xi)\,.$$

Thus, the general solution

$$u(x,t) \;=\; e^t + g(xe^{-t})\,.$$

Step 3. The initial data $u(x,0) = \sin x$ yields

$$1 + g(x) \;=\; \sin x \qquad \Longrightarrow \qquad g(x) \;=\; \sin x - 1\,.$$

The solution

$$u(x,t) + e^t + \sin(xe^{-t}) - 1\,.$$

□

Example 2. Solve the following Cauchy problem

$$\begin{cases} u_t(x,t) + 2tu_x(x,t) &= x\,, \\ u(x,0) &= e^{-x}\,. \end{cases}$$

Answer.

Step 1. Solve the ODE

$$\dot{x}(t) = 2t \qquad \Longrightarrow \qquad x(t) = t^2 + C \qquad \Longrightarrow \qquad x - t^2 = C\,.$$

Thus,

$$\xi = x - t^2 \qquad \text{and} \qquad x = \xi + t^2$$

Step 2. Set $u(x,t) = V(\xi,t)$. We have

$$V_t(\xi,t) = x = \xi + t^2\,.$$

This implies that

$$V(\xi,t) = \int \xi + t^2\,dt = \xi t + \frac{t^3}{3} + g(\xi)\,.$$

Thus, the general solution

$$\begin{aligned} u(x,t) &= V(x-t^2) = \frac{t^3}{3} + (x-t^2)t + g(x-t^2) \\ &= -\frac{2t^3}{3} + tx + g(x-t^2)\,. \end{aligned}$$

Step 3. The initial data $u(x,0) = e^{-x}$ yields

$$g(x) = e^{-x}\,.$$

The solution

$$u(x,t) = -\frac{2t^3}{3} + tx + e^{t^2-x}\,.$$

□

2.2 Nonlinear advection equations

Consider the first order nonlinear PDE

$$\begin{cases} u_t + c(u)\cdot u_x &= 0\,, \\ u(x,0) &= \Phi(x) \end{cases} \tag{2.6}$$

where

- $c(u)$ is a non constant speed which depends on u;
- Φ is a given initial data.

Goal: Find $u(x,t)$.

• **The method of characteristics.** Let $x(t)$ be the solution of

$$\dot{x}(t) = c(u(x(t),t)), \qquad x(0) = \beta .$$

The curve $(x(t),t)$ is called a characteristic curve.

Observe that

$$\begin{aligned} \frac{d}{dt}\, u(x(t),t) &= u_t(x(t),t) + \dot{x}(t) \cdot u_x(x(t),t) \\ &= u_t(x(t),t) + c(u(x(t),t)) \cdot u_x(x(t),t) = 0 . \end{aligned}$$

This implies that the function u is constant along the *characteristic curve* $(x(t),t)$. In particular, we have

$$u(x(t),t) = u(x(0),0) = \Phi(\beta) . \tag{2.7}$$

Hence,

$$c(u(x(t),t)) = c(\Phi(\beta)) ,$$

and it yields

$$x(t) = c(\Phi(\beta)) \cdot t + \beta .$$

Recalling (2.7), we obtain the general formula for the solution

$$u(\beta + c(\Phi(\beta))t, t) = \Phi(\beta) .$$

Remark. *The method can be applied as long as the solution is smooth.*

Example 1. Consider the Burger's equation with initial condition

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = 0 , \\ u(x,0) = x \end{cases}$$

Find $u(x,1)$.

Answer. Since $c(u) = u$ and $\Phi(x) = x$, one has

$$c(\Phi(\beta)) = \beta .$$

Thus,

$$u(\beta + \beta \cdot t, t) = \Phi(\beta) = \beta .$$

Set $x = \beta + \beta \cdot t$, we have

$$\beta = \frac{x}{1+t} .$$

The solution

$$u(x,t) = \frac{x}{t+1} .$$

In particular,

$$u(x,1) \;=\; \frac{x}{2}\,.$$

□

Example 2. Consider the Burger's equation with initial condition

$$\begin{cases} u_t + \left(\dfrac{u^4}{4}\right)_x &= \; 0\,, \\ u(x,0) &= \; x^{\frac{1}{3}} \end{cases}$$

Find $u(x,1)$.

Answer. Since $c(u) = u^3$ and $\Phi(x) = x^{\frac{1}{3}}$, one has

$$c(\Phi(\beta)) \;=\; \beta\,.$$

Thus,

$$u(\beta + \beta\cdot t, t) \;=\; \Phi(\beta) \;=\; \beta^{\frac{1}{3}}\,.$$

Set $x = \beta + \beta\cdot t$, we have

$$\beta \;=\; \frac{x}{1+t}\,.$$

The solution

$$u(x,t) \;=\; \left(\frac{x}{t+1}\right)^{\frac{1}{3}}\,.$$

□

3 Linear 1D Partial Differential Equations in unbounded domains

3.1 1D heat equation

The heat equation on a thin rod

$$\begin{cases} u_t(x,t) &= \; \alpha^2\cdot u_{xx}(x,t) + f(x,t)\,, \\ u(x,0) &= \; \Phi(x) \end{cases} \tag{3.1}$$

where

- α^2: a given positive constant which is the diffusivity of the rod;
- $\Phi(x)$: a given initial temperature at point x;
- $u(x,t)$: temperature at point x at time t

Goal: Find the presentation formula of u.

3.1.1 Derivation of the 1d heat equation

Consider 1-D rod of length L such that

- Temperature at all points of a cross section is constant;
- Heat flows only in the x-direction;
- made of a single homogeneous conducting material.

Let us denote by

- ρ: density fo the rod;
- A: cross-section area if the rod;
- c: thermal capacity of the rod (measures the ability of the rod to store heat);
- k: thermal conductivity of the rod (measures the ability to conduct heat);
- $g(x,t)$: external heat source.

Goal: Find $u(x,t)$ the temperature at location x at time t.

Given any two point a and b with $a < b$, the integral

$$\int_a^b c\rho A u(x,t)\ dx \ = \ \text{total amount heat in the interval } [a,b] \text{ at time } t.$$

We compute

$$\int_a^b c\rho A u_t(x,t)\ dx \ = \ \frac{d}{dt}\int_a^b c\rho A u(x,t)\ dx$$

$$= \text{[flux of heat crossing at } a] - \text{[flux of heat crossing at } b] + \text{[total heat generated in side } [a,b]]$$

"By using the Fourier's law"

$$= \kappa A[u_x(b,t) - u_x(a,t)] + A \cdot \int_a^b g(x,t) dx\,.$$

"By using the mean value theorem"

$$= kA\int_a^b u_{xx}(x,t)\ dx + A \cdot \int_a^b g(x,t) dt\,.$$

Thus,

$$\int_a^b u_t(x,t)\ dx \ = \ \frac{k}{c\rho}\int_a^b u_{xx}(x,t)\ dx + \frac{1}{c\rho}\cdot\int_a^b g(x,t)dt$$

for all $a < b$. This implies the second order linear PDEs

$$u_t \ = \ \alpha^2 \cdot u_{xx} + f(x,t)\,.$$

where

$$\alpha^2 \ \doteq \ \frac{k}{c\rho} \qquad \text{and} \qquad f(x,t) \ \doteq \ \frac{g(x,t)}{c\rho}\,.$$

3.1.2 Presentation formula of 1D heat equation without source

Consider the Cauchy problem

$$\begin{cases} u_t(x,t) &= \alpha^2 \cdot u_{xx}(x,t)\,, \qquad x \in \mathbb{R}, t > 0 \\ \\ u(x,0) &= \Phi(x) \qquad x \in \mathbb{R}\,. \end{cases} \tag{3.2}$$

Goal: Find the presentation formula of u.

1. Heat kernel or fundamental solution

$$u_t(x,t) \;=\; \alpha^2 \cdot u_{xx}(x,t)\,. \tag{3.3}$$

Observe that

- If u solves (3.3) then $w \doteq u_x$ also solves (3.3).
- If $u(x,t)$ solves (3.3) then $U(x,t) = u(\lambda \cdot x, \lambda^2 \cdot t)$ also solves (3.3) for every constant $\lambda \in \mathbb{R}$.

Thus, we will look for a solution with form

$$u(x,t) \;=\; v\left(\frac{x}{\sqrt{t}}\right).$$

A direct computation yields

$$u_t \;=\; -\frac{xt^{\frac{-3}{2}}}{2} \cdot v'\left(\frac{x}{\sqrt{t}}\right) \qquad \text{and} \qquad u_{xx} \;=\; \frac{1}{t} \cdot v''\left(\frac{x}{\sqrt{t}}\right).$$

From (3.2), we obtain that

$$v''\left(\frac{x}{\sqrt{t}}\right) + \frac{z}{2\alpha^2} \cdot v'\left(\frac{x}{\sqrt{t}}\right) \;=\; 0\,.$$

Set

$$z \;=\; \frac{x}{\sqrt{t}} \qquad \text{and} \qquad w(z) \;=\; v'(z)\,,$$

we have

$$w' + \frac{z}{2\alpha^2} w \;=\; 0 \qquad \Longrightarrow \qquad w(z) \;=\; Ce^{\frac{-z^2}{4\alpha^2}}\,.$$

Thus,

$$u_x(x,t) \;=\; w\left(\frac{x}{\sqrt{t}}\right) \;=\; Ce^{-\frac{x^2}{4\alpha^2 t}}\,.$$

The heat kernel (fundamental solution) is

$$G(x,t) \;=\; \frac{1}{\sqrt{4\pi\alpha^2 t}} \cdot e^{-\frac{x^2}{4\alpha^2 t}}\,.$$

Properties of heat kernel.

1. $G(x,t)$ solves (3.3);
2. For every $t > 0$, it holds
$$\int_{-\infty}^{\infty} G(x,t)\, dx \;=\; 1\,.$$
3. As $t \to 0+$, $G(\cdot,t)$ converges to Dirac delta function $\delta_0(\cdot)$.

Theorem 3.1 *Assume that Φ is bounded continuous function. The initial value problem (3.2) has a unique smooth solution $u(x,t)$ with*

$$\lim_{|x|\to+\infty} u(x,t) \;=\; 0 \qquad t>0\,.$$

Moreover, u can be presented by

$$u(x,t) \;=\; G(\cdot,t)*\Phi(x) \;=\; \int_{-\infty}^{\infty} G(x-y,t)\cdot\Phi(y)\;dy\,.$$

for all $(x,t)\in\mathbb{R}\times(0,+\infty)$.

Proof. 1. Let's first show that $u(x,t)$ solves (3.2. We compute

$$u_t(x,t) \;=\; \int_{-\infty}^{+\infty} G_t(x-y,t)\cdot\Phi(y)\;dy\,,$$

and

$$u_{xx}(x,t) \;=\; \int_{-\infty}^{+\infty} G_{xx}(x-y,t)\cdot\Phi(y)\;dy\,,$$

Since G is a fundamental solution of (3.3), we get

$$u_t(x,t) \;=\; \alpha^2\cdot u_{xx}(x,t)\,.$$

On the other hand, the third property (3) of G yields

$$u(x,0) \;=\; \lim_{t\to 0+} G(\cdot,t)*\Phi(x) \;=\; \int_{-\infty}^{\infty} G(x-y,t)\cdot\Phi(y)\;dy \;=\; \int_{-\infty}^{+\infty} \delta_0(x-y)\cdot\Phi(y)\;dy \;=\; \Phi(x)\,.$$

Thus, u is a solution of (3.2).

2. To complete the proof, we will show that (3.2) has at most one solution. Assume by a contradiction that (3.2) has two different solutions u_1 and u_2. Set

$$v(x,t) \;=\; u_2(x,t)-u_1(x,t)$$

Then, v is a solution of

$$v_t(x,t) \;=\; \alpha^2\cdot v_{xx}(x,t), \qquad\qquad v(x,0) \;=\; 0\,.$$

Let's consider the energy function

$$E(t) \;=\; \int_{-\infty}^{\infty} v^2(t,x)\;dx\,.$$

We compute

$$\frac{d}{dt}E(t) \;=\; 2\int_{-\infty}^{\infty} v(x,t)\cdot v_{xx}(x,t)\;dx \;=\; -2\int_{-\infty}^{\infty} v_x^2(x,t)\;dx \;\le\; 0\,.$$

The functon $E(t)$ is decreasing. In particular

$$0 \;\le\; E(t) \;\le\; E(0) \;=\; 0 \qquad \text{for all } x\in[0,\infty)\,.$$

Thus, $v(x,t) = 0$ for all $(x,t) \in \mathbb{R} \times [0,+\infty)$, and it yields a contradiction. $\square$

Example 1. Consider the initial value problem

$$\begin{cases} u_t(x,t) &= 4u_{xx}(x,t)\,, \qquad x \in \mathbb{R}, t > 0 \\ u(x,0) &= \sin x \end{cases}$$

Find the formula of the solution u.

Answer. We have

$$\alpha^2 = 4 \qquad \text{and} \qquad \Phi(x) = \sin x\,.$$

The heat kernel

$$G(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} \cdot e^{-\frac{x^2}{4\alpha^2 t}} = \frac{1}{4\sqrt{\pi t}} \cdot e^{-\frac{x^2}{16t}}\,.$$

The solution

$$u(x,t) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{16t}} \cdot \sin y\; dy\,.$$

$\square$

Example 2. Find the formula of the solution to the Cauchy problem

$$\begin{cases} u_t(x,t) - 2u &= 9u_{xx}(x,t)\,, \qquad x \in \mathbb{R}, t > 0 \\ u(x,0) &= e^{-x} \qquad x \in \mathbb{R}\,. \end{cases}$$

Answer. 1. Set $v = e^{-2t} \cdot u$. We compute

$$v_t(x,t) = e^{-2t} \cdot (u_t(x,t) - 2u(x,t)), \qquad v_{xx} = e^{-2t} \cdot u_{xx}$$

Thus, v is the solution to

$$\begin{cases} v_t(x,t) &= 9v_{xx}(x,t)\,, \qquad x \in \mathbb{R}, t > 0 \\ v(x,0) &= e^{-x}\,. \end{cases}$$

2. The heat kernel is

$$G(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} \cdot e^{-\frac{x^2}{4\alpha^2 t}} = \frac{1}{6\sqrt{\pi t}} \cdot e^{\frac{-x^2}{36t}}$$

Thus,

$$v(x,t) = \frac{1}{6\sqrt{\pi t}\cdot} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{36t}} \cdot e^{-y}\; dy = \frac{1}{6\sqrt{\pi t}\cdot} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{36t} - y}\; dy$$

The solution

$$u(x,t) = e^{2t} \cdot v(x,t) = \frac{e^{2t}}{6\sqrt{\pi t}} \cdot \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{36t} - y}\; dy\,.$$

$\square$

Example 3. Find the formula of the solution to the Cauchy problem

$$\begin{cases} u_t(x,t) + 2tu &= 4u_{xx}(x,t)\,, \qquad x \in \mathbb{R}, t > 0 \\ u(x,0) &= \dfrac{1}{1+x^2}\,. \end{cases}$$

Answer. 1. We compute

$$\mu(t) \;=\; \int_0^t 2s\; ds \;=\; t^2\,,$$

and set

$$v(x,t) \;=\; e^{t^2} \cdot u(x,t)\,.$$

Then, v is the solution to

$$\begin{cases} v_t(x,t) &= 4v_{xx}(x,t)\,, \qquad x \in \mathbb{R}, t > 0 \\ v(x,0) &= \dfrac{1}{1+x^2}\,. \end{cases}$$

2. The heat kernel is

$$G(x,t) \;=\; \frac{1}{\sqrt{4\pi\alpha^2 t}} \cdot e^{-\frac{x^2}{4\alpha^2 t}} \;=\; \frac{1}{4\sqrt{\pi t}} \cdot e^{\frac{-x^2}{16t}}\,.$$

Thus,

$$v(x,t) \;=\; \frac{1}{4\sqrt{\pi t}} \cdot \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{16t}} \cdot \frac{1}{1+y^2}\; dy\,.$$

The solution is

$$u(x,t) \;=\; \frac{e^{-t^2}}{4\sqrt{\pi t}} \cdot \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{16t}} \cdot \frac{1}{1+y^2}\; dy\,.$$

□

3.1.3 Semi-infinite domains

Consider the initial boundary value problem

$$\begin{cases} u_t(x,t) &= \alpha^2 \cdot u_{xx}(x,t)\,, \qquad x > 0, t > 0 \\ u(0,t) &= 0 \qquad t > 0 \\ u(x,0) &= \Phi(x) \qquad x > 0\,. \end{cases} \tag{3.4}$$

Goal: Find $u(x,t)$ for any $x, t > 0$.

Answer. Let's consider the *odd extension* of Φ which is defined as

$$\Psi(x) \;=\; \begin{cases} \Phi(x) & \text{for all } x > 0\,, \\ -\Phi(-x) & x < 0\,, \end{cases}$$

with $\Psi(0) = 0$.

Let v be the solution of

$$\begin{cases} v_t(x,t) &= \alpha^2 v_{xx}(x,t)\,, \qquad x > \mathbb{R}, t > 0 \\ v(x,0) &= \Psi(x)\,. \end{cases}$$

The heat Kernel

$$G(x,t) \;=\; \frac{1}{\sqrt{4\alpha^2 \pi t}} \cdot e^{\frac{-x^2}{4\alpha^2 t}}\,.$$

Thus,

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{+\infty} G(x-y,t)\cdot \Psi(y)\;dy \\ &= -\int_{-\infty}^{0} G(x-y,t)\Phi(-y)\;dy + \int_0^{\infty} G(x-y,t)\cdot\Phi(y)\;dy \\ &= \int_0^{\infty} \Big[G(x-y) - G(x+y)\big]\cdot\Phi(y)\;dy\,. \end{aligned}$$

Therefore, the solution of (3.4) is

$$u(x,t) \;=\; \int_0^{\infty} \Big[G(x-y) - G(x+y)\big]\cdot\Phi(y)\;dy \qquad \text{for all } x > 0, t > 0\,.$$

□

Example 1. Consider the initial boundary value problem

$$\begin{cases} u_t(x,t) &= 9\cdot u_{xx}(x,t)\,, \qquad x \in \mathbb{R}, t > 0\,, \\ u(0,t) &= 0 \qquad t > 0\,, \\ u(x,0) &= e^{-x} \qquad x > \mathbb{R}\,. \end{cases}$$

Find the presentation formula of $u(x,t)$.

Answer. We have

$$\alpha^2 = 9 \qquad \text{and} \qquad \Phi(x) = e^{-x}\,.$$

Thus, the heat kernel is

$$G(x,t) \;=\; \frac{1}{6\sqrt{\pi t}} \cdot e^{-\frac{x^2}{36t}}$$

The solution is

$$\begin{aligned} u(x,t) &= \int_0^{\infty} [G(x-y,t) - G(x+y,t)]\cdot\Phi(y)\;dy \\ &= \frac{1}{6\sqrt{\pi t}} \cdot \int_0^{\infty} \left[e^{-\frac{(x-y)^2}{36t}} - e^{-\frac{(x+y)^2}{36t}}\right]\cdot e^{-y}\;dy\,. \end{aligned}$$

□

3.1.4 Sources and Duhamel's principle

1. Duhamel's principle for ODEs. Consider the first order ODEs with sources

$$\begin{cases} y'(t) + a \cdot y(t) = F(t)\,, & t > 0\,, \\ y(0) = y_0 \end{cases} \tag{3.5}$$

where

- a and y_0 are given constant;
- $F(t)$ is a given external source;

Goal: Find the solution $u(t)$.

Answer. Observe that

$$\frac{d}{dt}\left[e^{at}y(t)\right] = e^{at}\cdot y'(t) + ae^{at}y(t) = e^{at}\cdot\left[y'(t) + a\cdot y(t)\right].$$

Thus,

$$\frac{d}{dt}\left[e^{at}y(t)\right] = e^{at}\cdot F(t)\,,$$

and this implies that

$$e^{at}\cdot y(t) - y_0 = \int_0^t e^{a\cdot s}F(s)\,ds$$

The solution of (3.5) is

$$y(t) = e^{-at}\cdot y_0 + \int_0^t e^{a(s-t)}\cdot F(s)\,ds$$

□

Example 1. Find the solution of the Cauchy problem

$$\begin{cases} y'(t) + y(t) = e^{2t}\,, & t > 0\,, \\ y(0) = 2\,. \end{cases}$$

Answer. We have

$$a = 1, \qquad y_0 = 2 \qquad \text{and} \qquad F(t) = e^{2t}\,.$$

Using the Duhamel's principle, the solution is

$$\begin{aligned} y(t) &= e^{-at}\cdot y_0 + \int_0^t e^{a(s-t)}\cdot F(s)\,ds = 2e^{-t} + \int_0^t e^{3s-t}\,ds \\ &= 2e^{-t} + \frac{1}{3}e^{-t}\cdot[e^{3t}-1] = \frac{5}{3}\cdot e^{-t} + \frac{1}{3}\cdot e^{2t}\,. \end{aligned}$$

□

2. Duhamel's principle for PDEs. Consider the linear PDEs

$$\begin{cases} u_t(x,t) + Au = f(x,t)\,, & t > 0, x \in \mathbb{R}\,, \\ u(x,0) = 0, & x \in \mathbb{R}\,, \end{cases} \tag{3.6}$$

where

- A is a linear differential operators;
- $f(x,t)$ is a given function of x and t;
- $u(x,t)$ is an unknown of x and t.

Theorem 3.2 *Let $w(x,t,s)$ be the solution of*

$$\begin{cases} w_t + Aw = 0\,, & t > 0, x \in \mathbb{R}\,, \\ w(x,0,s) = f(x,s), & x \in \mathbb{R}\,, \end{cases}$$

Then the function

$$u(x,t) = \int_0^t w(x,t-s,s)\; ds$$

is the solution of (3.6).

Proof. Using the fact that

$$\frac{d}{dt}\int_0^t K(t,s)\; ds = K(t,t) + \int_0^t K_t(t,s)\; ds\,,$$

we compute that

$$\begin{aligned} u_t(x,t) &= \frac{d}{dt}\int_0^t w(x,t-s,s)\; ds \\ &= w(x,0,t) + \int_0^t w_t(x,t-s,s)\; ds \\ &= f(x,t) + \int_0^t w_t(x,t-s,s)\; ds\,. \end{aligned}$$

On the other hand, the linear property of A implies that

$$Au(x,t) = A\int_0^t w(x,t-s,s)\; ds = \int_0^t Aw(x,t-s,s)\; ds\,.$$

Recalling that

$$w_t + Aw = 0\,,$$

we then have

$$u_t + Au = f(x,t) + \int_0^t w_t(x,t-s,s) + Aw(x,t-s,s)\; ds = f(x,t)\,.$$

On the other hand,

$$u(x,0) \;=\; \int_0^0 w(x,-s,s)\; ds = 0\,.$$

Thus, u is the solution to (3.6). $\square$

3. 1D Heat equation with sources. Consider the first order PDE with sources

$$\begin{cases} u_t(x,t) &=& \alpha^2 \cdot u_{xx}(x,t) + f(x,t)\,, \qquad t>0, x\in\mathbb{R}\,,\\ u(x,0) &=& 0, \qquad x\in\mathbb{R}\,, \end{cases} \tag{3.7}$$

where α is a given constant and $f(x.t)$ is a given function of x and t.

Goal: Find the $u(x,t)$ the temperature at point x at time t.

Answer. Rewrite the equation

$$\begin{cases} u_t(x,t) - \alpha^2 \cdot u_{xx}(x,t) &=& f(x,t)\,, \qquad t>0, x\in\mathbb{R}\,,\\ u(x,0) &=& 0, \qquad x\in\mathbb{R}\,. \end{cases} \tag{3.8}$$

In this case, we have

$$Au \;=\; -\,\alpha^2 \cdot u_{xx}\,.$$

Step 1. Let $w(x,t,\tau)$ be the solution of

$$\begin{cases} w_t - \alpha^2 \cdot w &=& 0\,, \qquad t>0, x\in\mathbb{R}\,,\\ w(x,0,s) &=& f(x,s), \qquad x\in\mathbb{R}\,. \end{cases}$$

We have

$$w(x,t,s) \;=\; \int_{-\infty}^{+\infty} G(x-y,t)\cdot f(y,s)\; dy\,.$$

where the heat kernel

$$G(x,t) \;=\; \frac{1}{\sqrt{4\alpha^2\pi t}} \cdot e^{-\frac{x^2}{4\alpha^2 t}}\,.$$

Step 2. Using the Duhamel's principle, we obtain that

$$\begin{aligned} u(x,t) &= \int_0^t w(x,t-s,s)\; ds\\ &= \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)\cdot f(y,s)\; dy\; ds\,. \end{aligned}$$

$\square$

Summary. The solution of (3.7) is

$$u(x,t) \;=\; \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)\cdot f(y,s)\; dy\; ds\,,$$

where

$$G(x,t) = \frac{1}{\sqrt{4\alpha^2\pi t}} \cdot e^{-\frac{x^2}{4\alpha^2 t}}.$$

Example 1. Find the presentation formula of the solution to

$$\begin{cases} u_t(x,t) = 4u_{xx}(x,t) + e^{-x}t\,, & t > 0, x \in \mathbb{R}\,, \\ u(x,0) = 0, & x \in \mathbb{R}\,, \end{cases}$$

Answer. We have

$$\alpha^2 = 4 \qquad \text{and} \qquad f(x,t) = e^{-x} \cdot t\,.$$

The heat kernel

$$G(x,t) = \frac{1}{4\sqrt{\pi t}} \cdot e^{\frac{-x^2}{16t}}.$$

The solution

$$\begin{aligned} u(x,t) &= \int_0^t \int_{-\infty}^{+\infty} G(x-y,t-s) \cdot f(y,s)\ ds \\ &= \int_0^t \int_{-\infty}^{+\infty} \frac{1}{4\sqrt{\pi(t-s)}} \cdot e^{\frac{-(x-y)^2}{16(t-s)}} \cdot e^{-y} \cdot s\ dy\ ds \\ &= \int_0^t \int_{-\infty}^{+\infty} \frac{s}{4\sqrt{\pi(t-s)}} \cdot e^{\frac{-(x-y)^2}{16(t-s)} - y}\,. \end{aligned}$$

□

4. More general case. Let's consider the equation

$$\begin{cases} u_t(x,t) = \alpha^2 \cdot u_{xx}(x,t) + f(x,t)\,, & t > 0, x \in \mathbb{R}\,, \\ u(x,0) = \phi(x), & x \in \mathbb{R}\,, \end{cases} \tag{3.9}$$

Goal: Find the $u(x,t)$ the temperature at point x at time t.

Answer. Using the superposition principle the solution

$$u = v + w$$

where v is the solution to

$$\begin{cases} v_t(x,t) = \alpha^2 \cdot v_{xx}(x,t) + f(x,t)\,, & t > 0, x \in \mathbb{R}\,, \\ v(x,0) = 0, & x \in \mathbb{R}\,, \end{cases}$$

and w is the solution to

$$\begin{cases} w_t(x,t) = \alpha^2 \cdot w_{xx}(x,t)\,, & t > 0, x \in \mathbb{R}\,, \\ w(x,0) = \phi(x), & x \in \mathbb{R}\,. \end{cases}$$

We have

$$v(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) \cdot f(y,s)\ dy\ ds\,,$$

and

$$w(x,t) = \int_{-\infty}^{\infty} G(x-y,t) \cdot \phi(y)\ dy\,.$$

The solution is

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) \cdot f(y,s)\ dy\ ds + \int_{-\infty}^{\infty} G(x-y,t) \cdot \phi(y)\ dy\,.$$

□

Example 2. Find the presentation formula for the solution of

$$\begin{cases} 4u_t(x,t) = 9u_{xx}(x,t) - 4\cos t\,, & t>0, x\in\mathbb{R}\,, \\ u(x,0) = \sin x, & x\in\mathbb{R}\,, \end{cases}$$

Answer. Rewrite the equation

$$u_t = \frac{9}{4} \cdot u_{xx} - \cos t\,.$$

We have

$$\alpha^2 = \frac{9}{4}, \qquad f(t) = -\cos t \qquad \text{and} \qquad \phi(x) = \sin x\,.$$

The heat kernel

$$G(x,t) = \frac{1}{3\sqrt{\pi t}} \cdot e^{\frac{-x^2}{9t}}\,.$$

The solution

$$\begin{aligned} u(x,t) &= \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) \cdot f(s)\ dy\ ds + \int_{-\infty}^{\infty} G(x-y,t) \cdot \phi(y)\ dy \\ &= -\int_0^t \int_{-\infty}^{\infty} \frac{1}{3\sqrt{\pi(t-s)}} \cdot e^{\frac{-(x-y)^2}{9(t-s)}} \cdot \cos s\ dy\ ds + \int_{-\infty}^{\infty} \frac{1}{3\sqrt{\pi t}} \cdot e^{\frac{-(x-y)^2}{t}} \cdot \sin y\ dy. \end{aligned}$$

□

3.2 1D wave equation

The motion equation of vibrating string

$$\begin{cases} u_{tt}(x,t) = c^2 \cdot u_{xx}(x,t) + f(x,t)\,, \\ u(x,0) = g(x)\,, \\ u_t(x,0) = h(x) \end{cases} \tag{3.1}$$

where

- c^2 is the wave number which is computed by

$$c^2 = \frac{T}{\rho}.$$

Here T is the tension of the string and ρ is the density such that $\rho\Delta x$ is the mass of the string segment.

- $f(x,t)$ is a given external force applied along the string at x at time t;
- $g(x)$ is the initial position of the string at point x;
- $h(x)$ is the initial standing velocity of the string at point x;

Goal: Find $u(x,t)$ the position of string at point x at time t.

3.2.1 General solution

Consider the 1D wave equation

$$u_{tt} = c^2 \cdot u_{xx} \tag{3.2}$$

Observe that the above equation can be rewritten as

$$\frac{d}{dt}[u_t - c\cdot u_x] + c\cdot\frac{d}{dx}[u_t - c\cdot u_x] = 0.$$

Set $w \doteq u_t - c\cdot u_x$. Then w solves the linear advection equation

$$w_t + c\cdot w_x = 0.$$

Thus,

$$w(x,t) = F_1(x-ct)$$

for some smooth function F_1. This implies that

$$u_t(x,t) - c\cdot u_x(x,t) = F_1(x-ct).$$

Similarly, we have that

$$u_t(x,t) + c\cdot u_x(x,t) = G_1(x+ct).$$

Thus,

$$u_t(x,t) = \frac{1}{2}\cdot[F_1(x-ct) + G_1(x+ct)].$$

Solving this equation, one gets

$$u(x,t) = G(x+ct) + F(x-ct).$$

Summary. The general solution of the wave equation

$$u_{tt}(x,t) = c^2\cdot u_{xx}(x,t)$$

is

$$u(x,t) = G(x+ct) + F(x-ct).$$

Here $G(x+ct)$ is the left traveling wave and $F(x-ct)$ is the right traveling wave with speed c.

Example 1. Find the general solution of

$$4 \cdot u_{tt}(x,t) - 9 \cdot u_{xx}(x,t) \;=\; 0\,.$$

Answer. Rewrite the equation

$$u_{tt}(x,t) \;=\; \frac{9}{4} \cdot u_{xx}(x,t) \qquad \Longrightarrow \qquad c \;=\; \frac{3}{2}\,.$$

The general solution is

$$u(x,t) \;=\; F(x - 3/2t) + G(x + 3/2t)$$

for some smooth function F and G. □

3.2.2 D'Alembert's formula

Consider the Cauchy problem

$$\begin{cases} u_{tt}(x,t) \;=\; c^2 \cdot u_{xx}(x,t), & \text{for all } x \in \mathbb{R}, t > 0 \\[2ex] u(x,0) \;=\; f(x), & \text{for all } x \in \mathbb{R}\,, \\[2ex] u_t(x,0) \;=\; g(x), & \text{for all } x \in \mathbb{R}\,, \end{cases} \tag{3.3}$$

where c is a given constant speed, f is a given initial position and g is a given initial standing velocity.

Goal: Find $u(x,t)$.

Answer. From the previous subsection, the general solution of the 1-D wave equation is

$$u(x,t) \;=\; F(x-ct) + G(x+ct)\,.$$

At time $t = 0$, we have

$$\begin{cases} u(x,0) \;=\; f(x) \\[2ex] u_t(x,0) \;=\; g(x) \end{cases} \qquad \Longrightarrow \qquad \begin{cases} F(x) + G(x) \;=\; f(x) \\[2ex] -cF'(x) + cG'(x) \;=\; g(x)\,. \end{cases}$$

for all $x \in \mathbb{R}$. This implies

$$\begin{cases} F(x-ct) + G(x-ct) \;=\; f(x-ct) \\[2ex] F(x+ct) + G(x+ct) \;=\; f(x+ct) \\[2ex] G'(x) - F'(x) \;=\; \dfrac{1}{c} \cdot g(x)\,. \end{cases}$$

Integrating both sides of the last ODE from $x - ct$ to $x + ct$, we have

$$\int_{x-ct}^{x+ct} G'(y) - F'(y)\ dy \ = \ \frac{1}{c} \cdot \int_{x-ct}^{x+ct} g(y)\ dy\,.$$

and it yields

$$G(x+ct) - G(x-ct) + F(x-ct) - F(x+ct) \ = \ \frac{1}{c} \cdot \int_{x-ct}^{x+ct} g(y)\ dy\,.$$

The D'Alembert's formula for u

$$u(x,t) \ = \ \frac{1}{2} \cdot [f(x+ct) + f(x-ct)] + \frac{1}{2c} \cdot \int_{x-ct}^{x+ct} g(y)\ dy\,.$$

□

Example 2. Solve the Cauchy problem

$$\begin{cases} 9u_{tt}(x,t) - 16u_{xx}(x,t) \ = \ 0, & x \in \mathbb{R}, t > 0\,, \\ u(x,0) \ = \ e^{-x}, & x \in \mathbb{R} \\ u_t(x,0) \ = \ x, & x \in \mathbb{R} \end{cases}$$

Answer. Rewrite the equation

$$u_{tt}(x,t) \ = \ \left(\frac{4}{3}\right)^2 \cdot u_{xx}(x,t).$$

We have

$$c \ = \ \frac{4}{3}, \qquad f(x) \ = \ e^{-x} \qquad \text{and} \qquad g(x) \ = \ x\,.$$

Using the D'Alembert's formula, we obtain

$$\begin{aligned} u(x,t) \ &= \ \frac{1}{2} \cdot [f(x-ct) + f(x+ct)] + \frac{1}{2c} \cdot \int_{x-ct}^{x+ct} g(y)\ dy \\ &= \ \frac{1}{2} \cdot \left[e^{x-\frac{4}{3}\cdot t} + e^{x+\frac{4}{3}\cdot t}\right] + \frac{3}{8} \cdot \int_{x-\frac{4}{3}\cdot t}^{x+\frac{4}{3}\cdot t} y\ dy \\ &= \ \frac{1}{2} \cdot \left[e^{x-\frac{4}{3}\cdot t} + e^{x+\frac{4}{3}\cdot t}\right] + xt\,. \end{aligned}$$

□

Example 3. Solve the Cauchy problem

$$\begin{cases} 4u_{tt}(x,t) - 25u_{xx}(x,t) \ = \ 0, & x \in \mathbb{R}, t > 0\,, \\ u(x,0) \ = \ \frac{1}{x^2+1}, & x \in \mathbb{R} \\ u_t(x,0) \ = \ xe^{-x^2}, & x \in \mathbb{R} \end{cases}$$

Answer.

$$u_{tt}(x,t) = \left(\frac{5}{2}\right)^2 \cdot u_{xx}(x,t).$$

We have

$$c = \frac{5}{2}, \qquad f(x) = \frac{1}{x^2+1} \qquad \text{and} \qquad g(x) = xe^{-x}.$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\cdot[f(x-ct)+f(x+ct)] + \frac{1}{2c}\cdot\int_{x-ct}^{x+ct} g(y)\, dy \\ &= \frac{1}{2}\cdot\left[\frac{1}{(x-5/2t)^2+1} + \frac{1}{(x+5/2t)^2+1}\right] + \frac{1}{5}\cdot\int_{x-\frac{5}{2}\cdot t}^{x+\frac{5}{2}\cdot t} ye^{-y^2}\, dy \\ &= \frac{1}{2}\cdot\left[\frac{1}{(x-5/2t)^2+1} + \frac{1}{(x+5/2t)^2+1}\right] - \frac{1}{10}\cdot\left[e^{-(x-5/2t)^2} - e^{-(x+5/2t)^2}\right]. \end{aligned}$$

□

2. Special case If the initial velocity $g = 0$ then the solution of (3.5) is

$$u(x,t) = \frac{1}{2}\cdot[f(x-ct)+f(x+ct)].$$

Example 4. Consider the IVP

$$\begin{cases} u_{tt}(x,t) - 4u_{xx}(x,t) = 0, & x\in\mathbb{R}, t>0, \\ u(x,0) = \begin{cases} 0, & x\notin[-1,1], \\ 1-|x| & x\in[-1,1], \end{cases} \\ u_t(x,0) = 0, & x\in\mathbb{R}. \end{cases}$$

(a) Find $u(x,1/4)$.

(b) Find u(x,1/2).

(c) Find u(x,3/4).

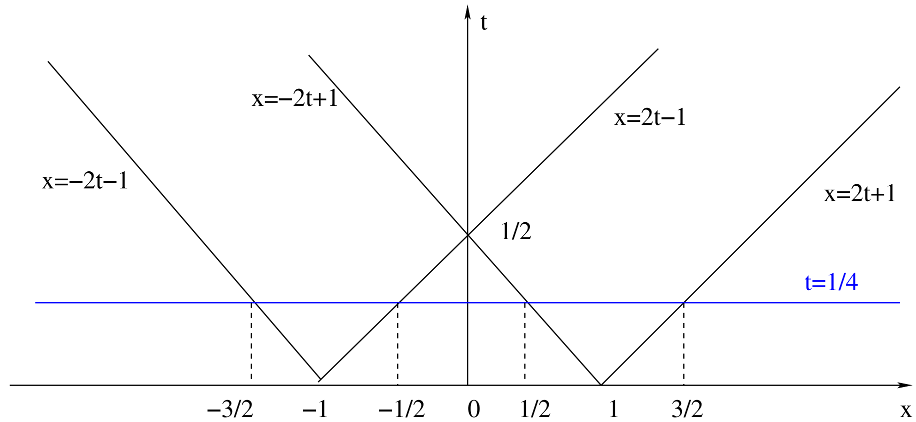
Answer. We have

$$c = 2, \qquad g(x) = 0 \qquad \text{and} \qquad f(x) = \begin{cases} 0, & x\notin[-1,1], \\ 1-|x| & x\in[-1,1], \end{cases}.$$

The solution

$$u(x,t) = \frac{1}{2}\cdot[f(x-2t)+f(x+2t)].$$

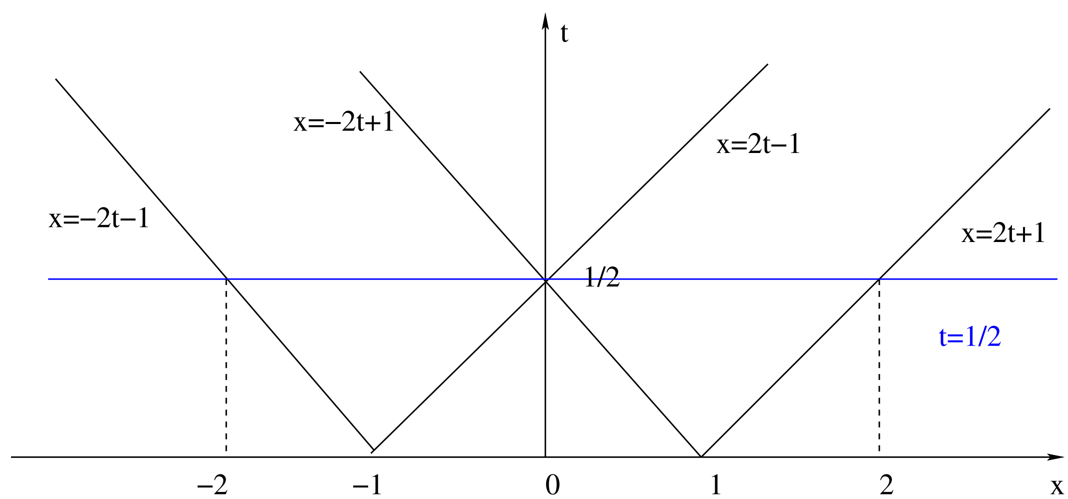
(a) Find $u(x,1/4)$.



The solution at time $t = 1/4$ is

$$u(x,1/4) = \frac{1}{2}\cdot[f(x-1/2)+f(x+1/2)]$$
$$= \begin{cases} 0, & x\in(-\infty,-3/2)\cup(3/2,\infty)\,, \\ \frac{1}{2}\cdot[1-|x+1/2|], & x\in[-3/2.-1/2) \\ \frac{1}{2}\cdot[2-|x+1/2|-|x-1/2|], & x\in[-1/2.1/2] \\ \frac{1}{2}\cdot[1-|x-1/2|], & x\in[1/2,3/2)\,. \end{cases}$$

(b) Find $u(x, 1/2)$.

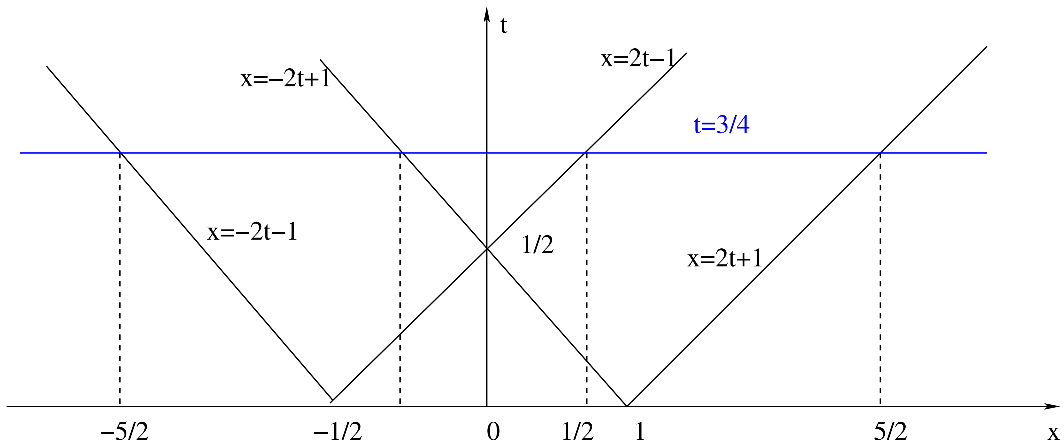


The solution at time $t = 1/2$ is

$$u(x,1/2) = \frac{1}{2}\cdot[f(x-1)+f(x+1)]$$

$$= \begin{cases} 0, & x \in (-\infty, -2) \cup (2, \infty)\,, \\ \frac{1}{2} \cdot [1 - |x+1|], & x \in [-2, 0] \\ \frac{1}{2} \cdot [1 - |x-1/2|], & x \in [0, 2]\,. \end{cases}$$

(c) Find $u(x, 3/4)$.



The solution at time $t = 3/4$ is

$$u(x, 3/4) = \frac{1}{2} \cdot [f(x - 3/2) + f(x + 3/2)]$$

$$= \begin{cases} 0, & x \in (-\infty, -3/2) \cup (-1/2, 1/2) \cup (3/2, \infty)\,, \\ \frac{1}{2} \cdot [1 - |x+3/2|], & x \in [-3/2, -1/2] \\ \frac{1}{2} \cdot [1 - |x-3/2|], & x \in [1/2, 3/2]\,. \end{cases}$$

□

3.2.3 1 D wave equation with sources

Consider the 1-D wave equation with sources

$$\begin{cases} u_{tt}(x,t) = c^2 \cdot u_{xx}(x,t) + f(x,t), & \text{for all } x \in \mathbb{R}, t > 0 \\ u(x,0) = u_t(x,0) = 0, & \text{for all } x \in \mathbb{R}\,. \end{cases} \tag{3.4}$$

Goal: Find $u(x,t)$.

Answer. Step 1. Fix $s \geq 0$, let $w(x,t,s)$ be the solution of

$$\begin{cases} w_{tt}(x,t,s) &= c^2 \cdot w_{xx}(x,t,s), \qquad \text{for all } x \in \mathbb{R}, t > 0 \\ w(x,0,s) &= 0, \qquad \text{for all } x \in \mathbb{R}, \\ w_t(x,0,s) &= f(x,s), \qquad \text{for all } x \in \mathbb{R}. \end{cases} \tag{3.5}$$

The D'Alembert's formula yields

$$w(x,t,s) = \frac{1}{2c} \cdot \int_{x-ct}^{x+ct} f(y,s)\, dy.$$

Step 2. Apply the Duhamel's principle, we obtain that

$$\begin{aligned} u(x,t) &= \int_0^t w(x,t-s,s)\, ds \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds. \end{aligned}$$

□

Example 1. Solve the initial value problem

$$\begin{cases} u_{tt}(x,t) &= 4 \cdot u_{xx}(x,t) + xe^t, \qquad \text{for all } x \in \mathbb{R}, t > 0 \\ u(x,0) &= u_t(x,0) = 0, \qquad \text{for all } x \in \mathbb{R}. \end{cases}$$

Answer. We have

$$c = 2 \qquad \text{and} \qquad f(x,t) = xe^t.$$

The solution is

$$\begin{aligned} u(x,t) &= \frac{1}{4} \cdot \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} ye^s\, dy\, ds \\ &= \frac{1}{4} \cdot \int_0^t e^s \cdot \left[\frac{1}{2} \cdot y^2\right]\Big|_{x-2(t-s)}^{x+2(t-s)} ds \\ &= x \cdot \int_0^t e^s (t-s)\, ds = x \cdot (e^t - t - 1). \end{aligned}$$

□

2. We are now ready to study the general case in (3.1)

$$\begin{cases} u_{tt}(x,t) &= c^2 \cdot u_{xx}(x,t) + f(x,t), \\ u(x,0) &= g(x), \\ u_t(x,0) &= h(x) \end{cases}$$

Goal. Find $u(x,t)$.

Answer. 1. The superposition-principle yields

$$u = v_1 + v_2$$

where v_1 is the solution of

$$\begin{cases} v_{tt}(x,t) = c^2 \cdot v_{xx}(x,t) + f(x,t), & \text{for all } x \in \mathbb{R}, t > 0 \\ v(x,0) = v_t(x,0) = 0, & \text{for all } x \in \mathbb{R}. \end{cases}$$

and v_2 is the solution of

$$\begin{cases} v_{tt}(x,t) = c^2 \cdot v_{xx}, & \text{for all } x \in \mathbb{R}, t > 0 \\ v(x,0) = g(x), \quad v_t(x,0) = h(x) & \text{for all } x \in \mathbb{R}. \end{cases}$$

2. From the previous results, we have

$$v_1(x,t) = \frac{1}{2c} \cdot \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\ dy\ ds$$

and

$$v_2(x) = \frac{1}{2} \cdot [g(x-ct) + g(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\ dy\,.$$

The solution is

$$u(x,t) = \frac{1}{2} \cdot [g(x-ct) + g(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\ dy + \frac{1}{2c} \cdot \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\ dy\ ds\,.$$

□

Example 2. Solve the following wave equation

$$\begin{cases} 4u_{tt}(x,t) = 9 \cdot u_{xx}(x,t) + x\,, & \text{for all } x \in \mathbb{R}, t \geq 0 \\ u(x,0) = 1\,, & \text{for all } x \in \mathbb{R}\,, \\ u_t(x,0) = e^{-x} & \text{for all } x \in \mathbb{R}\,. \end{cases}$$

Answer. Rewrite the equation

$$u_{tt} = \frac{9}{4} u_{xx} + \frac{x}{4}\,.$$

We have

$$c = \frac{3}{2}, \qquad f = x, \qquad g = 1 \qquad \text{and} \qquad h = e^{-x}\,.$$

The solution

$$u(x,t) = \frac{1}{2} \cdot [g(x-ct) + g(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\ dy + \frac{1}{2c} \cdot \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\ dy\ ds$$

$$
\begin{aligned}
&= 1+\frac{1}{3}\cdot\int_{x-3/2t}^{x+3/2t} e^{-y}\,dy+\frac{1}{3}\cdot\int_0^t\int_{x-3/2(t-s)}^{x-3/2(t-s)} y\,dy\,ds\\
&= 1+\frac{1}{3}\cdot\left[e^{3/2t-x}-e^{-x-3/2t}\right]+2x\cdot\int_0^t (t-s)\,ds\\
&= 1+\frac{1}{3}\cdot\left[e^{3/2t-x}-e^{-x-3/2t}\right]+xt^2\,.
\end{aligned}
$$

□

3.3 Laplace Transform

In this subsection, we will introduce an important transform which is a very powerful tool to convert ODEs into algebraic equation and PDEs into ODEs.

Definition 3.3 *Given a piecewise continuous function u such that*

$$|u(t)| \;\leq\; C\cdot e^{at}$$

for some constant a. The Laplace transform of u is defined as

$$\mathcal{L}\{u\}(s) \;=\; U(s) \;=\; \int_0^\infty u(t)e^{-st}\,dt$$

Inverse Laplace transform

$$\mathcal{L}^{-1}\{U(s)\} \;=\; u(t) \qquad \text{if} \qquad U(s) \;=\; \mathcal{L}\{u\}(s)\,.$$

Example 1. Find the Laplace transform of

$$u(t) \;=\; e^{at} \qquad \text{for all } t\in\mathbb{R}\,.$$

Answer. From the definition, we compute

$$
\begin{aligned}
U(s) &= \int_0^{+\infty} e^{at}\cdot e^{-st}\,dt \;=\; \int_0^{+\infty} e^{(a-s)t}\cdot\,dt\\
&= \frac{1}{a-s}\cdot e^{(a-s)t}\Big|_0^\infty \;=\; \frac{1}{s-a}
\end{aligned}
$$

for all $s>a$. Therefore, the Laplace transform

$$\mathcal{L}\{u\}(s) \;=\; U(s) \;=\; \frac{1}{s-a} \qquad \text{for all } s>a\,.$$

□

1. Properties of Laplace transform Given two functions u, v, the followings hold:

(i) *Linearity*

$$\mathcal{L}\{c_1\cdot u+c_2\cdot v\}(s) \;=\; c_1\cdot\mathcal{L}\{u\}(s)+c_2\cdot\mathcal{L}\{v\}(s)\,;$$

(ii) *First derivative*

$$\mathcal{L}\{u'\}(s) \;=\; s\cdot\mathcal{L}\{u\}(s) - u(0)\,;$$

Second derivative

$$\mathcal{L}\{u'\}(s) \;=\; s^2\cdot\mathcal{L}\{u\}(s) - su(0) - u'(0)\,;$$

(iii) *Shift theorem*

$$\mathcal{L}\{e^{at}\cdot u\} \;=\; U(s-a) \qquad \text{where} \qquad U(s) \;=\; \mathcal{L}\{u\}(s\,.$$

Theorem 3.4 (Convolution theorem) *Let u and v be piecewise continuous functions and*

$$|u(t)|, |v(t)| \;\leq\; e^{at} \qquad \textit{for all } t\in\mathbb{R}\,.$$

Denote by

$$(u*v)(t) \;=\; \int_0^t u(t-\tau)\cdot v(\tau)\;d\tau\,.$$

Then

$$\mathcal{L}\{u*v\}(s) \;=\; U(s)\cdot V(s) \qquad \text{where} \qquad U(s) \;=\; \mathcal{L}\{u\}, V(s) \;=\; \mathcal{L}\{v\}\,.$$

Moreover,

$$\mathcal{L}^{-1}\{U(s)V(s)\} \;=\; (u*v)(t)\,.$$

Proof. By the definition, we have

$$\begin{aligned}
\mathcal{L}\{u*v\}(s) \;&=\; \int_0^\infty (u*v)(t)\cdot e^{-st}\;dt\\
&=\; \int_0^\infty \left[\int_0^t u(t-\tau)\cdot v(\tau)\;d\tau\right]\;ds\\
&=\; \int_0^\infty \int_0^t \Big(u(t-\tau)\cdot e^{-s(t-\tau)}\Big)\cdot\big(v(\tau)\cdot e^{-s\tau}\big)\;d\tau dt\,.
\end{aligned}$$

Thanks to the Fubini's theorem, it holds

$$\begin{aligned}
&\int_0^\infty \int_0^t \Big(u(t-\tau)\cdot e^{-s(t-\tau)}\Big)\cdot\big(v(\tau)\cdot e^{-s\tau}\big)\,d\tau\;dt\\
&\qquad=\; \int_0^\infty \int_\tau^\infty \Big(u(t-\tau)\cdot e^{-s(t-\tau)}\Big)\cdot\big(v(\tau)\cdot e^{-s\tau}\big)\,dt\;d\tau\\
&\qquad=\; \left(\int_0^\infty v(\tau)\cdot e^{-s\tau}\;d\tau\right)\cdot\left(\int_0^t u(t)\cdot e^{-st}\;dt\right) \;=\; U(s)\cdot V(s)\,.
\end{aligned}$$

□

Example 2. Find inverse Laplace transform

$$F(s) \;=\; \frac{1}{s\cdot(s^2+1)}$$

Answer. Let's consider

$$U(s) \; = \; \frac{1}{s} \qquad \text{and} \qquad V(s) \; = \; \frac{1}{s^2+1}\,.$$

We have

$$\mathcal{L}^{-1}\{U(s)\} \; = \; \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \; = \; 1 \qquad \text{and} \qquad \mathcal{L}^{-1}\{V(s)\} \; = \; \mathcal{L}^{-1}\left\{\frac{1}{1+s^2}\right\}\,.$$

Using the convolution's theorem

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} \; &= \; \mathcal{L}^{U(s)\cdot V(s)} \; = \; (1*\sin(t))(t) \\ &= \; \int_0^t \sin(\tau)\, d\tau \; = \; -\cos(\tau)\Big|_0^t \; = \; 1-\cos(t)\,.\end{aligned}$$

□

Example 3. Find inverse Laplace transform

$$F(s) \; = \; \frac{1}{(s+1)\cdot(1+s^2)}\,.$$

Proof. Let's consider

$$U(s) \; = \; \frac{1}{s+1} \qquad \text{and} \qquad V(s) \; = \; \frac{1}{s^2+1}$$

We have

$$\mathcal{L}^{-1}\{U(s)\} \; = \; \mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\} \; = \; e^{-t} \qquad \text{and} \qquad \mathcal{L}^{-1}\{V(s)\} \; = \; \mathcal{L}^{-1}\left\{\frac{1}{1+s^2}\right\}\,.$$

Using the convolution's theorem

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} \; &= \; \mathcal{L}^{-1}\{U(s)\cdot V(s)\} \; = \; (e^{-t}*\sin(t))(t) \; = \; \int_0^t e^{t-\tau}\cdot\sin(\tau)\, d\tau \\ &= \; \frac{1-e^{-t}(1+\cos(t))}{2}.\end{aligned}$$

□

Example 4. (Application to ODEs) Using Laplace transform to solve the Cauchy problem

$$3u'(t)+2u(t) \; = \; \sin(t) \qquad \text{with} \qquad u(0) \; = \; 3\,.$$

Answer.

Step 1. Set $U(s) \doteq \mathcal{L}\{u\}$. By talking the Laplace transform in both side of the ODE, we have

$$\begin{aligned}\mathcal{L}\{\sin t\} \; &= \; \mathcal{L}\{3u'+2u\} \\ &= \; 3\cdot\mathcal{L}\{u'\}+2\cdot\mathcal{L}\{u\}\end{aligned}$$

$$= \quad 3 \cdot [s \cdot U(s) - u(0)] + 2 \cdot U(s) \;=\; 3sU(s) - 9 + 2U\,.$$

This implies that

$$U(s) \;=\; \frac{9}{3s+2} + \frac{F(s)}{3s+2} \qquad \text{where} \qquad F(s) \;=\; \mathcal{L}\{\sin t\}\,.$$

Step 2. Using the convolution's theorem, we recover the solution

$$\begin{aligned}
u(t) \;&=\; \mathcal{L}^{-1}\left\{\frac{9}{3s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{V(s)}{3s+2}\right\} \\
&=\; 3\cdot\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{2}{3}}\right\} + \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{V(s)\cdot\frac{1}{s+\frac{2}{3}}\right\} \\
&=\; 3\cdot e^{-\frac{2}{3}t} + \frac{1}{3}\cdot\left(e^{-\frac{2}{3}t} * \sin(t)\right)(t) \\
&=\; 3\cdot e^{-\frac{2}{3}t} + \frac{1}{3}\cdot\int_0^t e^{-\frac{2}{3}(t-\tau)}\cdot\sin(\tau)\,d\tau \\
&=\; \frac{42}{13}\cdot e^{-\frac{2}{3}t} + \frac{6\sin(t) - 9\cos(t)}{39}.
\end{aligned}$$

□

2. Heat equation in the semi-domain. Given $u(x,t)$, denote by

$$U(x,s) \;\doteq\; \mathcal{L}\{u(x,t)\} \;=\; \int_0^\infty u(x,t)\cdot e^{-st}\,dt$$

One has

$$\mathcal{L}\{u_x\} \;=\; U_x(x,s), \qquad\qquad \mathcal{L}\{u_{xx}\} \;=\; U_{xx}(x,s)$$

and

$$\mathcal{L}\{u_t\} \;=\; sU(x,s) - u(x,0)\,.$$

Example 5. Consider the heat equation with boundary condition

$$\begin{cases} u_t(x,t) \;=\; u_{xx}(x,t)\,, & \text{for all } x>0, t>0 \\ u(x,0) \;=\; 0\,, & \text{for all } x>0\,, \\ u(0,t) \;=\; f(t) & \text{for all } t>0\,. \end{cases}$$

Find a bounded solution u.

Answer. *Step 1.* Set $U(x,s) \doteq \mathcal{L}\{u(x,t)\}$. We have

$$\mathcal{L}\{u_t\} \;=\; \mathcal{L}\{u_{xx}\} \qquad \Longleftrightarrow \qquad sU(x,s) - u(x,0) \;=\; U_{xx}(x,s)\,.$$

Since $u(x,0) = 0$, we obtain the second order ODE

$$U_{xx}(x,s) - s\cdot U(x,s) \;=\; 0\,.$$

Solving the above equation, we obtain that

$$U(x,s) \;=\; a(s)\cdot e^{-\sqrt{s}\cdot x} + b(s)\cdot e^{\sqrt{s}\cdot x}\,.$$

On the other hand,

$$U(0,s) \;=\; \mathcal{L}\{f(t)\} \;\doteq\; F(s)\,.$$

This implies that

$$a(s)+b(s) \;=\; F(s)\,.$$

Since the solution u is bounded, we have $b(s)=0$ for all $s>0$ and it yields

$$a(s) \;=\; F(s) \qquad \text{for all } s>0\,.$$

Thus,

$$U(x,s) \;=\; F(s)\cdot e^{-\sqrt{s}\cdot x}\,.$$

Step 2. Recall that

$$\mathcal{L}^{-1}\left(e^{-\sqrt{s}\cdot x}\right) \;=\; \frac{x}{\sqrt{4\pi t^3}}\cdot e^{-\frac{x^2}{4t}} \;\doteq\; g(t)\,.$$

Using the convolution's theorem, we obtain

$$\begin{aligned} u(x,t) \;&=\; \mathcal{L}^{-1}\{U(x,s)\} \;=\; \mathcal{L}^{-1}\left(e^{-\sqrt{s}x}\cdot F(s)\right) \\ &=\; (g*f)(t) \;=\; \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}} e^{-\frac{x^2}{4(t-\tau)}}\cdot f(\tau)\,d\tau\,. \end{aligned}$$

□

3.4 The Fourier Transform

In this subsection, we will introduce several useful properties of Fourier transform and to apply them to solve linear PDEs.

Definition 3.5 *Let $f:\mathbb{R}\to\mathbb{R}$ be an integrable function. The Fourier function of f is denoted by*

$$\mathcal{F}\{f\}(\xi) \;=\; F(\xi) \;=\; \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} f(x)\cdot e^{-ix\xi}\,dx\,.$$

The new function F is defined on $(-\infty,\infty)$ and may or may not be a complex value function.

1. Common Fourier transforms.

- If $f(x)=\begin{cases} e^{-x}, & \text{for all } x\geq 0 \\ -e^{x}\,, & \text{for all } x<0 \end{cases}$ then $\mathcal{F}\{f\} = -i\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{\xi}{1+\xi^2}$;
- If $f(x)=\begin{cases} 1, & \text{for all } x\geq 0 \\ 0\,, & \text{for all } x<0 \end{cases}$ then $\mathcal{F}\{f\} = \sqrt{\frac{2}{\pi}}\cdot\frac{\sin\xi}{\xi}$;

- If $f(x) = e^{-x^2}$ then $F(\xi) = \frac{1}{\sqrt{2}} \cdot e^{-\frac{\xi^2}{4}}$

2. Properties of Fourier Transform. Given g and f two integrable functions, the followings hold:

(i) *Linearity.*

$$\mathcal{F}\{a \cdot f + b \cdot g\} = a \cdot \mathcal{F}\{f\} + b \cdot \mathcal{F}\{g\};$$

(ii) *First derivative*

$$\mathcal{F}\{f'\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f' \cdot e^{-ix\xi} \, dx = i\xi \mathcal{F}\{f\};$$

Second derivative

$$\mathcal{F}\{f''\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'' \cdot e^{-ix\xi} \, dx = -\xi^2 \mathcal{F}\{f\};$$

(iii) *Convolution's theorem.* Here, we denote by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x-y) g(y) \, dy \, .$$

Then

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\} \, .$$

Inverse Fourier Transform

$$\mathcal{F}^{-1}\{F\} = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} F(\xi) \cdot e^{ix\xi} \, d\xi = f(x)$$

where

$$f(x) = \mathcal{F}\{F\}(x) \, .$$

Thus,

$$\mathcal{F}^{-1}\{\mathcal{F}(f) \cdot \mathcal{F}\{g\}\}(x) = (f * g)(x) \, .$$

3. An application to PDEs. Let us use the Fourier Transform to derive a general formula for 1-D heat equation

$$u_t(x,t) = \alpha^2 \cdot u_{xx}(x,t) \qquad \text{for all } x \in \mathbb{R}, t > 0 \tag{3.6}$$

with the initial data

$$u(x,0) = \phi(x) \qquad \text{for all } x \in \mathbb{R} \, .$$

Step 1. Denote by

$$U(\xi, t) = \mathcal{F}\{u(x,t)\} = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} u(x,t) \cdot e^{-ix\xi} \, dx \, .$$

One has that

$$U_t(\xi,t) = \mathcal{F}\{u(x,t)\} \qquad \text{and} \qquad U_t(\xi,t) = -\xi^2 \cdot \mathcal{F}\{u_{xx}(x,t)\}$$

Taking the Fourier transform in both sides of (3.6), we get

$$U_t(\xi,t) \;=\; = -\alpha^2\xi^2 U(\xi,t), \qquad\qquad U(\xi,0) \;=\; \Phi(\xi)$$

where

$$\Phi(\xi) \;=\; \mathcal{F}\{\phi\}\,.$$

Step 2. Solving the above ODE, we obtain that

$$U(\xi,t) \;=\; \Phi(\xi)\cdot e^{-\alpha^2\xi^2 t}\,.$$

Step 3. The solution is

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}\{U(\xi,t)\}(x) \\
&= \mathcal{F}^{-1}\{e^{-\alpha^2\xi^2 t}\cdot\Phi(\xi)\}(x) \\
&= \mathcal{F}^{-1}\left\{\mathcal{F}\left\{\frac{1}{\alpha\sqrt{2t}}\cdot e^{-\frac{x^2}{4\alpha^2 t}}\right\}\mathcal{F}\{\phi\}\right\}(x) \\
&= \left(\frac{1}{\alpha\sqrt{2t}}\cdot e^{-\frac{x^2}{4\alpha^2 t}}\right) * \phi \\
&= \frac{1}{2\alpha\sqrt{\pi t}}\cdot\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha^2 t}}\cdot\phi(y)\;dy\,.
\end{aligned}$$

□

4 Orthogonal expansions

4.1 Inner product spaces and orthogonal basis

In this subsection, we study basic concepts in infinite dimensional vector space and definite the Fourier series.

I. Norm spaces.

Definition 4.1 *The set H is a real vector space if the followings holds*

(i) $\alpha\cdot f\in H$ *for all* $\alpha\in\mathbb{R}, f\in H$;

(ii) $f+g\in H$ *for all* $f,g\in H$.

Example 1. The sets

(a) $\mathbb{R}^n \;=\; \{v \mid v \text{ is a column real vector with } n \text{ components}\}$;

(b) $P_n \;=\; \{f(x) \mid f(x) \text{ is a a polynomial with degree } \le n\}$;

(c) $\mathbf{L}^1(a,b) \;=\; \left\{f(x)\;:\;\int_a^b |f(x)|\,dx \;<\; +\infty\right\}$

are vector spaces.

Inner product. We introduce $\langle \cdot, \cdot \rangle$ an inner product on H which satisfies the following properties:

(*i*) *Symmetry*

$$\langle f, g\rangle \;=\; \langle g, f\rangle \qquad \text{for all } f, g \in H\,;$$

(*ii*) *Linearity*

$$\langle \alpha \cdot f, g\rangle \;=\; \alpha \cdot \langle f, g\rangle \qquad \text{and} \qquad \langle f+g, h\rangle \;=\; \langle f, h\rangle + \langle g, h\rangle$$

for all $\alpha \in \mathbb{R}$ and $f, g, h \in H$;

(*iii*) *Positive-definiteness*

$$\langle f, f\rangle \;\geq\; 0 \qquad \text{for all } f \in H$$

and

$$\langle f, f\rangle \;=\; 0 \qquad \Longleftrightarrow \qquad f \;=\; 0\,.$$

Norm. The length of f is defined by

$$\|f\| \;=\; \sqrt{\langle f, f\rangle};$$

We say that f and g are orthogonal

$$f \perp g \qquad\qquad \text{if and only if} \qquad\qquad \langle f, g\rangle \;=\; 0\,.$$

Definition 4.2 *The subset $B \subset H$ is orthogonal if*

$$f \perp g \qquad\qquad \text{for all } f \neq g \in B\,.$$

Example 2. Consider

$$\mathbb{R}^3 \;=\; \;=\; \{v \mid v \text{ is a column real vector with 3 components}\}\,.$$

The inner product

$$\langle v, w\rangle \;=\; v \cdot w \;=\; \sum_{i=1}^{n} v_i w_i$$

where

$$v \;=\; \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \text{and} \qquad w \;=\; \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\,.$$

The norm of v is

$$\|v\| \;=\; \sqrt{\langle v, v\rangle} \;=\; \sqrt{v_1^2 + v_2^2 + v_3^2}\,.$$

The set $B = \{e_1, e_2, e_3\}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2 \;=\; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad e_3 \;=\; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are orthogonal.

Lemma 4.3 *Let* $\{f_1, f_2, ..., f_n\}$ *be orthogonal in* H*. If*

$$f \;=\; \sum_{i=1}^{n} \alpha_i \cdot f_i \;=\; \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2 + ... + \alpha_n \cdot f_n$$

then the coefficients α_i *are computed as*

$$\alpha_i \;=\; \frac{\langle f, f_i\rangle}{\langle f_i, f_i\rangle} \qquad \text{for all } i \in \{1, 2, ..., n\}\,.$$

Proof. Using the linearity property of the inner product, we have

$$\langle f, f_i\rangle \;=\; \left\langle \sum_{j=1}^{n} \alpha_j \cdot f_j, f_i \right\rangle \;=\; \sum_{j=1}^{n} \alpha_j \cdot \langle f_j, f_i\rangle\,.$$

Recalling that the set $\{f_1, f_2, ..., f_n\}$ is orthogonal, it holds

$$\langle f_j, f_i\rangle \;=\; 0 \qquad \text{for all } j \neq i\,.$$

Therefore,

$$\langle f, f_i\rangle \;=\; \alpha_i \cdot \langle f_i, f_i\rangle \qquad \text{for all } i \in \{1, 2, ..., n\}\,.$$

□

II. $\mathbf{L}^2(a,b)$ **space.** Given two real number $a < b$, we denote by

$$\mathbf{L}^2(a,b) \;\doteq\; \left\{ f : (a,b) \to \mathbb{R} \;:\; \int_a^b |f(x)|^2\, dx < +\infty \right\}.$$

It is clear that $\mathbf{L}^2(a,b)$ is a vector space. Indeed, for any $\alpha \in \mathbb{R}$ and $f, g \in \mathbf{L}^2(a,b)$, it holds

$$\int_a^b |\alpha f(x)|^2\, dx \;=\; |\alpha|^2 \cdot \int_a^b |f(x)|^2\, dx \;<\; +\infty$$

and it yields $\alpha \cdot f \in \mathbf{L}^2(a,b)$.

On the other hand, we have

$$\int_a^b |f(x) + g(x)|^2\, dx \;\leq\; 2 \cdot \left[\int_a^b |f(x)|^2 + |g(x)|^2\right] dx \;<\; +\infty\,.$$

By the definition, the function $(f+g)$ is $\mathbf{L}^2(a,b)$.

Let us now introduce the inner product for $L^2(\mathbb{R})$ space. Given f, g in $\mathbf{L}^2(a,b)$, the inner product of f and g is defined as

$$\langle f, g\rangle \;\doteq\; \int_a^b f(x)g(x)\, dx\,.$$

The $\mathbf{L}^2-$norm of f is

$$\|f\|_{\mathbf{L}^2} \;=\; \sqrt{\langle f, f\rangle} \;=\; \left(\int_a^b |f(x)|^2\, dx\right)^{\frac{1}{2}}.$$

Cauchy-Schwarz inequality.

$$\langle f, g\rangle \;\le\; \|f\|_{\mathbf{L}^2}\cdot\|g\|_{\mathbf{L}^2} \qquad \text{for all } f, g \in \mathbf{L}^2(a,b)\,.$$

Example 3. Consider three functions

$$f_1(x) \;=\; 1, \qquad f_2(x) \;=\; \sin(x) \qquad \text{and} \qquad f_3(x) \;=\; \cos(x)\,.$$

(a). Show that f_1, f_2, f_3 are in $\mathbf{L}^2(0,2\pi)$.

(b) Compute the $\mathbf{L}^2-$norm of f_i for $i\in\{1,2.3\}$.

(c) Is the set $\{f_1, f_2, f_3\}$ orthogonal?

Answer. *(a) and (b).* We compute that

$$\int_0^{2\pi} |f_1(x)|^2\;dx \;=\; \int_0^{2\pi} 1\;dx \;=\; 2\pi \;<\; +\infty\,.$$

Thus, f_1 is in $\mathbf{L}^2(0,2\pi)$ and

$$\|f_1\|_{\mathbf{L}^2} \;=\; \left(\int_0^{2\pi} |f_1(x)|^2\;dx\right)^{\frac{1}{2}} \;=\; \sqrt{2\pi}\,.$$

Similarly, we compute

$$\int_0^{2\pi} |f_2(x)|^2\;dx \;=\; \int_0^{2\pi}\sin^2 x\;dx \;=\; \frac{1}{2}\int_0^{2\pi}(1-\cos(2x))\;dx \;=\; \pi \;<\; +\infty$$

and

$$\int_0^{2\pi} |f_3(x)|^2\;dx \;=\; \int_0^{2\pi}\cos^2 x\;dx \;=\; \frac{1}{2}\int_0^{2\pi}(1+\cos(2x))\;dx \;=\; \pi \;<\; +\infty\,.$$

Thus, f_2 and f_3 are in $\mathbf{L}^2(0,2\pi)$ and

$$\|f_2\|_{\mathbf{L}^2} \;=\; \|f_3\|_{\mathbf{L}^2} \;=\; \sqrt{\pi}\,.$$

(c). We compute

$$\langle f_1, f_2\rangle \;=\; \int_0^{2\pi} f_1(x)\cdot f_2(x)\;dx \;=\; \int_0^{2\pi}\sin(x)\;dx \;=\; -\cos(x)\Big|_0^{2\pi} \;=\; 0\,,$$

$$\langle f_1, f_3\rangle \;=\; \int_0^{2\pi} f_1(x)\cdot f_3(x)\;dx \;=\; \int_0^{2\pi}\cos(x)\;dx \;=\; -\sin(x)\Big|_0^{2\pi} \;=\; 0\,,$$

and

$$\begin{aligned}\langle f_2, f_3\rangle \;&=\; \int_0^{2\pi} f_1(x)\cdot f_3(x)\;dx \;=\; \int_0^{2\pi}\sin(x)\cos(x)\;dx\\ &=\; \frac{1}{2}\cdot\int_0^{2\pi}\sin(2x)\;dx \;=\; -\frac{1}{4}\int_0^{2\pi}\cos(2x)\;dx \;=\; 0\,.\end{aligned}$$

This implies that

$$f_1 \perp f_2, \qquad f_1 \perp f_3 \qquad \text{and} \qquad f_2 \perp f_3 .$$

Therefore, the set $\{f_1, f_2, f_3\}$ is orthogonal. □

Example 4. Let $f(x) = x^2$ and $g(x) = 1 + x$ on $[0, 1]$.

(a) Compute $\|f\|^2_{\mathbf{L}^2}$, $\|g\|^2_{\mathbf{L}^2}$ and $\langle f, g\rangle$;

(b) Compute $\|2f + g\|_{\mathbf{L^2}}$.

Answer. (a). We compute

$$\|f\|^2_{\mathbf{L}^2} \;=\; \langle f, f\rangle \;=\; \int_0^1 x^4 \, dx \;=\; \frac{1}{5},$$

$$\|g\|^2_{\mathbf{L}^2} \;=\; \langle g, g\rangle \;=\; \int_0^1 (1+x)^2 \, dx \;=\; 1,$$

and

$$\langle f, g\rangle \;=\; \int_0^1 x^2(1+x) \, dx \;=\; \int_0^1 x^2 + x^3 \, dx \;=\; \frac{1}{3} + \frac{1}{4} \;=\; \frac{7}{12} .$$

(b). We have that

$$\begin{aligned} \|2f + g\|^2_{\mathbf{L^2}} &= \langle 2f + g, 2f + g\rangle \\ &= 4 \cdot \langle f, f\rangle + 4 \cdot \langle f, g\rangle + \langle g, g\rangle \\ &= \frac{4}{5} + 4 \cdot \frac{7}{12} + 1 \;=\; \frac{62}{15} . \end{aligned}$$

Thus, the norm

$$\|2f + g\|_{\mathbf{L}^2} \;=\; \sqrt{\frac{62}{15}} .$$

□

Definition 4.4 *The set of function $\{f_1, f_2, ..., f_n\} \subset \mathbf{L}^2(a, b)$ is called orthonormal system on the interval (a, b) if*

(i) the norm $\|f_i\|_{\mathbf{L}^2} = 1$ for all $i \in \{1, 2, ..., n\}$;

(ii) For any $i \neq j \in \{1, 2, ..., n\}$, it holds

$$\langle f_i \;,\; f_j\rangle \;=\; 0 .$$

Example 5. The set $\left\{\sqrt{\frac{2}{\pi}} \cdot \sin x, \sqrt{\frac{2}{\pi}} \cdot \sin 2x, ..., \sqrt{\frac{2}{\pi}} \cdot \sin nx\right\}$ is an orthonormal system on the interval $[0, \pi]$.

Answer. For any $k \in \{1, 2, ..., n\}$, we compute that

$$\|\sin kx\|^2_{\mathbf{L}^2} \;=\; \frac{2}{\pi} \cdot \int_0^\pi \sin^2 kx \;=\; \frac{1}{\pi} \cdot \int_0^\pi 1 - \cos(2kx) \, dx \;=\; 1$$

and it yields $\| \sin kx \|_{\mathbf{L}^2} = 1$.

On the other hand, for any $k \neq m \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} \langle \sin kx, \sin mx \rangle &= \int_0^\pi \sin(kx) \cdot \sin(mx) \, dx \\ &= \frac{1}{2} \cdot \int_0^\pi [\cos([k-m]x) - \cos([k+m]x)] \;=\; 0 \end{aligned}$$

and it yields that $\left(\sqrt{\frac{2}{\pi}} \cdot \sin kx \right)$ and $\left(\sqrt{\frac{2}{\pi}} \cdot \sin mx \right)$ are orthogonal. □

III. Orthogonal expasions. Given a orthonormal system of functions $\mathcal{F} = \{f_1, f_2, ..., f_n, ...\}$ in the space $\mathbf{L}^2(a, b)$. Can any function $f \in \mathbf{L}^2(a, b)$ be expanded in a infinite series of $\mathcal{F}$

$$f \;=\; \sum_{n=1}^{+\infty} c_n \cdot f_n$$

where c_n are real coefficients.

Theorem 4.5 *Let $f \in \mathbf{L}^2(a, b)$ and $\mathcal{F} = \{f_1, f_2, ..., f_n, ...\}$ be an orthonormal system of $\mathbf{L}^2(a, b)$. Assume that*

$$f \;=\; \sum_{n=1}^{+\infty} c_n \cdot f_n \,.$$

Then

$$c_n \;=\; \langle f, f_n \rangle \qquad \text{and} \qquad \|f\|_{\mathbf{L}^2}^2 \;=\; \sum_{n=1}^{+\infty} c_n^2 \,.$$

Proof. For any $n \in \{1, 2...\}$, it holds

$$\langle f_n \,,\, f_k \rangle \;=\; 0 \qquad \text{for all } n \neq k \,.$$

We have

$$\begin{aligned} \langle f_n \,,\, f \rangle &= \left\langle f_n \,,\, \sum_{k=1}^{\infty} c_k \cdot f_k \right\rangle \\ &= c_n \cdot \langle f_n \,,\, f_n \rangle + \sum_{n \neq k=1}^{\infty} c_k \cdot \langle f_n \,,\, f_k \rangle \;=\; c_n \cdot \|f_n\|_{\mathbf{L}^2}^2 \;=\; c_n \,. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f\|_{\mathbf{L}^2}^2 \;=\; \langle f \,,\, f \rangle &= \left\langle \sum_{k=1}^{\infty} c_k \cdot f_k \,,\, f \right\rangle \\ &= \sum_{k=1}^{\infty} c_k \cdot \langle f_k, f \rangle \;=\; \sum_{k=1}^{2} |c_k|^2 \,. \end{aligned}$$

□

Remark. *The series* $\sum_{n=1}^{+\infty} c_n \cdot f_n$ *is called the generalized Fourier series of* f *and* c_n *are called the Fourier coefficients.*

Definition 4.6 *An orthonomal system* $\{f_1, f_2, ...f_n, ...\} \subset \mathbf{L}^2(a.b)$ *is said complete if and only if*

$$\langle f, f_n \rangle \ = \ 0 \qquad \text{for all } n \qquad \Longrightarrow \qquad f \ = \ 0 \,.$$

Theorem 4.7 *(Fourier expansion) Assume that* $\{f_1, f_2, ..., f_n, ...\}$ *is a complete orthonormal system in* $\mathbf{L}^2(a, b)$*. Then for any* $f \in \mathbf{L}^2(a, b)$*, it holds*

$$f \ = \ \sum_{n=1}^{\infty} c_n \cdot f_n \,,$$

where the coefficient

$$c_n \ = \ \langle f, f_n \rangle \qquad \text{for all } n = 1, 2, ...$$

Proof. 1. Let's consider

$$S_n \ = \ \sum_{k=1}^{n} c_k \cdot f_k \,.$$

The orthogonal property of $\{f_1, f_2, ..., f_n, ...\}$ yields

$$\langle f, S_n \rangle \ = \ \|S_n\|^2_{\mathbf{L}^2} \ = \ \sum_{k=1}^{n} |c_k|^2 \,.$$

Using the Cauchy-Schwarz inequality, we have that

$$\|S_n\|^2_{\mathbf{L}^2} \ = \ \sum_{k=1}^{n} |c_k|^2 \ \leq \ \|f\|^2_{\mathbf{L}^2} \,.$$

Therefore, we can show that S_n converges to g in $\mathbf{L}^2(a, b)$ and it yields

$$g \ = \ \sum_{n=1}^{\infty} c_n \cdot f_n \,.$$

2. It remains to show that $f = g$. One can check that

$$\langle f - g, f_n \rangle \ = \ 0 \qquad \text{for all } n = 1, 2, ...$$

Thus, the completeness implies that $f - g = 0$. □

4.2 Classical Fourier series

1. Given $\ell > 0$, denote by

$$\mathbf{L}^2(-\ell,\ell) \;=\; \left\{ f : (-\ell,\ell) \to \mathbb{R} \;\Big|\; \int_{-\ell}^{\ell} |f(x)|^2 \, dx \right\}.$$

The following holds:

Lemma 4.8 *The trigonometric set*

$$\mathcal{F} \;=\; \left\{ 1, \sin\left(\frac{m\pi x}{\ell}\right), \cos\left(\frac{m\pi x}{\ell}\right) \;\Big|\; m = 1, 2, ... \right\}$$

is a complete orthogonal in $\mathbf{L}^2(-\ell,\ell)$.

From the above Lemma and Theorem 4.7, one can show that for any $f \in \mathbf{L}^2(-\ell,\ell)$, then f can be expressed by an infinite sum functions in $\mathcal{F}$. More precisely,

Definition 4.9 *For any function* $f \in \mathbf{L}^2(-\ell,\ell)$, *its Fourier series is*

$$f \;\simeq\; \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cdot \cos\frac{m\pi x}{\ell} + b_m \cdot \sin\frac{m\pi x}{\ell} \right)$$

where a_m *and* b_m *are Fourier coefficients and computed by*

$$a_n \;=\; \frac{1}{\ell} \cdot \int_{-\ell}^{\ell} f(x) \cdot \cos\frac{m\pi x}{\ell} \, dx$$

and

$$b_n \;=\; \frac{1}{\ell} \cdot \int_{-\ell}^{\ell} f(x) \cdot \sin\frac{m\pi x}{\ell} \, dx$$

for all $m = 0, 1, 2, ...$

Example 1. Find the Fourier series of the function

$$f(x) \;=\; \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \tag{4.1}$$

in $\mathbf{L}^2(-\pi,\pi)$.

Answer. We have $\ell = \pi$. The Fourier series of f in $\mathbf{L}^2(-\pi,\pi)$ is

$$f(x) \;\simeq\; \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cdot \cos mx + b_m \cdot \sin mx \right).$$

The Fourier coefficients are computed by

$$a_0 \;=\; \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \, dx \;=\; \frac{1}{\pi} \cdot \left[\int_{-\pi}^{0} -1 \, dx + \int_{0}^{\pi} 1 \, dx \right] \;=\; 0,$$

$$a_m \ = \ \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\cdot\cos mx\ dx \ = \ \frac{1}{\pi}\cdot\left[-\int_{-\pi}^{0}\cos mx\ dx + \int_{0}^{\pi}\cos mx\ dx\right] \ = \ 0\,,$$

and

$$\begin{aligned} b_m \ &= \ \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\cdot\sin mx\ dx \ = \ \frac{1}{\pi}\cdot\left[-\int_{-\pi}^{0}\sin x\ dx + \int_{0}^{\pi}\sin x\ dx\right] \\ &= \ \frac{1}{\pi}\cdot\left[-\int_{-\pi}^{0}\sin mx\ dx + \int_{0}^{\pi}\sin mx\ dx\right] \ = \ \frac{1}{m\pi}\cdot[2-\cos(-m\pi)-\cos(m\pi)] \\ &= \ \frac{2}{m\pi}\cdot[1-\cos(m\pi)] \ = \ \frac{2}{m\pi}\cdot[1-(-1)^m]\ . \end{aligned}$$

Therefore,

$$f(x) \ \simeq \ \sum_{m=1}^{\infty}\frac{2\cdot(1-(-1)^m)}{m\pi}\cdot\sin mx\,.$$

□

Example 2. Find a Fourier series for the function

$$f(x) \ = \ x \qquad\qquad \text{for all } x\in(-2,2)$$

in $\mathbf{L}^2(-2,2)$.

Answer. We have that $\ell = 2$. The Fourier series of f in $\mathbf{L}^2(-2,2)$ is

$$f(x) \ \simeq \ \frac{a_0}{2} + \sum_{m=1}^{+\infty}\left[a_m\cdot\cos\frac{m\pi x}{2} + b_m\cdot\sin\frac{m\pi x}{2}\right]\ .$$

The Fourier coefficients are computed by

$$a_m \ = \ \frac{1}{2}\cdot\int_{-2}^{2} x\cos\frac{m\pi x}{2}\ dx \ = \ 0$$

and

$$b_m \ = \ \frac{1}{2}\cdot\int_{-2}^{2} x\sin\frac{m\pi x}{2}\ dx \ = \ \frac{-4}{m\pi}\cdot\cos(m\pi) \ = \ \frac{4}{m\pi}\cdot(-1)^{m+1}$$

for all $m = 0, 1, \ldots.$ Thus,

$$f(x) \ \simeq \ \frac{4}{\pi}\cdot\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\cdot\sin\frac{m\pi x}{2}\,.$$

□

2. Fourier sine and Fourier cosine. Given a function $f:(-\ell,\ell)\to\mathbb{R}$ in $\mathbf{L}^2(-\ell,\ell)$, the followings hold:

- f is even if $f(x) = f(-x)$ for all $x\in(0,\ell)$. In this case, we have

$$\int_{-\ell}^{\ell} f(x)\ dx \ = \ 2\cdot\int_{0}^{\ell} f(x)\ dx\,.$$

- f is odd if $f(x) = -f(-x)$ for all $x \in (-0, \ell)$. In this case, we have

$$\int_{-\ell}^{\ell} f(x)\; dx \;=\; 0\,.$$

Fourier cosine. If the function f is even on $(-\ell, \ell)$, then

$$f(x) \;\simeq\; \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cdot \cos\frac{m\pi x}{\ell}$$

where the Fourier coefficients

$$a_m \;=\; \frac{2}{\ell} \cdot \int_0^{\ell} f(x) \cdot \cos\frac{m\pi x}{\ell}\; dx\,.$$

Fourier sine. If the function f is odd on $(-\ell, \ell)$, then

$$f(x) \;\simeq\; \sum_{m=1}^{\infty} b_m \cdot \cos\frac{m\pi x}{\ell}$$

where the Fourier coefficients

$$b_m \;=\; \frac{2}{\ell} \cdot \int_0^{\ell} f(x) \cdot \sin\frac{m\pi x}{\ell}\; dx\,.$$

3. Periodic functions on $\mathbb{R}$ and half-range expansion. Given a real function $f : \mathbb{R} \to \mathbb{R}$, we say that f is *periodic* with a period P if

$$f(x+P) \;=\; f(x) \qquad\qquad x \in \mathbb{R}\,.$$

Assume that $f : \mathbb{R} \to \mathbb{R}$ is periodic with period $2l$. Then the Fourier series of f in $\mathbf{L}^2(-\ell, \ell)$ is

$$f(x) \;\simeq\; \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\frac{m\pi x}{\ell} + b_m \sin\frac{m\pi x}{\ell}\right]$$

where a_m and b_m are Fourier coefficients and computed by

$$a_n \;=\; \frac{1}{\ell} \cdot \int_{-\ell}^{\ell} f(x) \cdot \cos\frac{m\pi x}{\ell}\; dx$$

and

$$b_n \;=\; \frac{1}{\ell} \cdot \int_{-\ell}^{\ell} f(x) \cdot \sin\frac{m\pi x}{\ell}\; dx$$

for all $m = 0, 1, 2, \ldots$.

Even periodic extension. Given $f : (0, \ell)$, we can extend f onto $(-\ell, \ell)$ such that

$$f(x) \;=\; f(-x) \qquad\qquad \text{for all } x \in (0, \ell)\,.$$

Then extend f into a periodic with period $P = 2\ell$, i.e.,

$$f(x) \;=\; f(x + 2\ell) \qquad \text{for all } x \in \mathbb{R}\,.$$

Odd periodic extension. Given $f : (0, \ell)$, we can extend f onto $\mathbb{R}$ such that

(i) (Odd function) $f(-x) = -f(x)$ for all $x \in (0, \ell)$;

(ii) (Periodic function) $f(x) = f(x + 2\ell)$ for all $x \in \mathbb{R}$.

Example 3. Let $f(x) = x$ for $x \in (0, 1)$. Sketch 3 periods of the even and the odd and compute the corresponding Fourier sine and cosine.

Answer. *1. Even extension.* We have

$$f_{even} \simeq \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cdot \cos m\pi x \,.$$

The Fourier coefficients are computed by

$$a_0 = 2 \cdot \int_0^1 x dx = 1 \,,$$

and

$$a_m = 2 \cdot \int_0^1 x \cdot \cos m\pi x \; dx = \frac{2 \cdot ((-1)^m - 1)}{m^2 \pi^2} \qquad \text{for all } m = 1, 2...$$

Therefore,

$$f_{even} \simeq \frac{1}{2} + \frac{2}{\pi^2} \cdot \sum_{m=1}^{\infty} \frac{(-1)^m - 1}{m^2} \cos m\pi x \,.$$

2. Odd extension. We have

$$f_{odd} \simeq \sum_{m=1}^{\infty} b_m \cdot \sin m\pi x \,.$$

The Fourier coefficients are computed by

$$b_m = 2 \cdot \int_0^1 x \cdot \sin(m\pi x) \; dx = \frac{2 \cdot (-1)^{m+1}}{m\pi} \,.$$

Therefore,

$$f_{odd} \simeq \frac{2}{\pi} \cdot \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cdot \sin(m\pi x) \,.$$

□

4. Properties of Fourier series. Given $f(x)$ and $g(x)$ in $\mathbf{L}^2(-\ell, \ell)$. Assume that

$$f \simeq \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cdot \cos \frac{m\pi x}{\ell} + b_m \cdot \sin \frac{m\pi x}{\ell} \right)$$

and

$$g \simeq \frac{c_0}{2} + \sum_{m=1}^{\infty} \left(c_m \cdot \cos \frac{m\pi x}{\ell} + d_m \cdot \sin \frac{m\pi x}{\ell} \right) \,.$$

Then the followings hold:

- For any $\alpha \in R$,

$$\alpha \cdot f \;\simeq\; \frac{\alpha a_0}{2} + \sum_{m=1}^{\infty} \left(\alpha a_m \cdot \cos \frac{m\pi x}{\ell} + \alpha b_m \cdot \sin \frac{m\pi x}{\ell} \right) .$$

- The Fourier series of the function $f+g$ is

$$f+g \;\simeq\; \frac{(a_0+c_0)}{2} + \sum_{m=1}^{\infty} \left[(a_m+c_m) \cdot \cos \frac{m\pi x}{\ell} + (b_m+d_m) \cdot \sin \frac{m\pi x}{\ell} \right] .$$

Theorem 4.10 *(Convergence theorem) Let f be in $L^2(-l,l)$ and piecewise smooth function and*

$$f \;\simeq\; \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cdot \cos \frac{m\pi x}{\ell} + b_m \cdot \sin \frac{m\pi x}{\ell} \right) .$$

Then it holds

(1). The Fourier series converges to $f(x)$ at all points x where f is continuous;

(2.) The Fourier series converges to

$$\frac{1}{2} \cdot [f(x-) + f(x+)]$$

at points x where f is discontinuous.

Example 4. Given the function $f : \mathbb{R} \to \mathbb{R}$ such that f is periodic with period 2π and

$$f(x) \;=\; \begin{cases} 0, & -\pi < x < 0 \\ 4, & 0 < x < \pi . \end{cases}$$

(a). Find the Fourier series of f in $\mathbf{L}^2(-\pi,\pi)$.

(b). Indicate the function that the Fourier series of f converges to.

Answer. (a) Let $g : (-\pi,\pi)$ be such that

$$g(x) \;=\; 2 \qquad \text{for all } x \in (-\pi,\pi) .$$

We have

$$h \;\doteq\; \frac{f-g}{2} \;=\; \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} .$$

From example 1, the Fourier series of h is

$$h \;\simeq\; \sum_{m=1}^{\infty} \frac{2 \cdot (1-(-1)^m)}{m\pi} \cdot \sin mx .$$

Recalling that $f = g + 2h$, the Fourier series of f is

$$f \simeq 2 + \sum_{m=1}^{\infty} \frac{4 \cdot (1 - (-1)^m)}{m\pi} \cdot \sin mx \,.$$

(b) Observe that f is continuous at $x \in \mathbb{R} \backslash k\pi$. Therefore, by using the convergence theorem

- The Fourier series of f converges to f at $x \in \mathbb{R} \backslash k\pi$;
- The Fourier series of f converges to 2 at $x = k\pi$ for all $k \in \mathbb{Z}$.

4.3 Sturm-Liouville problems

Let us consider a regular Sturm-Liouville system

$$[-p(x)y']' + q(x)y = \lambda w(x)y, \qquad x \in (a, b) \tag{4.2}$$

with boundary conditions

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases} . \tag{4.3}$$

Here

- α_i, β_i for $i \in \{1, 2\}$ are given constants such that $\alpha_i^2 + \beta_i^2 > 0$;
- $p(x), w(x) > 0$ and $q(x)$ are given functions.
- y and λ are unknown function and unknown constant.

Goal: Find $\lambda \in \mathbb{R}$ such that the ODE (4.2) with boundary conditions (4.3) has a non-trivial solution $y_\lambda(x)$.

This type of problem is called eigenvalue problem.

Does any $\lambda \in \mathbb{R}$, the ODE (4.2) with boundary conditions (4.3) always admits a non-trivial solution?

Example 1. Solve the two points boundary problem

$$\begin{cases} y'' + y = 0 \,, \\ y(0) = 0, \qquad y(\pi) = 2 \,. \end{cases}$$

Answer. Characteristic equation

$$r^2 + 1 = 0 \,.$$

Two complex conjugate roots

$$r_1 = i \qquad \text{and} \qquad r_2 = -i \,.$$

The general solution

$$y(x) = c_1 \cdot \cos(x) + c_2 \cdot \sin(x)\,.$$

The first boundary condition $y(0) = 0$ implies that $c_1 = 0$ and it yields

$$y(x) = c_2 \cdot \sin(x)\,.$$

The second boundary condition $y(\pi) = 2$ implies that

$$2 = y(\pi) = c_2 \cdot \sin(\pi) = 0$$

and it yields a contradiction. Thus, the ODE does not have any solution. □

Definition 4.11 *Assume that with $\lambda \in \mathbb{R}$, the ODE (4.2) with boundary conditions (4.3) has a nontrivial solution $y_\lambda(x)$. Then*

- *λ is called an eigenvalue;*
- *$y_\lambda(x)$ is called an corresponding eigenfunction.*

(λ, y_λ) is called an eigen-pair of (4.2)-(4.3).

1. Two points boundary problems with constant coefficients. Let's consider the second order linear ODE with constant coefficients

$$\begin{cases} y'' + \lambda \cdot y = 0, \\ \alpha_1 \cdot y(a) + \alpha_2 \cdot y'(a) = 0, \\ \beta_1 \cdot y(b) + \beta_2 \cdot y'(b) = 0. \end{cases}$$

Goal: *Find all eigenpairs the above two points boundary problem.*

Example 2. Consider the linear second order ODE

$$y''(x) + \lambda \cdot y(x) = 0$$

with Dirichlet boundary condition

$$y(0) = y(\pi) = 0\,.$$

Find all eigenvalues and corresponding eigenfunctions.

Answer. The characteristic equation

$$r^2 + \lambda = 0$$

Three cases are consider:

- If $\lambda < 0$ then
$$r_1 \;=\; \sqrt{|\lambda|} \qquad \text{and} \qquad r_2 \;=\; -\sqrt{|\lambda|}\,.$$
The general solution
$$y \;=\; c_1 \cdot e^{-\sqrt{|\lambda|}\cdot x} + c_2 \cdot e^{\sqrt{|\lambda|}\cdot x}\,.$$
The boundary conditions $y(0) \;=\; y(\pi) \;=\; 0$ implies that
$$c_1 + c_2 \;=\; 0 \qquad \text{and} \qquad c_1 \cdot e^{-\sqrt{|\lambda|}\cdot \pi} + c_2 \cdot e^{\sqrt{|\lambda|}\cdot \pi}$$
and it yields $c_1 = c_2 = 0$. Thus, $y = 0$ (trivial solution).

- If $\lambda = 0$ then
$$y'' \;=\; 0 \qquad \Longrightarrow y \;=\; c_1 \cdot x + c_2\,.$$
The boundary conditions $y(0) \;=\; y(\pi) \;=\; 0$ implies that
$$c_2 \;=\; 0 \qquad \text{and} \qquad c_1 \cdot \pi + c_2 \;=\; 0$$
and it yields $c_1 = c_2 = 0$. Thus, $y = 0$ (trivial solution).

- If $\lambda > 0$ then $\lambda = k^2$ for $k > 0$. The characteristic equation admit two complex roots
$$r_1 \;=\; k \cdot i \qquad \text{and} \qquad r_2 \;=\; -\,k \cdot i\,.$$
The general solution
$$y(x) \;=\; c_1 \cdot \cos(kx) + c_2 \cdot \sin(kx)\,.$$
Boundary conditions
$$y(0) \;=\; 0 \qquad \Longrightarrow \qquad c_1 \;=\; 0 \qquad \Longrightarrow \qquad y(x) \;=\; c_2 \cdot \sin(kx)$$
and thus
$$y(\pi) \;=\; 0 \qquad \Longrightarrow \qquad c_2 \cdot \sin(k\pi) \;=\; 0\,.$$
Since we are looking for nontrivial solution, we have
$$\sin(k\pi) \;=\; 0 \qquad \Longrightarrow \qquad k \;=\; n \qquad \text{for all } n = 1, 2, ...$$
Thus,
$$\lambda \;=\; n^2 \qquad \text{and} \qquad y(x) \;=\; c_2 \cdot \sin(nx) \qquad n \;=\; 1, 2, ...$$

Eigenvalues and eigenfunctions
$$\begin{cases} \lambda_n \;=\; n^2 \\ y_n(x) \;=\; \sin(nx) \end{cases} \qquad \text{for } n = 1, 2, ...$$

□

Example 3. Consider the linear second order ODE
$$y''(x) - \lambda \cdot y(x) \;=\; 0$$

with Neumann boundary condition

$$y'(0) \;=\; y'(2) \;=\; 0\,.$$

Find all eigenvalues and corresponding eigenfunctions.

Answer. The characteristic equation

$$r^2 - \lambda \;=\; 0$$

It is quite similar to the previous example, one show that if $\lambda > 0$ then the above ODE has only a trivial solution.

If $\lambda = 0$ then the solution

$$y(x) \;=\; 1 \qquad \text{for all } x \in [0,2]\,.$$

We only need to consider $\lambda < 0$. In this case, one can write

$$\lambda \;=\; -k^2 \qquad \text{for } k > 0\,.$$

The general solution

$$y(x) \;=\; c_1 \cdot \cos(kx) + c_2 \cdot \sin(kx)\,.$$

Boundary conditions

$$y'(0) \;=\; 0 \qquad \Longrightarrow \qquad c_2 \;=\; 0 \qquad \Longrightarrow \qquad y(x) \;=\; c_1 \cdot \cos(kx)$$

and thus

$$y'(2) \;=\; 0 \qquad \Longrightarrow \qquad -c_2 k \cdot \sin(2k) \;=\; 0 \qquad \Longrightarrow \qquad \sin(2k) \;=\; 0\,.$$

Therefore,

$$2k \;=\; n\pi \qquad \text{for all } n = 1, 2...$$

Eigenvalues and eigenfunctions

$$\begin{cases} \lambda_n \;=\; -\dfrac{n^2\pi^2}{4} \\ y_n(x) \;=\; \cos\left(\frac{n\pi}{2} \cdot x\right) \end{cases} \qquad \text{for } n = 0, 1, 2...$$

□

Example 4. Find all positive eigenvalues and corresponding eigenfunctions

$$\begin{cases} y'' + \lambda \cdot y \;=\; 0\,, \\ y'(0) \;=\; 0, \qquad y(\pi) + y'(\pi) \;=\; 0\,. \end{cases}$$

Answer. Set $\lambda = k^2$. The general solution

$$y(x) \;=\; c_1 \cdot \cos(kx) + c_2 \cdot \sin(kx)\,.$$

Boundary conditions

$$y'(0) \;=\; 0 \qquad \Longrightarrow \qquad c_2 \;=\; 0 \qquad \Longrightarrow \qquad y(x) \;=\; c_1 \cdot \cos(kx)$$

and thus

$$y(\pi) + y'(\pi) \;=\; 0 \qquad \Longrightarrow \qquad c_1 \cos(k\pi) - c_1 k \cdot \sin(k\pi) \;=\; 0\,.$$

This implies that

$$\frac{1}{k} \;=\; \tan(k\pi)\,.$$

Eigenvalues and eigenfunctions

$$\begin{cases} \lambda_n &= \rho_n^2 \\ y_n(x) &= \cos(\rho_n x) \end{cases} \qquad \text{for } n = 1, 2, ...$$

where ρ_n are positive solutions of the equation $\dfrac{1}{\rho} = \tan(\rho\pi)$ □

2. General theory of Sturm-Liouville problems. Let's reconsider the regular Sturm-Liouville system

$$[-p(x)y']' + q(x)y \;=\; \lambda w(x)y, \qquad x \in (a,b) \tag{4.4}$$

with boundary conditions

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) \;=\; 0 \\ \beta_1 y(b) + \beta_2 y'(b) \;=\; 0. \end{cases} \tag{4.5}$$

The second order linear differential operator

$$L[y] \;=\; \frac{1}{w(x)} \cdot \Big(- \big[p(x)y'\big]' + q(x) \cdot y \Big)\,.$$

The ODE (4.4) can be rewritten as

$$L[y] \;=\; \lambda \cdot y\,.$$

Denote by

$$\mathcal{H} \;=\; \Big\{ f \in \mathbf{L}^2(a,b) \;\Big|\; f \text{ satisfies the boundary condition (4.5)} \Big\}\,.$$

Lemma 4.12 *The operator L is a self-adjoint operator on $\mathcal{H}$, i.e.,*

$$\langle L[y_1], y_2 \rangle \;=\; \langle y_1, L[y_2] \rangle \qquad \textit{for all } y_1, y_2 \in \mathcal{H}\,.$$

Answer. By using the integrating by parts, a direct computation yields

$$\int_a^b L[y_1](x) \cdot y_2(x)\; dx \;=\; \int_a^b y_1(x) \cdot L[y_2](x)\; dx$$

□

Lemma 4.13 *Let* (λ_1, y_1) *and* (λ_2, y_2) *be eigenpairs of (4.4)-(4.5). If* $\lambda_1 \neq \lambda_2$ *then* y_1 *are* y_2 *are orthogonal in* $\mathcal{H}$.

Answer. By the definition of eigen-pairs, we have

$$L[y_1] \;=\; \lambda_1 \cdot y_1 \qquad \text{and} \qquad L[y_2] \;=\; \lambda_2 \cdot y_2$$

In particular, this implies that

$$\langle L[y_1], y_2 \rangle \;=\; \lambda_1 \cdot \langle y_1, y_2 \rangle$$

and

$$\langle y_1, L[y_2] \rangle \;=\; \lambda_2 \cdot \langle y_1, y_2 \rangle \,.$$

Since L is self-adjoint, we obtain

$$\lambda_1 \cdot \langle y_1, y_2 \rangle \;=\; \lambda_2 \cdot \langle y_1, y_2 \rangle$$

and this yields $\langle y_1, y_2 \rangle = 0$. □

Lemma 4.14 *An eigenvalue* λ *has a unique corresponding eigenfunction up to a constant multiple, i.e., if* y_1 *and* y_2 *are corresponding eigenfunctions of* λ *then*

$$y_2 \;=\; c \cdot y_1 \qquad \textit{for all } x \in (a, b)\,. \tag{4.6}$$

Answer. Introduce the Wronskian of two functions

$$W[y_1, y_2] \;=\; y_1 y_2' - y_1' y_2,$$

we compute that

$$\frac{d}{dx}\left(\frac{y_1}{y_2}\right) \;=\; \frac{y_2' y_1 - y_1 y_2'}{y_1^2} \;=\; \frac{W[y_1, y_2]}{y_1^2}\,.$$

On the other hand, a direct computation yields

$$\begin{aligned} \frac{d}{dx}\;(p \cdot W) \;&=\; [p y_1 y_2']' - [p y_1' y_2]' \\ &=\; [p y_2']' \cdot y_1 - [p y_1']' \cdot y_2 \\ &=\; (q \cdot y_2 - L[y_2]) \cdot y_1 - (q \cdot y_1 - L[y_1]) \cdot y_2 \\ &=\; y_2 \cdot L[y_1] - y_1 L[y_2]\,. \end{aligned}$$

Thus, if y_1 and y_2 are corresponding eigenfunctions of λ then

$$\frac{d}{dx}\,(p \cdot W)\,(x) \;=\; y_2 \cdot L[y_1] - y_1 L[y_2] \;=\; 0,$$

and this yields

$$(p \cdot W)(x) \;=\; \text{constant} \;=\; c \qquad \text{for all } x \in (a, b).$$

However, the Wronskian of these function

$$W[y_1, y_2](a) \;=\; y_1(a) y_2'(a) - y_1'(a) y_2(a) \;=\; 0$$

because y_1 and y_2 satisfies the same boundary condition at a. Thus,

$$W[y_1, y_2](x) \;=\; 0 \qquad \text{for all } x \in (a, b),$$

the two functions must be linearly dependent. □

We conclude this subsection with a main theorem.

Theorem 4.15 *Consider the Sturm-Liouville problems*

$$[-p(x)y']' + q(x)y = \lambda w(x)y, \qquad x \in (a,b)$$

with boundary conditions

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y(a) = 0 \\ \beta_1 y(b) + \beta_2 y(b) = 0 \end{cases}.$$

with $\alpha_1^2 + \alpha_2^2 \neq 0$ *and* $\beta_1^2 + \beta_2^2 \neq 0$. *Then the followings hold:*

(i) *There are countably infinite number of real eigenvalues*

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ... \qquad \text{and} \qquad \lim_{n\to\infty} \lambda_n = +\infty .$$

(ii) *For each eigenvalue* λ_i, *there is a unique corresponding eigenfunction up to a constant multiple.*

(iii) *Given* λ_i *and* λ_j *such that* $\lambda_i \neq \lambda_j$, *the corresponding eigenfunctions* y_i *and* y_j *are orthogonal.*

(iv) *For any* $u \in \mathcal{H}$, *one has*

$$u = \sum_{n=1}^{+\infty} c_n \cdot y_n$$

where the coefficients are computed by

$$c_n = \frac{\langle u, y_n \rangle}{\|y_n\|^2} \qquad \text{for all } n \in \mathbb{Z}^+ .$$

5 Linear Partial differential equations on bounded domains

Consider a linear PDEs on a bounded domain in $\mathbb{R}^2$

$$A(u(x,y)) = 0 \qquad \text{for all } (x,y) \in \Omega \subset \mathbb{R}^2 . \tag{5.1}$$

where

- A is a given linear differential operator
- u is an unknown of variables x and y.

Our goal is to derive the general formula of solution u to (5.1) by using the method of separation of variables.

Method of separation of variables.

- *Step 1:* Seek for solutions of the form

  $$u(x,y) = F(x) \cdot G(y)$$

  where F is an unknown of x and G an unknown of y.

  Plug $u = FG$ into the PDE (5.1), one obtains ODEs for F and G. Together with boundary conditions, the ODE becomes Sturm-Liouville problems.

- *Step 2:* Solve Sturm-Liouville problems to obtain eigen-functions F_n and G_n. Thus, particular solution of (5.1) is

$$u_n(x,y) \;=\; F_n(x)\cdot G_n(y)\,.$$

- *Step 3:* The set of particular solutions $\{u_1, u_2, ..., u_n, \dots\}$ is a complete and orthogonal in a suitable space. Therefore, the general solution is

$$u \;=\; \sum_{n=1}^{+\infty} c_n \cdot u_n$$

  where the constant c_n will be the coefficients of the Fourier series of initial data or boundary data.

5.1 1-D heat equation on bounded domain

1. Dirichlet boundary condition. Consider the 1-D heat equation with Dirichlet boundary condition

$$\begin{cases} u_t(x,t) \;=\; c^2\cdot u_{xx}(x,t)\,, & x\in(0,L), t>0 \\ u(0,t) \;=\; u(L,t) \;=\; 0, & t>0\,, \\ u(x,0) \;=\; f(x), & \text{for all } x\in[0,L]\,, \end{cases}$$

where

- c is a given constant which is the diffusivity of the rod;
- L is the length of the rod;
- f is the given initial temperature.

Goal: Find $u(x,t)$ the temperature at point $x \in (0, L)$ at time $t > 0$.

Answer. It is divided into several steps:

Step 1: (Separating variable) Seek solutions for the form

$$u(x,t) \;=\; F(x)\cdot G(t)\,.$$

We compute

$$u_t \;=\; F(x)\cdot G'(t), \qquad u_{xx} \;=\; F''(x)\cdot G(t)\,.$$

Plug these into the heat equation, we obtain

$$F(x)\cdot G'(t) \;=\; c^2\cdot F''(x)\cdot G(t)\,.$$

This implies that

$$\frac{F''(x)}{F(x)} \;=\; \frac{G'(t)}{c^2 G(t)} \;=\; \text{constant} \;=\; -\,\lambda\,.$$

The ODEs of F and G

$$\begin{cases} F''(x) + \lambda \cdot F(x) = 0, & x \in (0, L), \\ G'(t) + \lambda c^2 G(t) = 0, & t \geq 0. \end{cases}$$

Step 2: Solve for F and G. The boundary conditions

$$u(0,t) = F(0) \cdot G(t) = 0 \quad \Longrightarrow \quad F(0) = 0,$$

and

$$u(L,t) = F(L) \cdot G(t) = 0 \quad \Longrightarrow \quad F(L) = 0.$$

Two points boundary problem (Sturm-Liouville problem)

$$\begin{cases} F''(x) + \lambda \cdot F(x) = 0, & x \in (0, L), \\ F(0) = F(L) = 0. \end{cases}$$

Eigenvalues and corresponding eigenfunctions

$$\begin{cases} \lambda_n = \dfrac{n^2 \pi^2}{L^2} \\ F_n(x) = \sin\left(\dfrac{n\pi}{L} \cdot x\right) \end{cases} \qquad \text{for } n = 1, 2, ...$$

For any $n \geq 1$, the equation

$$G'(t) + \lambda_n c^2 \cdot G(t) = 0.$$

and the general solution

$$G_n(t) = e^{-c^2 \lambda_n t}.$$

Step 3. (Find the general solution). Particular solutions of the above 1-D heat equation

$$u_n(x,t) = F_n(x) \cdot G_n(t) = e^{-\dfrac{n^2 c^2 \pi^2}{L^2} \cdot t} \cdot \sin\left(\frac{n\pi}{L} \cdot x\right).$$

The general solution

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n \cdot u_n(x,t) \\ &= \sum_{n=1}^{\infty} c_n \cdot e^{-\dfrac{n^2 c^2 \pi^2}{L^2} \cdot t} \cdot \sin\left(\frac{n\pi}{L} \cdot x\right). \end{aligned}$$

Step 4: Find c_n by the initial conditions

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \cdot \sin\left(\frac{n\pi}{L} \cdot x\right).$$

The c_n are coefficients of Fourier sine for the odd extension of f

$$c_n = \frac{2}{L} \cdot \int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right)\, dx\,.$$

$\square$

Summary. 1-*D heat equation with Dirichlet boundary condition*

$$\begin{cases} u_t(x,t) = c^2 \cdot u_{xx}(x,t)\,, & x \in (0,L), t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0\,, \\ u(x,0) = f(x), & x \in [0,L]\,, \end{cases}$$

The general solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cdot e^{-\frac{n^2c^2\pi^2}{L^2} \cdot t} \cdot \sin\left(\frac{n\pi}{L} \cdot x\right)$$

where the coefficients are computed by

$$f(x) = \sum_{n=1}^{\infty} c_n \cdot \sin\left(\frac{n\pi}{L} \cdot x\right)\,,$$

or

$$c_n = \frac{2}{L} \cdot \int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right)\, dx\,.$$

Discussion of the solution

- The solution is harmonic oscillation in x and exponential decay in t.
- As time t goes to $+\infty$, the solution $u(t,x)$ goes to 0 for all $x \in \mathbb{R}$.

Example 1. Solve the following 1-D heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t)\,, & x \in (0,1), t > 0 \\ u(0,t) = u(1,t) = 0, & t > 0\,, \\ u(x,0) = 10\sin(\pi x) + 5\sin(3\pi x), & \text{for all } x \in [0,1]\,. \end{cases}$$

Answer. We have

$$c = 1, \quad L = 1 \qquad \text{and} \quad f(x) = 10\sin(\pi x) + 5\sin(3\pi x)\,.$$

The general solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cdot e^{-n^2\pi^2 t} \cdot \sin(n\pi x)\,.$$

The initial data implies that

$$10\sin(\pi x) + 5\sin(3\pi x) \;=\; f(x) \;=\; \sum_{n=1}^{\infty} c_n \cdot \sin(n\pi x)$$

Comparing the coefficients, we obtain

$$c_1 \;=\; 10, \qquad c_3 \;=\; 5 \qquad \text{and} \qquad c_n \;=\; 0 \qquad \text{for all } n \neq 1, 3\,.$$

The solution is

$$u(x,t) \;=\; 10 \cdot e^{-\pi^2 \cdot t} \cdot \sin(\pi x) + 5 \cdot e^{-9\pi^2 t} \cdot \sin(3\pi x)\,.$$

□

Example 2. Consider the 1-D heat equation

$$\begin{cases} u_t(x,t) - u \;=\; 4u_{xx}(x,t)\,, & x \in (0, 3\pi), t > 0 \\ u(0,t) \;=\; u(3\pi, t) \;=\; 0, & t > 0\,, \\ u(x,0) \;=\; \sin x - 2\sin 2x + 3\sin 3x, & \text{for all } x \in [0, 3\pi]\,. \end{cases}$$

Find the temperature at $x = \frac{\pi}{2}$ at time $t = 1$.

Answer. 1. Set

$$v \;=\; e^{-t} \cdot u\,.$$

We compute

$$v_t \;=\; e^{-t} \cdot [u_t - u] \qquad \text{and} \qquad v_{xx} \;=\; e^{-t} \cdot u_{xx}\,.$$

Thus, v is the solution of

$$\begin{cases} v_t(x,t) \;=\; 4v_{xx}(x,t)\,, & x \in (0, 3\pi), t > 0 \\ v(0,t) \;=\; v(3\pi, t) \;=\; 0, & t \geq 0\,, \\ v(x,0) \;=\; \sin x - 2\sin 2x + 3\sin 3x, & \text{for all } x \in [0, 3\pi]\,. \end{cases}$$

2. Solve for v. We have

$$c \;=\; 2, \qquad L \;=\; 3\pi \qquad \text{and} \qquad f(x) \;=\; \sin x - 2\sin 2x + 3\sin 3x\,.$$

The general solution is

$$v(x,t) \;=\; \sum_{n=1}^{\infty} c_n \cdot e^{-\frac{4n^2}{9} \cdot t} \cdot \sin\frac{nx}{3}\,.$$

The initial condition implies that

$$\sum_{n=1}^{\infty} c_n \cdot \sin\frac{nx}{3} \;=\; \sin x - 2\sin 2x + 3\sin 3x\,.$$

Compare the coefficients, we obtain that

$$c_3 = 1, \qquad c_6 = -2, \qquad c_9 = 3 \qquad \text{and} \qquad c_n = 0 \qquad \text{for all } n \neq 3,6,9\,.$$

Thus,

$$v(x,t) = e^{-4t}\cdot\sin x - 2e^{-16t}\cdot\sin(2x) + 3e^{-36t}\cdot\sin 3x\,.$$

3. The solution is

$$u(x,t) = e^t\cdot v(x,t) = e^{-3t}\cdot\sin x - 2e^{-15t}\cdot\sin(2x) + 3e^{-35t}\cdot\sin 3x\,.$$

In particular,

$$u(\pi/2,1) = e^{\frac{-3}{2}} - 3e^{-\frac{35}{2}}\,.$$

□

2. Neumann boundary condition. Consider the 1-D heat equation with Neumann boundary condition

$$\begin{cases} u_t(x,t) = c^2\cdot u_{xx}(x,t)\,, & x\in(0,L), t>0 \\ u_x(0,t) = u_x(L,t) = 0, & t>0\,, \\ u(x,0) = f(x), & x\in[0,L]\,, \end{cases}$$

Goal: Find $u(x,t)$.

Answer. 1. Seek for solutions for the form

$$u(x,t) = F(x)\cdot G(t)\,.$$

ODEs for F and G

$$\begin{cases} F''(x)+\lambda\cdot F(x) = 0, & x\in(0,L), \\ G'(t)+\lambda c^2 G(t) = 0, & t\geq 0\,. \end{cases}$$

2. Solve for F and G. The boundary conditions

$$u_x(0,t) = F'(0)\cdot G(t) = 0 \qquad\Longrightarrow\qquad F'(0) = 0\,,$$

and

$$u_x(L,t) = F'(L)\cdot G(t) = 0 \qquad\Longrightarrow\qquad F'(L) = 0\,.$$

Two points boundary problem (Sturm-Liouville problem)

$$\begin{cases} F''(x)+\lambda\cdot F(x) = 0, & x\in(0,L), \\ F'(0) = F'(L) = 0\,. \end{cases}$$

Eigenvalues and corresponding eigenfunctions

$$\begin{cases} \lambda_n = \dfrac{n^2\pi^2}{L^2} \\ F_n(x) = \cos\left(\dfrac{n\pi}{L}\cdot x\right) \end{cases} \qquad \text{for } n = 0,1,2,...$$

Solve for G. For any $n \in \mathbb{N}$,

$$G'(t) + c^2 \lambda_n G(t) \;=\; 0\,.$$

Thus,

$$G_n(t) \;=\; e^{-\dfrac{n^2\pi^2c^2}{L^2}\cdot t}, \qquad n = 0, 1, 2, ...$$

3. Particular solutions of the above heat equation

$$u_n(x,t) \;=\; F_n(x) \cdot G_n(t) \;=\; e^{-\dfrac{n^2\pi c^2}{L^2}\cdot t} \cdot \cos\left(\frac{n\pi}{L}\cdot x\right), \qquad n = 0, 1, 2, ...$$

The general solution

$$u(x,t) \;=\; c_0 + \sum_{n=1}^{+\infty} c_n \cdot e^{-\dfrac{n^2\pi^2c^2}{L^2}\cdot t} \cdot \cos\left(\frac{n\pi}{L}\cdot x\right).$$

4. The initial condition implies that

$$f(x) \;=\; c_0 + \sum_{n=1}^{+\infty} c_n \cdot \cos\left(\frac{n\pi}{L}\cdot x\right)$$

and it yields

$$c_0 \;=\; \frac{1}{L} \cdot \int_0^L f(x)\; dx$$

and

$$c_n \;=\; \frac{2}{L} \cdot \int_0^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)\; dx\,.$$

□

Summary. *1-D heat equation with Neumann boundary condition*

$$\begin{cases} u_t(x,t) \;=\; c^2 \cdot u_{xx}(x,t)\,, & x \in (0,L), t > 0 \\ u_x(0,t) \;=\; u_x(L,t) \;=\; 0, & t \geq 0\,, \\ u(x,0) \;=\; f(x), & x \in [0,L]\,, \end{cases}$$

The general solution

$$u(x,t) \;=\; c_0 + \sum_{n=1}^{+\infty} c_n \cdot e^{-\frac{n^2\pi c^2}{L^2}\cdot t} \cdot \cos\left(\frac{n\pi}{L}\cdot x\right)$$

where the coefficients can be computed by

$$f(x) \;=\; c_0 + \sum_{n=1}^{+\infty} c_n \cdot \cos\left(\frac{n\pi}{L}\cdot x\right)$$

or

$$c_0 = \frac{1}{L} \cdot \int_0^L f(x)\ dx$$

and

$$c_n = \frac{2}{L} \cdot \int_0^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)\ dx\,.$$

Discussion of the solution

- The solution is harmonic oscillation in x and exponential decay in t.
- As time t goes to $+\infty$, the solution $u(t,x)$ goes to the average value of the initial temperature

$$\lim_{t\to+\infty} u(t,x) = \frac{1}{L} \cdot \int_0^L f(x)\ dx\,.$$

Example 3. Solve the heat equation with Neumann boundary condition

$$\begin{cases} u_t(x,t) = 9 \cdot u_{xx}(x,t)\,, & x \in (0,2\pi), t > 0 \\ u_x(0,t) = u_x(2\pi,t) = 0, & t \geq 0\,, \\ u(x,0) = 2 + \frac{1}{2} \cdot \cos x - 3 \cdot \cos 3x, & x \in [0,L]\,, \end{cases}$$

Answer. We have

$$c = 3, \qquad L = 2\pi \qquad \text{and} \qquad f(x) = 2 + \frac{1}{2} \cdot \cos x - 3\cos 3x\,.$$

The general solution is

$$u(x,t) = c_0 + \sum_{n=1}^{+\infty} c_n \cdot e^{-\frac{9n^2}{4}t} \cdot \cos\left(\frac{n}{2} \cdot x\right)$$

Initial condition

$$f(x) = c_0 + \sum_{n=1}^{+\infty} c_n \cdot \cos\left(\frac{n}{2} \cdot x\right) = 2 + \frac{1}{2} \cdot \cos x - 3\cos 3x\,.$$

Compare the coefficients, we get

$$c_0 = 2, \qquad c_2 = \frac{1}{2}, \qquad c_6 = -3 \qquad \text{and} \qquad c_n = 0 \qquad \text{for all } n \neq 0,2,6\,.$$

The solution is

$$u(x,t) = 2 + \frac{1}{2}e^{-9t}\cos x - 3e^{-81t}\cos 3x\,.$$

□

3. Steady state of heat equation. Consider the 1-D heat equation

$$\begin{cases} u_t(x,t) = c^2 \cdot u_{xx}(x,t)\,, & x \in (0,L), t > 0 \\ \text{Boundary Conditions}\,. \end{cases}$$

As $t \to +\infty$, solution does not change in time anymore, as it reaches a steady state. Call it $U(x)$. Informally,

$$U(x) \;=\; \lim_{t\to+\infty} u(t,x) \qquad \text{for all } x \in [0,L]\,.$$

Goal: How to find $U(x)$?

Since U does not depend on time t and satisfies the heat equation, one has

$$U_t \;=\; 0 \qquad \text{and} \qquad U_{xx} \;=\; 0\,.$$

Thus,

$$U(x) \;=\; Ax + B$$

where constants A and B are identified by boundary conditions.

Example 4. Find the steady state of the heat equation

$$\begin{cases} \qquad\qquad u_t(x,t) \;=\; 4\cdot u_{xx}(x,t)\,, \qquad x \in (0,2), t > 0 \\[2ex] u(0,t) \;=\; 1 \qquad\qquad u(2,t) \;=\; 3\,. \end{cases}$$

Answer. We have

$$U(x) \;=\; Ax + B\,.$$

The boundary conditions imply that

$$U(0) \;=\; 1, \qquad \text{and} \qquad U(2) \;=\; 3\,.$$

Thus,

$$B \;=\; 1 \qquad \text{and} \qquad 2A + B \;=\; 3$$

and it yields $B = 1$ and $A = 1$. The steady state is

$$U(x) \;=\; x + 1\,.$$

□

Example 5. Find the steady state of the heat equation

$$\begin{cases} \qquad\qquad u_t(x,t) \;=\; 4\cdot u_{xx}(x,t)\,, \qquad x \in (0,1), t > 0 \\[2ex] u(0,t) + u'(0,t) \;=\; 1 \qquad\qquad u(1,t) - u'(1,t) \;=\; 2\,. \end{cases}$$

Answer. We have

$$U(x) \;=\; Ax + B\,.$$

The boundary conditions imply that

$$U(0) + U'(0) \;=\; 1, \qquad \text{and} \qquad U'(1) - U(1) \;=\; 2\,.$$

Thus,

$$A + B = 1 \qquad \text{and} \qquad - B = 2$$

and it yields $A = 3$ and $B = -2$. The steady state is

$$U(x) = 3x - 2.$$

□

4. Non-homogeneous boundary conditions. Let's consider the heat equation with non-homogeneous boundary conditions

$$\begin{cases} u_t(x,t) = c^2 \cdot u_{xx}(x,t), & x \in (0,L), t > 0 \\ \qquad \text{Nonhomogeneous boundary conditions}, \\ u(x,0) = f(x), & x \in [0,L], \end{cases}$$

How to solve?

Step 1: Find the steady state $U(x)$.

Step 2. Set $v(x,t) = u(x,t) - U(x)$. Then v is the solution of

$$\begin{cases} v_t(x,t) = c^2 \cdot v_{xx}(x,t), & x \in (0,L), t > 0 \\ \qquad \text{Homogeneous boundary conditions}, \\ v(x,0) = f(x) - U(x), & x \in [0,L], \end{cases}$$

Solve for v.

Step 3. The solution is

$$u(x,t) = U(x) + v(x,t).$$

□

Example 6. Solve the heat equation with non-homogeneous condition

$$\begin{cases} u_t(x,t) = 4 \cdot u_{xx}(x,t), & x \in (0,\pi), t > 0 \\ u(0,t) = 1, \quad u(\pi,t) = 3, & t > 0, \\ u(x,0) = \frac{2}{\pi} \cdot x + e^x + 1, & x \in [0,\pi], \end{cases}$$

Answer. Step 1. Find a steady state

$$U(x) = ax + b.$$

Initial condition implies that

$$U(0) = 1 \qquad \Longrightarrow \qquad b = 1$$

and

$$U(\pi) \;=\; 3 \qquad \Longrightarrow \qquad a\pi + b \;=\; 3\,.$$

Thus, $a = \dfrac{2}{\pi}, b = 1$ and the steady state

$$U(x) \;=\; \frac{2}{\pi}\cdot x + 1\,.$$

Step 2. Set $v(x,t) \doteq u(x,t) - U(x)$. Then v is the solution of the heat equation with Dirichlet boundary conditions

$$\begin{cases} v_t(x,t) \;=\; 4\cdot v_{xx}(x,t)\,, & x\in(0,\pi), t>0 \\[2mm] u(0,t) \;=\; u(\pi,t) \;=\; 0, & t>0\,, \\[2mm] v(x,0) \;=\; u(x,0) - U(x) \;=\; e^x, & x\in[0,\pi]\,. \end{cases}$$

We have

$$c \;=\; 2 \qquad \text{and} \qquad f(x) \;=\; e^x\,.$$

The general solution is

$$v(x,t) \;=\; \sum_{n=1}^{\infty} c_n\cdot e^{-4n^2t}\cdot \sin nx$$

where

$$c_n \;=\; \frac{2}{\pi}\cdot\int_0^{\pi} e^x\cdot\sin(nx)\;dx, \qquad \text{for all } n = 1,2,...$$

Step 3. The solution is

$$u(x,t) \;=\; U(x) + v(x,t) \;=\; U(x) \;=\; \frac{2}{\pi}\cdot x + 1 + \sum_{n=1}^{\infty} c_n\cdot e^{-4n^2t}\cdot\sin nx$$

where

$$c_n \;=\; \frac{2}{\pi}\cdot\int_0^{\pi} e^x\cdot\sin(nx)\;dx \;=\; \frac{n(1-e^{\pi}\cdot(-1)^n)}{n^2+1}, \qquad \text{for all } n = 1,2,...$$

□

Example 7. Find the solution of the 1-D heat equation with non-homogeneous boundary condition

$$\begin{cases} u_t(x,t) \;=\; u_{xx}(x,t)\,, & x\in(0,2), t>0 \\[2mm] u_x(0,t) \;=\; u_x(2,t) = \;1, & t\geq 0\,, \\[2mm] u(x,0) \;=\; \dfrac{\cos(\pi x)}{\pi} + 2\cos(2\pi x) + x + 1, & x\in[0,2]\,, \end{cases}$$

Answer. 1. Find a steady state

$$U(x) \;=\; ax + b\,.$$

Initial condition implies that

$$U_x(0) \;=\; U_x(2) \;=\; 1 \qquad \Longrightarrow \qquad a \;=\; 1\,.$$

Thus,

$$U(x) \;=\; x+b\,.$$

Choose $b=0$, we have that $U(x)=x$.

2. Set $v(x,t) \doteq u(x,t)-U(x)$. Then v is the solution of the heat equation with Neumnann boundary condition

$$\begin{cases} v_t(x,t) \;=\; v_{xx}(x,t)\,, \qquad x\in(0,1), t>0 \\[2mm] v_x(0,t) \;=\; v_x(2,t) \;=\; 0, \qquad t>0\,, \\[2mm] v(x,0) \;=\; u(x,0)-U(x) \;=\; \dfrac{\cos(\pi x)}{\pi}+2\cos(2\pi x), \qquad x\in[0,2]\,. \end{cases}$$

We have

$$c \;=\; 1 \qquad \text{and} \qquad f(x) \;=\; 1+\frac{\cos(\pi x)}{\pi}+2\cos(2\pi x)\,.$$

The general solution

$$v(x,t) \;=\; c_0+\sum_{n=1}^{\infty} c_n e^{-\dfrac{n^2\pi^2}{4}t}\cdot\cos\left(\frac{n\pi}{2}x\right)\,.$$

Initial condition implies that

$$1+\frac{\cos(\pi x)}{\pi}+2\cos(2\pi x) \;=\; c_0+\sum_{n=1}^{\infty} c_n\cdot\cos\left(\frac{n\pi}{2}x\right)$$

and it yields

$$c_0 \;=\; 1, \qquad c_2 \;=\; \frac{1}{\pi}, \qquad c_4 \;=\; 2 \qquad \text{and} \qquad c_n \;=\; 0 \qquad \text{for all } n\neq 0,2,4\,.$$

Thus,

$$w(x,t) \;=\; 1+\frac{1}{\pi}e^{-\pi^2 t}\cos(\pi x)+2^{-4\pi^2 t}\cos(2\pi x)\,.$$

3. The solution is

$$u(x,t) \;=\; w(x,t)+U(x) \;=\; 1+x+\frac{1}{\pi}e^{-\pi^2 t}\cos(\pi x)+2^{-4\pi^2 t}\cos(2\pi x)\,.$$

□

5.2 1-D Wave equation on bounded domain

Consider 1-D wave equation in an interval $[0, L]$

$$\begin{cases} u_{tt}(x,t) = c^2 \cdot u_{xx}(x,t), & \text{for all } x \in [0,L], t > 0 \\ u(0,t) = u(L,t) = 0, & \text{for all } t \geq 0\,, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) & \text{for all } x \in [0,L]\,, \end{cases} \tag{5.2}$$

where

- L is the length of the string;
- $c^2 = \dfrac{T}{\rho}$ with tensor T and density ρ.

Find $u(x,t)$.

How to solve?

Step 1. (Separate variables) Look for a solution of form

$$u(x,t) = F(x) \cdot G(t)\,.$$

We compute

$$u_{tt} = F(x) \cdot G''(t) \qquad \text{and} \qquad u_{xx} = F''(x) \cdot G(t)$$

Plug these into (5.2), we get

$$F(x) \cdot G''(t) = c^2 \cdot F''(x) \cdot G(t)$$

and it yields

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 \cdot G(t)} = -\lambda\,.$$

Thus, F and G are solutions of the ODEs

$$\begin{cases} F''(x) + \lambda \cdot F(x) = 0, & x \in (0,L), \\ G''(t) + \lambda c^2 G(t) = 0, & t \geq 0\,. \end{cases}$$

Step 2. Solve for F and G. The boundary conditions

$$u(0,t) = F(0) \cdot G(t) = 0 \qquad \Longrightarrow \qquad F(0) = 0\,,$$

and

$$u(L,t) = F(L) \cdot G(t) = 0 \qquad \Longrightarrow \qquad F(L) = 0\,.$$

Two points boundary problem (Sturm-Liouville problem)

$$\begin{cases} F''(x) + \lambda \cdot F(x) = 0, & x \in (0,L), \\ F(0) = F(L) = 0\,. \end{cases}$$

Eigenvalues and corresponding eigenfunctions

$$\begin{cases} \lambda_n &= \dfrac{n^2\pi^2}{L^2} \\ F_n(x) &= \sin\left(\dfrac{n\pi}{L}\cdot x\right) \end{cases} \qquad \text{for } n = 1, 2, ...$$

Solve for G. For any n, we have

$$G''(t) + \frac{n^2c^2\pi^2}{L^2}\cdot G(t) \;=\; 0\,.$$

Thus,

$$G_n(t) \;=\; c_n\cdot\cos\left(\frac{nc\pi}{L}\cdot t\right) + d_n\cdot\sin\left(\frac{nc\pi}{L}\cdot t\right)\,.$$

Particular solution

$$u_n(x,t) \;=\; F_n(x)\cdot G_n(t) \;=\; \left[c_n\cdot\cos\left(\frac{nc\pi}{L}\cdot t\right) + d_n\cdot\sin\left(\frac{nc\pi}{L}\cdot t\right)\right]\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)\,.$$

Step 3. General solution

$$u(x,t) \;=\; \sum_{n=1}^{+\infty}\left[c_n\cdot\cos\left(\frac{nc\pi}{L}\cdot t\right) + d_n\cdot\sin\left(\frac{nc\pi}{L}\cdot t\right)\right]\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)$$

where

$$f(x) \;=\; \sum_{n=1}^{\infty} c_n\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)$$

and

$$g(x) \;=\; \sum_{n=1}^{\infty}\frac{nc\pi}{L}\cdot d_n\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)$$

Therefore,

$$c_n \;=\; \frac{2}{L}\cdot\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)\,dx \qquad \text{and} \qquad d_n \;=\; \frac{2}{nc\pi}\cdot\int_0^L g(x)\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)\,dx\,.$$

□

Remark. If $g = 0$ then $d_n = 0$ and

$$u(x,t) \;=\; \sum_{n=1}^{+\infty} c_n\cdot\cos\left(\frac{nc\pi}{L}\cdot t\right)\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)\,.$$

If $f = 0$ then $c_n = 0$ and

$$u(x,t) \;=\; \sum_{n=1}^{+\infty} d_n\cdot\sin\left(\frac{nc\pi}{L}\cdot t\right)\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)\,.$$

Example 1. Find the solution of the following equation

$$\begin{cases} u_{tt}(x,t) &= 9\cdot u_{xx}(x,t), \qquad \text{for all } x\in[0,\pi], t>0 \\ u(0,t) &= u(\pi,t) = 0, \qquad \text{for all } t\geq 0\,, \\ u(x,0) &= \sin x - \sin(3x) \qquad \text{for all } x\in[0,\pi]\,, \\ u_t(x,0) &= \sin(2x)+5\sin(4x) \qquad \text{for all } x\in[0,\pi]\,. \end{cases}$$

Answer. We have

$$c = 3, \quad L = \pi, \quad f(x) = \sin x - \sin(3x) \quad \text{and} \quad g(x) = \sin(2x)+5\sin(4x)\,.$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} [c_n\cdot\cos(3nt)+d_n\sin(3nt)]\cdot\sin(nx)]\,.$$

The coefficients are computed by

$$\sin(x)-\sin(3x) = \sum_{n=1}^{\infty} c_n\cdot\sin(nx)$$

and

$$\sin(2x)+5\sin(4x) = \sum_{n=1}^{+\infty} 3nd_n\cdot\sin(nx)\,.$$

This implies

$$c_1 = 1, \qquad c_3 = -1, \qquad c_n = 0 \qquad \text{for all } n\neq 1,3$$

and

$$d_2 = \frac{1}{6}, \qquad d_4 = \frac{5}{12}, \qquad d_n = 0 \qquad \text{for all } n\neq 2,4\,.$$

The solution is

$$u(x,t) = \cos(3t)\cdot\sin x+\frac{1}{6}\sin(6t)\sin(2x)-\cos(9t)\sin(3x)+\frac{5}{12}\sin(12t)\sin(4x)\,.$$

□

Example 2. Find the solution of the following equation

$$\begin{cases} u_{tt}(x,t) &= 9\cdot u_{xx}(x,t)+t, \qquad \text{for all } x\in[0,\pi], t>0, \\ u(0,t) &= u(\pi,t) = 0 \qquad \text{for all } t\geq 0\,, \\ u(x,0) &= \sin x - \sin(3x) \qquad \text{for all } x\in[0,\pi]\,, \\ u_t(x,0) &= -\dfrac{x(x-\pi)}{18}+\sin(2x)+5\sin(4x) \qquad \text{for all } x\in[0,\pi]\,. \end{cases}$$

Answer. Set $v = u + \dfrac{x(x-\pi)}{18} \cdot t$, we compute

$$v_{tt} \;=\; u_{tt} \qquad \text{and} \qquad v_{xx} \;=\; u_{xx} + \frac{t}{9},$$

$$v(0,t) \;=\; u(0,t) \;=\; 0, \qquad v(\pi,t) \;=\; u(\pi,t) \;=\; 0,$$

and

$$v_t(x,0) \;=\; u_t(x,0) + \frac{x(x-\pi)}{18} \;=\; \sin(2x) + 5\sin(4x), \qquad v(x,0) \;=\; \sin x - \sin(3x).$$

Thus, v solves the equation

$$\begin{cases} v_{tt}(x,t) \;=\; 9 \cdot v_{xx}(x,t), & \text{for all } x \in [0,\pi], t > 0 \\ v(0,t) \;=\; 0, \quad v(\pi,t) \;=\; 0 & \text{for all } t \geq 0\,, \\ v(x,0) \;=\; \sin x - \sin(3x) & \text{for all } x \in [0,\pi]\,, \\ v_t(x,0) \;=\; \sin(2x) + 5\sin(4x) & \text{for all } x \in [0,\pi]\,. \end{cases}$$

Thus,

$$v(x,t) \;=\; \cos(3t) \cdot \sin x + \frac{1}{6}\sin(6t)\sin(2x) - \cos(9t)\sin(3x) + \frac{5}{12}\sin(12t)\sin(4x)$$

and this yields

$$u(x,t) \;=\; -\frac{x(x-\pi)t}{18} + \cos(3t) \cdot \sin x + \frac{1}{6}\sin(6t) \cdot \sin(2x) - \cos(9t) \cdot \sin(3x) + \frac{5}{12}\sin(12t) \cdot \sin(4x).$$

□

Example 3. Solve the nonhomogeneous PDE with given boundary and initial conditions

$$\begin{cases} u_{tt}(x,t) \;=\; u_{xx}(x,t) + x, & \text{for all } x \in [0,1], t > 0 \\ u(0,t) \;=\; 0, \quad u(1,t) \;=\; 0 & \text{for all } t > 0\,, \\ u(x,0) \;=\; -\dfrac{x^3}{6} + \dfrac{x}{6} + \sin(\pi x) - 2\sin(3\pi x), & u_t(x,0) \;=\; 0 \end{cases}$$

Answer. 1. By superposition principle, we have

$$u(x,t) \;=\; v(x,t) + w(x)$$

where w is the solution of

$$\begin{cases} w''(x) \;=\; -x, \\ w(0) \;=\; w(1) \;=\; 0. \end{cases}$$

and v is the solution of

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t), & \text{for all } x \in [0,1], t > 0 \\ v(0,t) = 0, \quad v(1,t) = 0 & \text{for all } t > 0\,, \\ v(x,0) = u(0,x) - w(x), \quad v_t(x,0) = 0 & \text{for all } x \in [0,1] \end{cases}$$

2. Solve for w, we get

$$w(x) = -\frac{x^3}{6} + \frac{x}{6} \qquad \text{for all } x \in [0,1].$$

To solve for v, we have

$$c = 1, \quad L = 1, \quad g(x) = 0 \qquad \text{and} \qquad f(x) = u(0,x) - w(x) = \sin(\pi x) - 2\sin(3\pi x)$$

The general solution is

$$v(x,t) = \sum_{n=1}^{+\infty} c_n \cdot \cos(n\pi t) \cdot \sin(n\pi x)$$

with

$$\sin(\pi x) - 2\sin(3\pi x) = \sum_{n=1}^{+\infty} c_n \cdot \sin(n\pi x)\,.$$

Compare the coefficients, we get

$$c_1 = 1, \qquad c_3 = -2 \qquad \text{and} \qquad c_n = 0 \qquad \text{for all } n \neq 1, 3,$$

and this yields

$$v(x,t) = \cos(\pi t) \cdot \sin(\pi x) - 2\cos(3\pi t) \cdot \sin(3\pi x)\,.$$

Thus, the solution is

$$u(x,t) = -\frac{x^3}{6} + \frac{x}{6} + \cos(\pi t) \cdot \sin(\pi x) - 2\cos(3\pi t) \cdot \sin(3\pi x)$$

□

Nonhomogenous wave equations. In general, to solve the nonhomogeneous PDE

$$\begin{cases} u_{tt}(x,t) = \alpha^2 \cdot u_{xx}(x,t) + k(x) & \text{for all } x \in [0,L], t > 0 \\ u(0,t) = a, \quad u(L,t) = b & \text{for all } t > 0\,, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & \end{cases}$$

we will use the superposition principle

$$u(x,t) = v(x,t) + w(x)$$

where $w(x)$ solves the equation

$$\begin{cases} w''(x) &= -\dfrac{k(x)}{\alpha^2} \qquad \text{for all } x \in (0, L) \\ w(0) &= a, \qquad w(L) \;=\; b, \end{cases}$$

and v solves the homogeneous PDE

$$\begin{cases} v_{tt}(x,t) &= \alpha^2 \cdot v_{xx}(x,t) \qquad\qquad \text{for all } x \in [0,L], t > 0 \\ \\ u(0,t) &= 0, \qquad u(L,t) \;=\; 0 \qquad\qquad \text{for all } t > 0\,, \\ \\ u(x,0) &= f(x) - w(x), \qquad u_t(x,0) \;=\; g(x). \end{cases}$$

Example 4. Solve the following nonhomogeneous PDE

$$\begin{cases} u_{tt}(x,t) &= u_{xx}(x,t) + x \qquad\qquad \text{for all } x \in [0,1], t > 0 \\ \\ u(0,t) &= 1, \qquad u(1,t) \;=\; 2 \qquad\qquad \text{for all } t > 0\,, \\ \\ u(x,0) &= -\dfrac{x^3}{6} + \dfrac{7x}{6} + 1, \qquad u_t(x,0) \;=\; -\sin(\pi x) + 2\sin(3\pi x). \end{cases}$$

Answer. 1. By superposition principle, we have

$$u \;=\; v + w(x)$$

where is the solution of

$$\begin{cases} w''(x) &= -x, \\ \\ w(0) &= 1, \qquad w(1) \;=\; 2, \end{cases}$$

and v is the solution of

$$\begin{cases} v_{tt}(x,t) &= v_{xx}(x,t), \qquad\qquad \text{for all } x \in [0,1], t > 0 \\ \\ v(0,t) &= 0, \qquad v(1,t) \;=\; 0 \qquad\qquad \text{for all } t > 0\,, \\ \\ v(x,0) &= u(0,x) - w(x), \qquad v_t(x,0) \;=\; -\sin(\pi x) + 2\sin(3\pi x). \end{cases}$$

2. Solve for w, we get

$$w(x) \;=\; -\frac{x^3}{6} + \frac{7x}{6} + 1 \qquad \text{for all } x \in [0,1].$$

To solve for v, we have

$$c \;=\; 1, \qquad L \;=\; 1, \qquad f(x) \;=\; 0 \qquad \text{and} \qquad g(x) \;=\; -\sin(\pi x) + 2\sin(3\pi x).$$

The general solution is

$$u(x,t) \;=\; \sum_{n=1}^{+\infty} d_n \cdot \sin(n\pi t) \cdot \sin(n\pi x)\;.$$

with

$$-\sin(\pi x) + 2\sin(3\pi x) \;=\; \sum_{n=1}^{+\infty} n\pi d_n \cdot \sin(n\pi x)\,.$$

Comparing the coefficients, we get

$$d_1 \;=\; -\frac{1}{\pi}, \qquad d_3 \;=\; \frac{2}{3\pi} \qquad \text{and} \qquad d_n \;=\; 0 \qquad \text{for all } n \neq 1,3.$$

Thus,

$$v(x,t) \;=\; -\frac{1}{\pi} \cdot \sin(\pi t)\sin(\pi x) + \frac{2}{3\pi} \cdot \sin(3\pi t)\sin(3\pi x),$$

and this yields

$$u(x,t) \;=\; -\frac{x^3}{6} + \frac{7x}{6} + 1 - \frac{1}{\pi} \cdot \sin(\pi t)\sin(\pi x) + \frac{2}{3\pi} \cdot \sin(3\pi t)\sin(3\pi x)$$

□

Example 5. Solve the following nonhomogeneous PDE

$$\begin{cases} u_{tt}(x,t) \;=\; u_{xx}(x,t) + x + 2t & \text{for all } x \in [0,1], t > 0 \\ u(0,t) \;=\; 1, \qquad u(1,t) \;=\; 2 & \text{for all } t > 0\,, \\ u(x,0) \;=\; -\dfrac{x^3}{6} + \dfrac{7x}{6} + 1, \qquad u_t(x,0) \;=\; -x(x-1) - \sin(\pi x) + 2\sin(3\pi x). \end{cases}$$

Answer. Set $v = u + x(x-1)t$, we compute

$$v_{tt} \;=\; u_{tt} \qquad \text{and} \qquad v_{xx} \;=\; u_{xx} + 2t,$$

$$v(0,t) \;=\; u(0,t) \;=\; 1, \qquad v(1,t) \;=\; u(1,t) \;=\; 2, \qquad v(x,0) \;=\; -\frac{x^2}{2} + \frac{3x}{2} + 1,$$

and

$$v_t(x,0) \;=\; u_t(x,0) + x(x-1) \;=\; -\sin(\pi x) + 2\sin(3\pi x).$$

Thus, v solves the equation

$$\begin{cases} v_{tt}(x,t) \;=\; v_{xx}(x,t) + x & \text{for all } x \in [0,1], t > 0 \\ v(0,t) \;=\; 1, \qquad v(1,t) \;=\; 2 & \text{for all } t > 0\,, \\ v(x,0) \;=\; -\dfrac{x^3}{6} + \dfrac{7x}{6} + 1, \qquad v_t(x,0) \;=\; -\sin(\pi x) + 2\sin(3\pi x). \end{cases}$$

From example 4, we know that

$$v(x,t) \;=\; -\frac{x^2}{2}+\frac{3x}{2}+1-\frac{1}{\pi}\cdot\sin(\pi t)\sin(\pi x)+\frac{2}{3\pi}\cdot\sin(3\pi t)\sin(3\pi x).$$

Thus, the solution is

$$u(x,t) \;=\; -x(x-1)t-\frac{x^3}{6}+\frac{7x}{6}+1-\frac{1}{\pi}\cdot\sin(\pi t)\sin(\pi x)+\frac{2}{3\pi}\cdot\sin(3\pi t)\sin(3\pi x).$$

□

5.3 Laplace equation in 2D

Consider the Laplace equation

$$\Delta u(x,y) \;=\; 0 \qquad \text{for all } (x,y)\in\Omega\subseteq\mathbb{R}^2$$

with

$$\Delta u \;=\; u_{xx}+y_{yy}.$$

The above equation is the steady state of the 2D heat equation

$$u_t(x,y,t) \;=\; c^2\cdot\Delta u(t,x,y) \qquad \text{for all } t\geq 0, (x,y)\in\Omega\subseteq\mathbb{R}^2$$

and its solution is a harmonic function.

5.3.1 Laplace equation in rectangular domain

Given positive constant a,b, consider the Laplace equation

$$\Delta u(x,y) \;=\; 0 \qquad \text{for all } (x,y)\in(0,a)\times(0,b)$$

with the boundary conditions

$$\begin{cases} u(0,y) \;=\; g_1(y), \qquad u(a,y) \;=\; g_2(y) \qquad \text{for all } y\in(0,b) \\[2ex] u(x,0) \;=\; f_1(x), \qquad u(b,x) \;=\; f_2(x) \qquad \text{for all } y\in(0,a). \end{cases}$$

Goal: Given f_1,f_2,g_1 and g_2, can we find u?

By using a superposition principle of a linear PDE and a change of variables, one can reduce the study to the following case:

CASE 1:

$$g_1 \;=\; 0, \qquad g_2 \;=\; 0 \qquad \text{and} \qquad f_1 \;=\; 0.$$

In this case, a solution can be found by using the method of separation of variable.

Step 1. (Separate variables) Look for a solution of form

$$u(x,y) \;=\; F(x)\cdot G(y),$$

we derive the ODEs for F and G

$$F''(x) + \lambda \cdot F(x) \;=\; 0 \qquad \text{and} \qquad G''(y) - \lambda \cdot G(y) \;=\; 0$$

Step 2. Solve for F. Since $u(0,y) = u(a,y) = 0$, one has that

$$F(0) \;=\; F(a) \;=\; 0.$$

The two points boundary problem

$$\begin{cases} F''(x) + \lambda \cdot F(x) \;=\; 0 \\ F(0) \;=\; F(a) \;=\; 0. \end{cases}$$

has eigen-pairs

$$\lambda_n \;=\; \frac{n^2\pi^2}{a^2}, \qquad F_n(x) \;=\; \sin\left(\frac{n\pi x}{a}\right) \qquad \text{for all } n = 1, 2, \ldots.$$

For every $n \geq 1$, solve the corresponding ODE for G

$$G''(y) - \frac{n^2\pi^2}{a^2} \cdot G(y) \;=\; 0,$$

we get

$$G_n(y) \;=\; \frac{A_n}{2} \cdot e^{\frac{n\pi}{a} \cdot y} + \frac{B_n}{2} \cdot e^{-\frac{n\pi}{a} \cdot y}$$

The boundary condition implies that

$$G_n(0) \;=\; 0 \qquad \Longrightarrow \qquad B_n \;=\; -A_n.$$

Thus,

$$G_n(y) \;=\; A_n \cdot \frac{e^{\frac{n\pi}{a} \cdot y} - e^{-\frac{n\pi}{a} \cdot y}}{2} \;=\; A_n \cdot \sinh\left(\frac{n\pi}{a} \cdot y\right).$$

Step 3. The solution is

$$u(x,y) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi}{a} \cdot y\right) \cdot \sin\left(\frac{n\pi}{a} \cdot x\right).$$

with

$$f_2(x) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi b}{a}\right) \cdot \sin\left(\frac{n\pi}{a} \cdot x\right)$$

and

$$A_n \;=\; \frac{2}{a \cdot \sinh(\frac{n\pi b}{a})} \cdot \int_0^a f_2(x) \cdot \sin\left(\frac{n\pi}{a} \cdot x\right) dx.$$

□

Example 1. Solve the Laplace equation

$$\Delta u(x,y) \;=\; 0 \qquad \text{for all } (x,y) \in (0,1) \times (0,1)$$

with boundary conditions

$$\begin{cases} u(0,y) \;=\; u(1,y) \;=\; 0 \qquad \text{for all } y \in (0,1) \\ \\ u(x,0) \;=\; 0, \qquad u(1,x) \;=\; x(1-x) \qquad \text{for all } y \in (0,1). \end{cases}$$

Answer. We have

$$a \;=\; 1, \qquad b \;=\; 1 \qquad \text{and} \qquad f_2(x) \;=\; x(1-x)$$

The general solution is

$$u(x,y) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh(n\pi y) \cdot \sin(n\pi x).$$

Here, the coefficients are computed by

$$\begin{aligned} A_n \;&=\; \frac{2}{\sinh(n\pi)} \cdot \int_0^1 x(1-x)\sin(n\pi x)dx \\ &=\; \frac{4}{\sinh(n\pi)} \cdot \frac{1-\cos(n\pi)}{n^3\pi^3} \;=\; \frac{4}{\sinh(n\pi)} \cdot \frac{1-(-1)^n}{n^3\pi^3}. \end{aligned}$$

Thus, the solution is

$$u(x,y) \;=\; 4 \cdot \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3\pi^3 \sinh(n\pi)} \cdot \sinh(n\pi y) \cdot \sin(n\pi x).$$

□

Summary. The Laplace equation

$$\begin{cases} \Delta u(x,y) \;=\; 0 \qquad (x,y) \in (0,a) \times (0,b) \\ u(0,y) \;=\; u(a,y) \;=\; 0 \qquad y \in (0,b) \\ u(x,0) \;=\; 0, \qquad u(x,b) \;=\; f(x) \qquad x \in (0,a) \end{cases}$$

has the solution

$$u(x,y) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi}{a} \cdot y\right) \cdot \sin\left(\frac{n\pi}{a} \cdot x\right).$$

with

$$f(x) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi b}{a}\right) \cdot \sin\left(\frac{n\pi}{a} \cdot x\right)$$

and

$$A_n \;=\; \frac{2}{a \cdot \sinh(\frac{n\pi b}{a})} \cdot \int_0^a f(x) \cdot \sin\left(\frac{n\pi}{a} \cdot x\right) dx.$$

□

CASE 2: Let us now consider the Laplace equation

$$\begin{cases} \Delta u(x,y) \;=\; 0 & (x,y) \in (0,a)\times(0,b) \\ u(0,y) \;=\; u(a,y) \;=\; 0 & y\in(0,b) \\ u(x,0) \;=\; f(x), \quad u(x,b) \;=\; 0 & x\in(0,a) \end{cases}$$

In this case, the function

$$v(x,y) \;=\; u(x,b-y) \qquad \text{for all } (x,y)\in(0,a)\times(0,b)$$

solve the equation

$$\begin{cases} \Delta v(x,y) \;=\; 0 & (x,y) \in (0,a)\times(0,b) \\ v(0,y) \;=\; v(a,y) \;=\; 0 & y\in(0,b) \\ v(x,0) \;=\; 0, \quad v(x,b) \;=\; f(x) & x\in(0,a) \end{cases}$$

From case 1, we have

$$v(x,y) \;=\; \sum_{n=1}^{\infty} A_n\cdot\sinh\left(\frac{n\pi}{a}\cdot y\right)\cdot\sin\left(\frac{n\pi}{a}\cdot x\right)$$

with

$$f(x) \;=\; \sum_{n=1}^{\infty} A_n\cdot\sinh\left(\frac{n\pi b}{a}\right)\cdot\sin\left(\frac{n\pi}{a}\cdot x\right)$$

and

$$A_n \;=\; \frac{2}{a\cdot\sinh(\frac{n\pi b}{a})}\cdot\int_0^a f(x)\cdot\sin\left(\frac{n\pi}{a}\cdot x\right)dx.$$

Thus, the solution is

$$u(x,y) \;=\; v(x,b-y) \;=\; \sum_{n=1}^{\infty} A_n\cdot\sinh\left(\frac{n\pi}{a}\cdot(b-y)\right)\cdot\sin\left(\frac{n\pi}{a}\cdot x\right).$$

CASE 3: Consider the Laplace equation

$$\begin{cases} \Delta u(x,y) \;=\; 0 & (x,y) \in (0,a)\times(0,b) \\ u(0,y) \;=\; 0, \quad u(a,y) \;=\; f(y) & y\in(0,b) \\ u(x,0) \;=\; 0, \quad u(x,b) \;=\; 0 & x\in(0,a) \end{cases}$$

In this case, we set

$$v(y,x) \;=\; u(x,y) \qquad \text{for all } (x,y)\in(0,a)\times(0,b).$$

Then v define on $(0,b)\times(0,a)$ solves the equation

$$\begin{cases} \Delta v(x,y) \;=\; 0 & (x,y) \in (0,b)\times(0,a) \\ v(0,y) \;=\; 0, \quad v(b,y) \;=\; 0 & y\in(0,a) \\ v(x,0) \;=\; 0, \quad u(x,a) \;=\; f(x) & x\in(0,b). \end{cases}$$

From the case 1, we have

$$v(x,y) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi}{b}\cdot y\right)\cdot \sin\left(\frac{n\pi}{b}\cdot x\right)$$

with

$$f(x) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi a}{b}\right)\cdot \sin\left(\frac{n\pi}{b}\cdot x\right)$$

and

$$A_n \;=\; \frac{2}{b\cdot \sinh\left(\frac{n\pi a}{b}\right)}\cdot \int_0^b f(x)\cdot \sin\left(\frac{n\pi}{b}\cdot x\right)dx.$$

Thus, the solution is

$$u(x,y) \;=\; v(y,x) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi}{b}\cdot x\right)\cdot \sin\left(\frac{n\pi}{b}\cdot y\right)$$

CASE 4: Similarly, one can show that the Laplace equation

$$\begin{cases} \Delta u(x,y) \;=\; 0 & (x,y)\in(0,a)\times(0,b) \\ u(0,y) \;=\; f(y), \quad u(a,y) \;=\; 0 & y\in(0,b) \\ u(x,0) \;=\; 0, \quad u(x,b) \;=\; 0 & x\in(0,a) \end{cases}$$

has the solution

$$u(x,y) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi}{b}\cdot (a-x)\right)\cdot \sin\left(\frac{n\pi}{b}\cdot y\right)$$

with

$$f(x) \;=\; \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi a}{b}\right)\cdot \sin\left(\frac{n\pi}{b}\cdot x\right)$$

and

$$A_n \;=\; \frac{2}{b\cdot \sinh\left(\frac{n\pi a}{b}\right)}\cdot \int_0^b f(x)\cdot \sin\left(\frac{n\pi}{b}\cdot x\right)dx.$$

□

Using a superposition principle, we can solve Laplace equation with general boundary condition.

Example 2. Solve the Laplace equation

$$\Delta u(x,y) \;=\; 0 \qquad \text{for all } (x,y)\in(0,1)\times(0,1)$$

with boundary conditions

$$\begin{cases} u(0,y) \;=\; u(1,y) \;=\; 1 & \text{for all } y\in(0,1) \\[2em] u(x,0) \;=\; x, \quad u(1,x) \;=\; 1-x & \text{for all } y\in(0,1). \end{cases}$$

Answer. The solution u is computed by

$$u(x,y) \;=\; u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)$$

where

- u_1 is the solution to

$$\begin{cases}\Delta u(x,y) \;=\; 0 \qquad (x,y)\in(0,1)\times(0,1)\\ u(0,y) \;=\; u(1,y) \;=\; 0 \qquad y\in(0,1)\\ u(x,0) \;=\; 0, \qquad u(x,1) \;=\; (1-x) \qquad x\in(0,1)\end{cases}$$

- u_2 is the solution to

$$\begin{cases}\Delta u(x,y) \;=\; 0 \qquad (x,y)\in(0,1)\times(0,1)\\ u(0,y) \;=\; u(1,y) \;=\; 0 \qquad y\in(0,1)\\ u(x,0) \;=\; x, \qquad u(x,1) \;=\; 0 \qquad x\in(0,1)\end{cases}$$

- u_3 is the solution to

$$\begin{cases}\Delta u(x,y) \;=\; 0 \qquad (x,y)\in(0,1)\times(0,1)\\ u(0,y) \;=\; 1, \qquad u(1,y) \;=\; 0 \qquad y\in(0,1)\\ u(x,0) \;=\; u(x,1) \;=\; 0 \qquad x\in(0,1)\end{cases}$$

- u_4 is the solution to

$$\begin{cases}\Delta u(x,y) \;=\; 0 \qquad (x,y)\in(0,1)\times(0,1)\\ u(0,y) \;=\; 0, \qquad u(1,y) \;=\; 1 \qquad y\in(0,1)\\ u(x,0) \;=\; u(x,1) \;=\; 0 \qquad x\in(0,1)\end{cases}$$

From case 1 and case 2, we have

$$u_1(x,y) \;=\; \sum_{n=1}^{\infty} A_n\cdot\sinh(n\pi y)\cdot\sin(n\pi x)$$

and

$$u_2(x,y) \;=\; \sum_{n=1}^{\infty} B_n\cdot\sinh(n\pi(1-y))\cdot\sin(n\pi x)$$

with

$$\begin{cases}A_n \;=\; \dfrac{2}{\sinh(n\pi)}\cdot\displaystyle\int_0^1(1-x)\sin(n\pi x)dx \;=\; \dfrac{2}{n\pi\sinh(n\pi)}\\[2ex] B_n \;=\; \dfrac{2}{\sinh(n\pi)}\cdot\displaystyle\int_0^1 x\sin(n\pi x)dx \;=\; \dfrac{2(-1)^{n+1}}{n\pi\sinh(n\pi)}.\end{cases}$$

From case 3 and case 4, we have

$$u_3(x,y) \;=\; \sum_{n=1}^{\infty} C_n\cdot\sinh\left(n\pi x\right)\cdot\sin\left(n\pi y\right)$$

and

$$u_4(x,y) \;=\; \sum_{n=1}^{\infty} D_n \cdot \sinh\left(n\pi(1-x)\right) \cdot \sin\left(n\pi y\right)$$

with

$$C_n \;=\; D_n \;=\; \frac{2}{\sinh(n\pi)} \cdot \int_0^1 \sin(n\pi x)dx \;=\; \frac{2\cdot(1-(-1)^n)}{n\pi\sinh(n\pi)}$$

Therefore, the solution is

$$\begin{aligned} u(x,y) \;=\; \sum_{n=1}^{\infty} \frac{2}{n\pi\sinh(n\pi)} \cdot \Big[\big(\sinh(n\pi y) + (-1)^{n+1}\cdot\sinh(n\pi(1-y))\big)\cdot\sin(n\pi x) \\ + (1-(-1)^n)\cdot\left(\sinh\left(n\pi x\right) + \sinh\left(n\pi(1-x)\right)\right)\cdot\sin\left(n\pi y\right)\Big] \end{aligned}$$

for $(x,y) \in [0,1]\times[0,1]$. □

5.3.2 Temperature in a disk

Consider the Laplace equation

$$\begin{cases} \Delta u &= 0 \qquad \text{in } B(0,R) \\ u &= f \qquad \text{on } \partial B(0,R). \end{cases}$$

Polar coordinate: By a change of variables

$$\begin{cases} x \;=\; r\cdot\cos(\theta) \\ \\ y \;=\; r\cdot\sin(\theta) & \qquad \text{for all } 0\le r\le R, 0\le\theta\le 2\pi \\ \\ v(r,\theta) \;=\; u(r\cdot\cos\theta, r\sin\theta), \end{cases}$$

we compute

$$v_r \;=\; u_x\cdot\cos\theta + u_y\cdot\sin\theta,$$

$$\begin{aligned} v_{rr} &= [u_{xx}\cdot\cos\theta + u_{xy}\cdot\sin\theta]\cdot\cos\theta + [u_{xy}\cdot\cos\theta + u_{yy}\cdot\sin\theta]\cdot\sin\theta \\ &= u_{xx}\cdot\cos^2\theta + 2\cdot u_{xy}\sin\theta\cdot\cos\theta + u_{yy}\cdot\sin^2\theta, \end{aligned}$$

and

$$v_\theta \;=\; -r\cdot\sin\theta\cdot u_x + r\cdot\cos\theta\cdot u_y,$$

$$\begin{aligned} v_{\theta\theta} &= r^2\cdot\left[u_{xx}\cdot\sin^2\theta - 2\cdot u_{xy}\sin\theta\cdot\cos\theta + u_{yy}\cdot\cos^2\theta\right] - r\cdot[u_x\cdot\cos\theta + u_y\cdot\sin\theta] \\ &= r^2\cdot\left[u_{xx}\cdot\sin^2\theta - 2\cdot u_{xy}\sin\theta\cdot\cos\theta + u_{yy}\cdot\cos^2\theta\right] - r\cdot v_r. \end{aligned}$$

Thus, v solves the equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial v}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 v}{\partial\theta^2} \;=\; 0 \qquad \text{for all } (r,\theta)\in(0,R)\times(0,2\pi)$$

with boundary conditions

$$\begin{cases} v(r,0) \;=\; v(r,2\pi) & x \in [0,R] \\ \\ v(R,\theta) \;=\; g(\theta) \;=\; f(R\cos\theta, R\sin\theta) & \theta \in [0,2\pi]. \end{cases}$$

Goal: Given R and g, find v in $[0,R] \times [0,2\pi]$.

1. Using the method of separation of variables, we seek particular solutions of form

$$v(r,\theta) \;=\; F(r) \cdot G(\theta).$$

From the PDEs, one derive the ODEs for F and G

$$\begin{cases} G''(\theta) + \lambda \cdot G(\theta) \;=\; 0 \\ \\ r^2 F''(r) + rF'(r) - \lambda F(r) \;=\; 0 \end{cases}$$

2. From the boundary condition, we solve the two points boundary problem

$$G''(\theta) + \lambda \cdot G(\theta) \;=\; 0, \qquad G(0) \;=\; G(2\pi).$$

and get eigenpairs

$$\lambda_n \;=\; n^2, \qquad G_n(\theta) \;=\; a_n \cdot \cos(n\theta) + b_n \cdot \sin(n\theta) \qquad \text{for all } n = 0,1,2,\ldots.$$

For every $n = 0,1,\ldots,$ the corresponding ODEs for F

$$r^2 F''(r) + rF'(r) - n^2 F(r) \;=\; 0$$

has the general solution

$$F_n(r) \;=\; c_n \cdot \left(\frac{r}{R}\right)^n.$$

3. Finally, the solution v is

$$v(r,\theta) \;=\; A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cdot [A_n \cdot \cos(n\theta) + B_n \sin(n\theta)]$$

with

$$A_0 \;=\; \frac{1}{2\pi} \cdot \int_0^{2\pi} g(\theta)d\theta, \qquad A_n \;=\; \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cdot \cos(n\theta)d\theta$$

and

$$B_n \;=\; \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cdot \sin(n\theta)d\theta$$

for all $n \geq 1$. □

Example 1. Solve the Laplace equation

$$\begin{cases} \Delta u \;=\; 0 & \text{in } B(0,1) \\ u \;=\; f & \text{on } \partial B(0,1). \end{cases}$$

where

$$f(\cos\theta, \sin\theta) \;=\; 1 + \sin\theta + \frac{1}{2}\sin(3\theta) + \cos(4\theta) \qquad \theta \in [0, 2\pi]$$

Answer. We have

$$R \;=\; 1 \qquad \text{and} \qquad g(\theta) \;=\; 1 + \sin\theta + \frac{1}{2}\sin(3\theta) + \cos(4\theta).$$

The general solution is

$$v(r,\theta) \;=\; A_0 + \sum_{n=1}^{\infty} r^n \cdot [A_n \cdot \cos(n\theta) + B_n \sin(n\theta)] \qquad \text{for all } 0 < r \leq 1, \theta \in [0, 2\pi]$$

From the boundary condition, one has

$$1 + \sin\theta + \frac{1}{2}\sin(3\theta) + \cos(4\theta) \;=\; A_0 + \sum_{n=1}^{\infty} [A_n \cdot \cos(n\theta) + B_n \sin(n\theta)]$$

and this yields

$$A_0 \;=\; 1, \qquad B_1 \;=\; 1, \qquad B_3 \;=\; \frac{1}{3} \qquad \text{and} \qquad A_4 \;=\; 1.$$

Thus, the solution is

$$v(r,\theta) \;=\; 1 + r\sin\theta + \frac{r^3}{2}\sin(3\theta) + r^4\cos(4\theta).$$

□

Poisson integral formula. Consider Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} \;=\; 0 \qquad \text{for all } (r,\theta) \in (0,R) \times (0, 2\pi)$$

with boundary conditions

$$u(R,\theta) \;=\; g(\theta) \qquad \text{for all } \theta \in [0, 2\pi).$$

The separation of variables solution is

$$u(r,\theta) \;=\; A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cdot [A_n \cdot \cos(n\theta) + B_n \sin(n\theta)]$$

with

$$A_0 \;=\; \frac{1}{2\pi}\cdot\int_0^{2\pi} g(\theta)d\theta, \qquad A_n \;=\; \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cdot\cos(n\theta)d\theta$$

and

$$B_n \;=\; \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cdot\sin(n\theta)d\theta \qquad \text{for all } n \geq 1.$$

We compute that

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\cdot\int_0^{2\pi} g(\alpha)d\alpha + \frac{1}{\pi}\cdot\left[\frac{r}{R}\right)^n\cdot\int_0^{2\pi} g(\alpha)\cdot(\cos(n\alpha)\cos(n\theta)+\sin(n\alpha)\sin(n\theta)]\,d\alpha \\
&= \frac{1}{2\pi}\cdot\int_0^{2\pi}\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cdot\cos[n(\theta-\alpha)]\right]\cdot g(\alpha)d\alpha \\
&= \frac{1}{2\pi}\cdot\int_0^{2\pi}\left[1+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cdot\left(e^{in(\theta-\alpha)}+e^{-in(\theta-\alpha)}\right)\right]\cdot g(\alpha)d\alpha \\
&= \frac{1}{2\pi}\cdot\int_0^{2\pi}\left[1+\frac{re^{i(\theta-\alpha)}}{R-re^{i(\theta-\alpha)}}+\frac{re^{-i(\theta-\alpha)}}{R-re^{-i(\theta-\alpha)}}\right]\cdot g(\alpha)d\alpha \\
&= \frac{1}{2\pi}\cdot\int_0^{2\pi}\left[\frac{R^2-r^2}{R^2-2rR\cos(\theta-\alpha)+r^2}\right]\cdot g(\alpha)d\alpha.
\end{aligned}$$

The last equation is the Poisson Integral formula of the Laplace equation

$$u(r,\theta) = \frac{1}{2\pi}\cdot\int_0^{2\pi}\left[\frac{R^2-r^2}{R^2-2rR\cos(\theta-\alpha)+r^2}\right]\cdot g(\alpha)d\alpha.$$

5.3.3 Exterior Dirichlet problem and the Dirichlet problem in an Annulus

1. Exterior Dirichlet problem Consider Laplace equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} = 0 \qquad \text{for all } (r,\theta)\in(R,\infty)\times(0,2\pi)$$

with boundary conditions

$$u(R,\theta) = g(\theta) \qquad \text{for all } \theta\in[0,2\pi).$$

By using the same argument in the previous one, we obtain that

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n\cdot[A_n\cos(n\theta)+B_n\sin(n\theta)]$$

with

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta, \qquad B_0 = 0$$

and

$$A_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta, \qquad B_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta$$

for all $n\geq 1$.

Example 1. The Exterior problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} = 0 \qquad \text{for all } (r,\theta)\in(1,\infty)\times(0,2\pi)$$

with boundary conditions

$$u(1,\theta) \;=\; 1+\sin(\theta)+\cos(3\theta) \qquad \text{for all } \theta \in [0,2\pi).$$

has the solution

$$u(r,\theta) \;=\; 1+\frac{1}{r}\cdot\sin(\theta)+\frac{1}{r^3}\cdot\sin(3\theta).$$

□

2. Dirichlet problem in an Annulus. Consider the Laplace equation between two circles

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} \;=\; 0 \qquad R_1<r<R_2$$

with boundary condition

$$u(R_1,\theta) \;=\; g_1(\theta) \qquad \text{and} \qquad u(R_2,\theta) \;=\; g_2(\theta) \quad \text{for all } \theta\in[0,2\pi).$$

By using the method of separation of variable, one gets

$$u(r,\theta) \;=\; a_0+b_0\ln r+\sum_{n=1}^{\infty}\left[a_n r^n+b_n r^{-n}\right]\cdot\cos(n\theta)+\left[c_n r^n+d_n r^{-n}\right]\cdot\sin(n\theta)$$

where

$$\begin{cases} a_0+b_0\ln R_1 \;=\; \dfrac{1}{2\pi}\cdot\displaystyle\int_0^{2\pi} g_1(s)ds \\[2ex] a_0+b_0\ln R_2 \;=\; \dfrac{1}{2\pi}\cdot\displaystyle\int_0^{2\pi} g_2(s)ds \end{cases}$$

and

$$\begin{cases} a_n R_1^n+b_n R_1^{-n} \;=\; \dfrac{1}{\pi}\cdot\displaystyle\int_0^{2\pi} g_1(s)\cdot\cos(ns)ds \\[2ex] a_n R_2^n+b_n R_2^{-n} \;=\; \dfrac{1}{\pi}\cdot\displaystyle\int_0^{2\pi} g_2(s)\cdot\cos(ns)ds \end{cases}$$

and

$$\begin{cases} c_n R_1^n+d_n R_1^{-n} \;=\; \dfrac{1}{\pi}\cdot\displaystyle\int_0^{2\pi} g_1(s)\cdot\sin(ns)ds \\[2ex] c_n R_2^n+d_n R_2^{-n} \;=\; \dfrac{1}{\pi}\cdot\displaystyle\int_0^{2\pi} g_2(s)\cdot\sin(ns)ds \end{cases}$$

Example 1. Solve the Laplace equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} \;=\; 0 \qquad 1<r<2$$

with boundary condition

$$u(1,\theta) \;=\; 0 \qquad \text{and} \qquad u(2,\theta) \;=\; \sin\theta \quad \text{for all } \theta\in[0,2\pi).$$

Answer. We have

$$R_1 \;=\; 1, \qquad R_2 \;=\; 2, \qquad g_1(\theta) \;=\; 0, \qquad g_2(\theta) \;=\; \sin\theta.$$

A direct computation yields

$$a_n \;=\; b_n \;=\; 0 \qquad \text{for all } n \geq 0$$

and

$$c_n \;=\; d_n \;=\; 0 \qquad \text{for all } n \geq 2.$$

It remains to compute c_1 and d_1. Since

$$\frac{1}{\pi} \cdot \int_0^{2\pi} \sin^2(s)ds \;=\; 1,$$

one has

$$c_1 + d_1 \;=\; 0 \qquad \text{and} \qquad 2c_1 + \frac{d_1}{2} \;=\; 1.$$

and this yields

$$c_1 \;=\; 2/3 \qquad \text{and} \qquad d_1 \;=\; -2/3.$$

Thus,

$$u(r,\theta) \;=\; \frac{2}{3} \cdot \left(r - \frac{1}{r}\right) \cdot \sin\theta \qquad \text{for all } (r,\theta) \in [1,2] \times [0,2\pi].$$

□

Example 2. Solve the Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} \;=\; 0 \qquad 1 < r < 2$$

with boundary condition

$$u(1,\theta) \;=\; 3 \qquad \text{and} \qquad u(2,\theta) \;=\; 5 \quad \text{for all } \theta \in [0,2\pi).$$

Answer. We have

$$R_1 \;=\; 1, \qquad R_2 \;=\; 2, \qquad g_1(\theta) \;=\; 3, \qquad g_2(\theta) \;=\; 5.$$

It is clear that

$$a_n \;=\; b_n \;=\; c_n \;=\; d_n \;=\; 0 \qquad \text{for all } n \geq 1$$

and

$$\begin{cases} a_0 & = \dfrac{1}{2\pi} \cdot \displaystyle\int_0^{2\pi} 3ds \;=\; 3 \\ a_0 + b_0 \ln 2 & = \dfrac{1}{2\pi} \cdot \displaystyle\int_0^{2\pi} 5ds \;=\; 5 \end{cases} \qquad \Longrightarrow \qquad a_0 \;=\; 3, \qquad b_0 \;=\; \frac{2}{\ln 2}.$$

Thus, the solution is

$$u(r,\theta) \;=\; 2 + \frac{2}{\ln 2} \cdot \ln r.$$

□

Example 3. Solve the Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} \;=\; 0 \qquad 1 < r < 2$$

with boundary condition

$$u(1,\theta) \;=\; 0 \qquad \text{and} \qquad u(2,\theta) \;=\; \sin\theta \quad \text{for all } \theta \in [0,2\pi).$$

Answer. We have

$$R_1 \;=\; 1, \qquad R_2 \;=\; 2, \qquad g_1(\theta) \;=\; \sin\theta, \qquad g_2(\theta) \;=\; \sin\theta.$$

The coefficients $a_0, b_0, a_n, b_n, c_n, d_n$ are zero excepts for c_1, d_1. We have

$$\begin{cases} c_1 + d_1 & = \dfrac{1}{\pi} \cdot \displaystyle\int_0^{2\pi} \sin^2 s ds \;=\; 1 \\ 4c_1 + \dfrac{1}{4} d_1 & = \dfrac{1}{\pi} \cdot \displaystyle\int_0^{2\pi} \sin^2(s) ds \;=\; 1. \end{cases}$$

and this yields

$$c_1 \;=\; \frac{1}{3} \qquad \text{and} \qquad d_1 \;=\; \frac{2}{3}.$$

Thus, the solution is

$$u(r,\theta) \;=\; \left(\frac{r}{3} + \frac{2}{3r}\right) \cdot \sin\theta.$$

□