Lecture note on Analysis I

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1 Introduction

Functional analysis is an abstract branch of mathematics analysis. It is a necessary background is any area of analysis (e.g., real and complex analysis, convex analysis, measure theory, numerical analysis, ...), and important in various fields of mathematics and its applications (e.g, PDEs, calculus of variations, approximation theory, optimal control, game theory, ...) .

Functional analysis is the study of

- infinite dimensional vector space over $\mathbb{R}$ and $\mathbb{C}$.
- linear maps between them.

A key idea is to regard functions

$$f : \mathbb{R}^n \to \mathbb{R}$$

as points in an abstract space. All information of f is considered in one single number $\|f\|$-norm of f (size of f).

Basic spaces

- Metric spaces
- Normed vector spaces
- Banach spaces
- Hilbert spaces

Major and foundational results

- Contraction mapping theorem
- Uniform boundedness principle
- Open-mapping theorem
- Closed-graph theorem
- Baire category theorem
- Arzela-Ascoli theorem
- Hahn-Banach theorem
- ...

2 Metric spaces

Given a set X, we wish to introduce a distance $d(\cdot,\cdot)$ between 2 points in X. This distance will allows us to define limits, convergent sequences, series and continuous maps.

Definition 2.1 *A metric on X is a nonnegative function $d : X \times X \to [0,+\infty)$ satisfying the following properties:*

(i). $d(x,y) = 0 \quad \Longleftrightarrow \quad x = y$ *(Identity of indiscernibles)*

(ii). $d(x,y) = d(y,x)$ *(Symmetry)*

(iii). $d(x,y) \leq d(x,z) + d(z,y)$ *(Triangle inequality)*

These conditions express intuitive notions about the concept of distance. Here, we will call that

- $d(x,y)$ is a distance from x to y.
- (X,d) is a metric space (equipped with metric d).

Examples of metric spaces.

(1). Real line $X = \mathbb{R}$

$$d(x,y) = |x-y| \qquad \text{for all } x,y \in \mathbb{R} \qquad (\text{Euclidean metric in } \mathbb{R}).$$

(2). n dimensional space ($X = \mathbb{R}^n$)

$$d(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \qquad (\text{Euclidean metric in } \mathbb{R}^n).$$

Notice that the metric is not unique. In this case, one can construct several different metrics in $\mathbb{R}^n$, e.g.,

$$d_1(x,y) = \max_{i \in \{1,2,\ldots,n\}} |x_i - y_i|$$

and

$$d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i| \qquad (\text{Taxicab metric}).$$

Question: is $d(x,y) = |x-y|^2$ a metric in $\mathbb{R}$? NO.

(3). Discrete metric on X

$$d(x,y) \;=\; \begin{cases} 1 & \text{if} \quad x \neq y\,, \\ 0 & \text{if} \quad x = y\,. \end{cases}$$

(4). On the space of continuous function from $[a,b]$ to $\mathbb{R}$

$$C([a,b],\mathbb{R}) \;=\; \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}\;.$$

For any $f, g \in C([a,b],\mathbb{R})$, denote by

$$d(f,g) \;=\; \max_{t\in[a,b]} |f(t) - g(t)|\,.$$

Then $(C([a,b],\mathbb{R}), d)$ is a metric space.

Question: Is

$$d_1(f,g) \;=\; \int_a^b |f(t) - g(t)|\, dt \qquad\qquad \text{for all } f,g \in C([a,b],\mathbb{R})$$

a metric on $(C([a,b],\mathbb{R}), d)$? YES.

(5). **Space of sequences:** For any $p \geq 1$, denote by

$$X \;:=\; \ell^p \;=\; \left\{ x = (x_1, x_2, \dots) \;\middle|\; \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

and

$$d_p(x,y) \;=\; \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p} \qquad\qquad x, y \in l^p\,.$$

In the case $p = 2$, we call that ℓ^2 is a Hilbert sequence space.

In the case $p = \infty$, we have the space of bounded sequences

$$\ell^\infty \;=\; \{x = \{x_i\}_{i\geq 1} \mid |x_i| \leq C_x \quad \text{for all } i \geq 1\}\;.$$

The function

$$d_\infty(x,y) \;=\; \sup_{i\in\{1,2,\dots\}} |x_i - y_i| \qquad \text{for all } x, y \in \ell^\infty$$

is a metric on ℓ^∞.

Question: Is (ℓ^p, d_p) a metric space?

In order to answer the above question, let us introduce the following classical inequalities.

2.1 Classical inequalities

1. Convex function: *The function* $f : [a,b] \to \mathbb{R}$ *is convex, i.e.,*

$$t \cdot f(x) + (1-t) \cdot f(y) \ \geq \ f(tx + (1-t)y) \qquad \text{for all } t \in [0,1] \, .$$

It is clear that the epigraph of f

$$\mathrm{Epi}(f) \ = \ \{(x,\beta) \mid x \in [a,b], \beta \geq f(x)\}$$

is convex. Moreover, if f *is a* C^2 *function and* f'' *is non-negative then* f *is convex.*

An application: (Young's inequality) Given a constant $p > 1$, let q be its conjugate, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$u \cdot v \ \leq \ \frac{u^p}{p} + \frac{v^q}{q} \qquad \text{for all } u, v \geq 0. \tag{2.1}$$

Proof. Since $u, v > 0$, one can write

$$u \ = \ e^{\frac{a}{p}} \qquad \text{and} \qquad v \ = \ u \ = \ e^{\frac{b}{q}}$$

for some $a, b \in \mathbb{R}$. Since e^x is convex, we have

$$\frac{1}{p} \cdot e^a + \frac{1}{q} \cdot e^b \ \geq \ e^{\frac{1}{p} \cdot a + \frac{1}{q} \cdot b}$$

and it yields (2.1). □

2. Hölder inequality for sums: *Let* $p, q > 0$ *be conjugate. For any* $x = \{x_i\}_{i \geq 1} \in \ell^p$ *and* $y = \{y_i\}_{i \geq 1}$ *in* ℓ^q*, it holds*

$$\sum_{i=1}^{\infty} |x_i y_i| \ \leq \ \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{1/q} . \tag{2.2}$$

Proof. 1. For any $i = 1, 2, \ldots,$ set

$$u_i \ := \ \frac{x_i}{\left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}} \qquad \text{and} \qquad v_i \ := \ \frac{y_i}{\left(\sum_{j=1}^{\infty} |y_j|^q \right)^{1/q}} ,$$

we have that

$$\sum_{i=1}^{\infty} |u_i|^p \ = \ \sum_{i=1}^{\infty} |v_i|^q \ = \ 1 \, . \tag{2.3}$$

Moreover, (2.2) is equivalent to

$$\sum_{i=1}^{\infty} |u_i \cdot v_i| \leq 1 .$$

2. From the above inequality, one has

$$|u_i \cdot v_i| \leq \frac{|u_i|^p}{p} + \frac{|v_i|^q}{q} \qquad \text{for all } i \in \{1, 2, \dots\} .$$

Recalling (2.3), we obtain that

$$\sum_{i=1}^{\infty} |u_i \cdot v_i| \leq \sum_{i=1}^{\infty} \frac{|u_i|^p}{p} + \frac{|v_i|^q}{q} = 1 .$$

The proof is complete. □

In the case $p = q = 2$, (2.2) implies the Cauchy-Schwarz inequality

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{1/2} . \tag{2.4}$$

3. Minkowski inequality for sums: *Given $p \geq 1$, for every $x, y \in \ell^p$, it holds*

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{1/p} \tag{2.5}$$

Proof. If $p = 1$ then (2.5) follows from the triangle inequality in $\mathbb{R}$. Otherwise, let q be such that $\frac{1}{p} + \frac{1}{q}$. The inequality (2.5) is equivalent to

$$\sum_{i=1}^{\infty} |x_i + y_i|^p \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/q} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/q} .$$

Using the Hölder inequality, we get

$$\left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/q} \geq \sum_{i=1}^{\infty} |x_i| \cdot |x_i + y_i|^{p/q},$$

and

$$\left(\sum_{i=1}^{\infty} |y_i|^p \right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/q} \geq \sum_{i=1}^{\infty} |y_i| \cdot |x_i + y_i|^{p/q} .$$

Thus,

$$\left(\sum_{i=1}^{\infty} \ |x_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} \ |x_i+y_i|^p\right)^{1/q} + \left(\sum_{i=1}^{\infty} \ |y_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{\infty} \ |x_i+y_i|^p\right)^{1/q}$$

$$\geq \ \left(\sum_{i=1}^{\infty} |x_i+y_i|\right)^{1+p/q} \ = \ \left(\sum_{i=1}^{\infty} |x_i+y_i|\right)^{p},$$

and this yields (2.5). □

Recalling that

$$\ell^p \ = \ \left\{ x = (x_1, x_2, \dots) \ \middle| \ \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

and

$$d_p(x,y) \ = \ \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} \qquad\qquad x, y \in l^p\,.$$

We will show that

Lemma 2.2 *(ℓ^p, d) is a metric space for all $p \geq 1$.*

Proof. We first show that d_p is finite. By Minkowski inequality, we estimate

$$d_p(x,y) \ = \ \left(\sum_{i=1}^{\infty} \ |x_i - y_i|^p\right)^{1/p} \ \leq \ \left(\sum_{i=1}^{\infty} \ |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} \ |y_i|^p\right)^{1/p} \ < \ +\infty$$

for all $x, y \in \ell^p$.

It is obvious that d is nonnegative and

$$\begin{cases} d_p(x,y) \ = \ d_p(y,x) & \text{if and only if} \quad x = y \in \ell^p\,, \\ d_p(x,y) \ = \ d_p(y,x) & \text{for every} \quad x, y \in \ell^p\,. \end{cases}$$

Thus, we only need to check that d satisfies the triangle inequality, i.e.,

$$d_p(x,y) \ \leq \ d_p(x,z) + d_p(z,y) \qquad \text{for all } x, y, z \in \ell^p\,.$$

Equivalently,

$$\sum_{i=1}^{\infty} (|x_i - y_i|)^{1/p} \ \leq \ \sum_{i=1}^{\infty} (|x_i - z_i|)^{1/p} + \sum_{i=1}^{\infty} (|z_i - y_i|)^{1/p}\,.$$

The Minkowski inequality yields the above inequality. □

To complete this subsection, let us prove an increasing property of ℓ^p.

Lemma 2.3 (Embedding) *For any $1 < p_1 < p_2 < \infty$, the following relation holds*

$$\ell_1^p \subset \ell^{p_2} .$$

Proof. Given any $x \in \ell^{p_1}$, we need to show that $x \in \ell^{p_2}$. Equivalently,

$$\left(\sum_{i=1}^{\infty} |x_i|^{p_1} \right)^{1/p_1} < \infty \qquad \Longrightarrow \qquad \left(\sum_{i=1}^{\infty} |x_i|^{p_2} \right)^{1/p_2} < \infty .$$

This follows from Jensen's sum inequality

$$\left(\sum_{i=1}^{\infty} |x_i|^{p_1} \right)^{1/p_1} \geq \left(\sum_{i=1}^{\infty} |x_i|^{p_2} \right)^{1/p_2} \qquad \text{for all } 1 < p_1 < p_2 < \infty . \tag{2.6}$$

To complete the proof, we will prove that Jensen's sum inequality. Set

$$v_i := |x_i|^{p_1} \qquad \text{and} \qquad p := \frac{p_2}{p_1} > 1 .$$

The inequality (2.6) can read as

$$\sum_{i=1}^{\infty} |v_i| \geq \left(\sum_{i=1}^{\infty} |v_i|^p \right)^{\frac{1}{p}} .$$

Thus, it suffices to show that

$$\sum_{i=1}^{n} |v_i| \geq \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}} \qquad \text{for all } n \in \mathbb{N} . \tag{2.7}$$

For $n = 2$, it holds

$$|v_1| + |v_2| \geq (|v_1|^p + |v_2|^p)^{\frac{1}{p}} \qquad \text{(homogeneous function of degree 1)} .$$

Indeed, set $0 < t := \dfrac{v_2}{v_1}$, we can rewrite the above inequality as

$$(1+t)^p \geq 1 + t^p . \tag{2.8}$$

Consider the function $f(t) := (1+t)^p - 1 - t^p$. We compute

$$f(0) = 0 \qquad \text{and} \qquad f'(t) = p \cdot (1+t)^{p-1} - p \cdot t^{p-1} > 0 \qquad \text{for all } t > 0 .$$

Thus, $f(t) > 0$ for all $t > 0$ and it yields (2.8).

Assume that (2.7) holds for $n = k \geq 2$, we need to show that it holds for $n = k+1$. Indeed,

$$\sum_{i=1}^{k+1} |v_i| \;\geq\; |v_{k+1}| + \left(\sum_{i=1}^{k} |v_i|^p\right)^{1/p} \;\geq\; \left(\sum_{i=1}^{k+1} |v_i|^p\right)^{1/p}.$$

By induction, one obtains (2.7). □

More concepts.

- *(Metric subspaces).* Given a metric space (X, ρ), let Y be a nonempty subset of X. Then the restriction of ρ to $Y \times Y$ defines a metric on Y and we call $(Y, \rho_{|Y})$ is a metric subspace of (X, ρ).

- *(Metric products).* Given two metrics (X_1, ρ_1) and (X_2, ρ_2), define

  $$X \doteq X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$$

  and $\tau : X \times X \to [0, \infty)$ such that

  $$\tau\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{\rho_1^2(x_1, y_1) + \rho_2^2(x_2, y_2)}$$

  for all $(x_1, x_2), (y_1, y_2) \in X$.

The following relation of equivalence between metrics on a given set X is useful.

Definition 2.4 (Equivalence) *Two metrics ρ_1 and ρ_2 on X are equivalent if any only if there exist $\lambda_1, \lambda_2 > 0$ such that*

$$\lambda_1 \cdot \rho_1(x_1, x_2) \;\leq\; \rho_2(x_1, x_2) \;\leq\; \lambda_2 \cdot \rho_1(x_1, x_2) \qquad \text{for all } (x_1, x_2) \in X \times X.$$

Definition 2.5 (Isometry) *A mapping $f : (X, \rho) \to (Y, \sigma)$ is said to be isometry if any only if*

$$\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2) \qquad \text{for all } (x_1, x_2) \in X \times X.$$

In addition, if $f(X) = Y$ then (X, ρ) and (Y, σ) are isometric.

2.2 Open sets, closed sets and neighborhood

Given a metric space (X, d), let us first introduce fundamental subsets of X associated to the distance d.

Definition 2.6 *Given a point $a \in X$ and $r > 0$, we denote by*

$$B(a,r) = \{y \in X \mid d(a,y) < r\} \qquad \textit{(open ball)},$$

$$\overline{B}(a,r) = \{y \in X \mid d(a,y) \leq r\} \qquad \textit{(closed ball)},$$

and

$$S(a,r) = \overline{B}(a,r) \backslash B(a,r) = \{y \in X \mid d(a,y) = r\} \qquad \textit{(sphere)}.$$

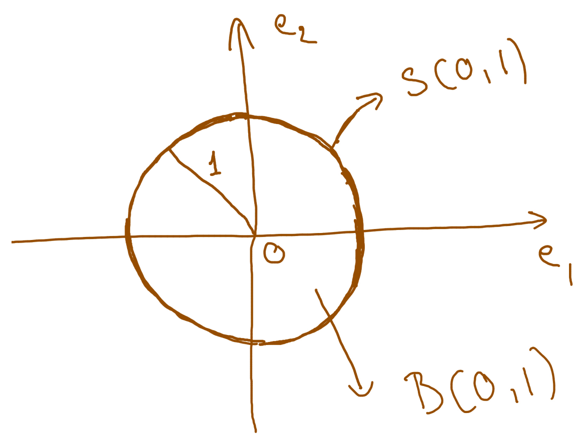
It is clear that

$$B(a,r) \subseteq \overline{B}(a,r) \subseteq B(a,r_1) \quad \text{for all } 0 < r < r_1 .$$

Examples 1. Let $X = \mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$.

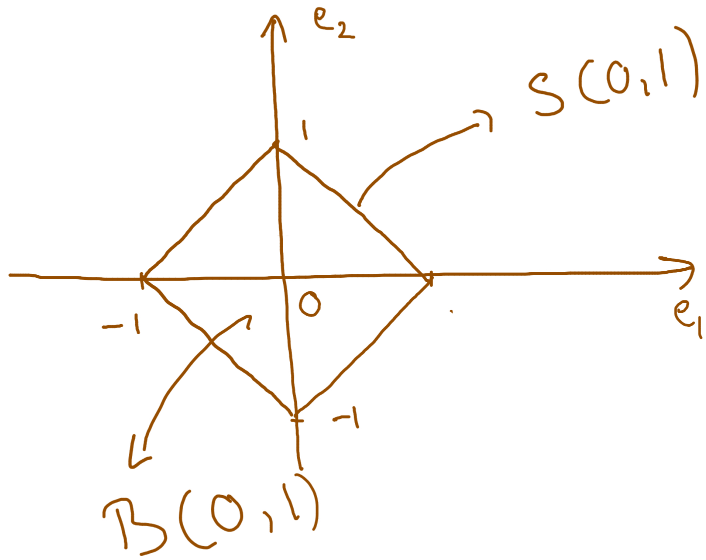
(a) If $d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ for all $x, y \in \mathbb{R}^2$ then

$$\begin{cases} B(0,1) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \\ S(0,1) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}. \end{cases}$$



(b) If $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x, y \in \mathbb{R}^2$ then

$$\begin{cases} B(0,1) = \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| + |x_2| < 1\} \\ S(0,1) = \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| + |x_2| = 1\}. \end{cases}$$



Example 2. Let (X, d) be is discrete metric space

$$d(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y\,, \\ 0 & \text{if} \quad x = y\,. \end{cases}$$

For any $a \in X$, we have

$$B(a, 1) = \{a\}, \qquad \overline{B}(0, 1) = X, \qquad S(a, 1) = X \backslash \{a\}$$

and

$$B(a, 1/2) = \{a\}, \qquad B(a, 3/2) = X, \qquad S(x, 1/2) = \emptyset\,.$$

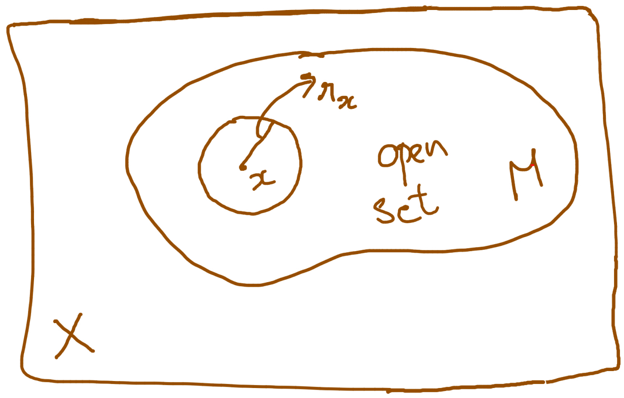
Definition 2.7 (Open and closed sets) *Let M be a subset of X. We say that*

- *M is a open set in (X, d) if and only if for every $x \in M$*

$$B(x, r_x) \subseteq M \quad \textit{for some } r_x > 0\,.$$

- *M is a closed set in (X, d) if and only if*

$$M^c = X \backslash M = \{x \in X \mid x \notin M\} \textit{ is open}.$$



It is clear that $\varnothing$ and X are open and closed. By the definition, it is sufficient to show that X is open.

$$B(a,1) \subseteq X \qquad \text{for all } a \in X$$

and X is open.

Claim 1. *Given $a \in X$ and $r > 0$, the ball $B(a,r)$ is open.*

Proof. Let $b \in B(a,r)$, we have

$$r_b := r - d(a,b) > 0 .$$

We will show that

$$B(b,r_b) \subset B(a,r) \tag{2.9}$$

and this implies that $B(a,r)$ is open.

For any $y \in B(b,r_b)$, by the triangle inequality, it holds

$$d(a,y) \leq d(a,b) + d(b,y) < d(a,b) + r_b = r$$

and it yields $y \in B(a,r)$. Thus, (2.9) holds. □

Main properties: *Given (X,d) a metric space, the followings holds* *(i) If $\{U_\alpha\}$ is a collection of open sets in (X,d) then $U = \bigcup_\alpha U_\alpha$ is open.*

(ii) If U and V are open then $U \cap V$ is also open.

Proof. (i). For any $x \in U$, there exists α_0 such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r_x > 0$ such that

$$B(x, r_x) \ \subset \ U_{\alpha_0} \ \subseteq \ U\,.$$

By the definition of open set, one has that the set U is open.

(ii). For any $x \in U \cap V$, we need to find $r_x > 0$ such that

$$B(x, r_x) \ \subset \ U \cap V\,. \tag{2.10}$$

Since U and V are open, there exists $r_1, r_2 > 0$ such that

$$B(x, r_1) \ \subset \ U \qquad \text{and} \qquad B(x, r_2) \ \subset \ V\,.$$

Set $r_x = \min\{r_1, r_2\} > 0$, we obtain (2.10). □

Definition 2.8 (Topological space) *Given X, let $\mathcal{T}$ be a collection of subsets of X such that*

(1) $\varnothing$ and X are in $\mathcal{T}$;

(2) If $U_\alpha \in X$ for $\alpha \in \mathcal{I}$, then $\bigcup_{\alpha\in\mathcal{I}} U_\alpha \in \mathcal{T}$;

(3) If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.

Then, $(X, \mathcal{T})$ is a topological space.

The following holds:

Proposition 2.8.1 *A metric space (X,d) is a topological space, i.e., the collection of open sets for the metric d is a topology for X.*

2.3 Continuous maps on metric space

Given two metric spaces (X,ρ) and (Y,σ), consider a map

$$\begin{array}{rccc} f: & (X,\rho) & \longrightarrow & (Y,\sigma) \\ & x \in X & \longmapsto & f(x) \in Y\,. \end{array}$$

Question *What does it means for f to be continuous at $x \in X$?*

Recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point $x \in X$ if any only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| \;<\; \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| \;\leq\; \varepsilon\,. \tag{2.11}$$

Set $d(x, y) = |x - y|$. The condition for continuity (2.11) can be written as

$$d(x, y) \;<\; \delta \qquad \Longrightarrow \qquad d(f(x), f(y)) \;\leq\; \varepsilon\,.$$

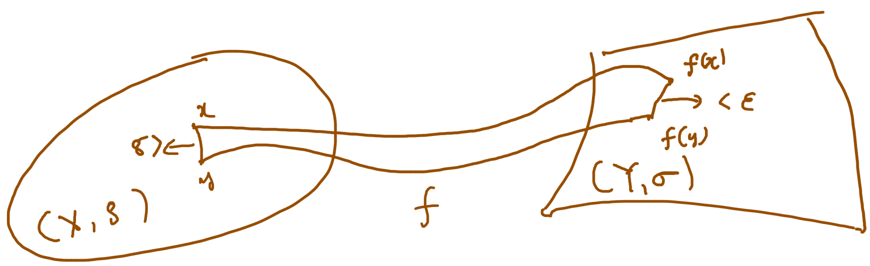
This leads to the following definition.

Definition 2.9 *The map $f : (X, \rho) \longrightarrow (Y, \sigma)$ is continuous at x if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\rho(x, y) \;<\; \delta \qquad \Longrightarrow \qquad \sigma(f(x), f(y)) \;\leq\; \varepsilon\,.$$

The above condition can be rewritten as

$$f(y) \;\in\; B(f(x), \varepsilon) \qquad \text{for all } y \in B(x, \delta)\,.$$



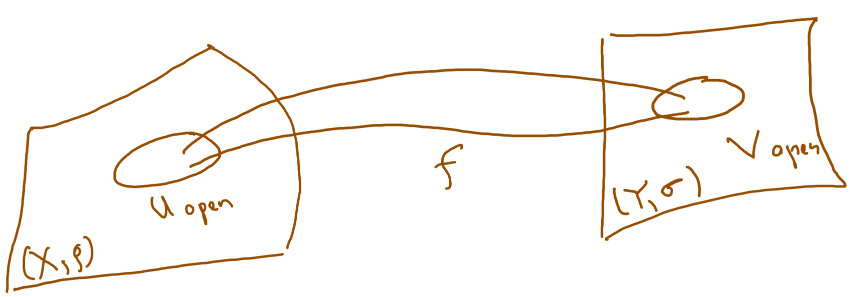
Equivalently,

$$f(B(x, \delta)) \;\subset\; B(f(x, \varepsilon))\,.$$

We say that *the function f is continuous on $A \subseteq X$ if and only if f is continuous at all $x \in A$.*

Theorem 2.10 *A map $f : (X, \rho) \to Y(, \sigma)$ is continuous of X if and only if*

$$U \;=\; f^{-1}(V) \textit{ is open in } X \textit{ for all } V \textit{ open in } Y\,.$$



Proof. ($\Longrightarrow$) Assume that f is continuous on X, we need to show that

$$V \subset Y \ \text{ open} \qquad \Longrightarrow \qquad f^{-1}(V) \ \text{ open}\,.$$

For any given $x \in f^{-1}(V)$, we need to find $\delta > 0$ such that $B(x,\delta) \subset f^{-1}(V)$. Equivalently,

$$f(B(x,\delta)) \ \subset \ V\,. \tag{2.12}$$

Since $f(x) \in V$, there exists $\varepsilon > 0$ such that $B(f(x),\varepsilon) \subset V$. By the continuity property of f, there exists $\delta > 0$ such that

$$f(B(x,\delta)) \ \subset \ B(f(x),\varepsilon)$$

and it yields (2.12).

($\Longleftarrow$) Assume that

$$V \subset Y \ \text{ open} \qquad \Longrightarrow \qquad f^{-1}(V) \ \text{ open}\,,$$

we show that f is continuous.

Fix $x \in X$, for every $\varepsilon > 0$, find $\delta > 0$ such that

$$f(B(x,\delta)) \ \subset \ B(f(x),\varepsilon)\,.$$

Equivalently,

$$B(x,\delta) \ \subset \ f^{-1}(B(f(x,\varepsilon)))\,. \tag{2.13}$$

Since $B(f(x),\varepsilon)$ is open in Y, the set $f^{-1}(B(f(x,\varepsilon)))$ is open in X. Observe that $x \in f^{-1}(B(f(x,\varepsilon)))$, there exists $\delta > 0$ such that (2.13) holds. □

Some concepts. We say that

- M is a neighborhood of x if there exists $\varepsilon > 0$ such that

  $$B(x,\varepsilon) \ \subset \ M\,.$$

- x is an interior of M if M is a neighborhood of x. The set

  $$\text{Int}(M) \ := \ M^o \ = \ \{x \in M \mid x \text{ is an interior of } M\}\,.$$

  is the interior of M

Claim. *M^o is the largest open set contained in M .*

Proof. *Homework problem.* □

Denote by

$$\overline{M} \;=\; X\backslash(X\backslash M)^o \;\text{ is the closure of } M\,,$$

and

$$\partial M \;=\; \overline{M}\backslash M^o \;\text{ is the boundary of } M$$

It holds

$$M^o \;\subseteq\; M \;\subseteq\; \overline{M}\,.$$

Moreover, the following holds

Claim. *$\overline{M}$ is the smallest closed set which contains M .*

Proof. *Homework problem.* □

As a consequence of the above claims, one has that

- *If M is an open set in (X,d) then $M^o = M$;*
- *If M is a closed set in (X,d) then $\overline{M} = M$.*

Let's now introduce concept of dense sets and separable property of a metric space.

Definition 2.11 (Dense set and separable metric space) *The subset $M \subset X$ is dense in (X,d) if*

$$\overline{M} \;=\; X\,.$$

The metric space (X,d) is called "separable" if X has a countable dense subset.

Notice that if M is dense in (X,d) then every nonempty open subset of X contains at least one point of M. Indeed, assume by a contradiction, there exists an nonempty open set U of X such that

$$U \cap M \;=\; \varnothing\,.$$

By the definition of open set, there exists a small ball $B(x,\varepsilon) \subset U$ and it yields that

$$B(x,\varepsilon) \;\subset\; X\backslash M\,.$$

Thus, x is in $(X\backslash M)^o$ and it yields a contradiction.

Some examples.

- $\mathbb{Q}$ is dense in $\mathbb{R}$.
- $\mathbb{R}^n$ with Euclican distance is separable.
- Is (ℓ^p, d_p) separable for all $p \geq 1$?

Proposition 2.11.1 *A metric space is separable if any only if there exists a countable collection $\{\mathcal{O}_n\}_{n\geq 1}$ of open subsets of X such that any open subset of X is the union of a subcollection of $\{\mathcal{O}_n\}_{n\geq 1}$.*

Proof. 1. Assume that X is separable, i.e.,

$$X = \overline{\{x_1, x_2, \ldots, x_n, \ldots\}}.$$

Consider the a countable collection of open balls $\{B(x_n, 1/m)\}_{n,m\geq 1}$ in X. Let $U \subset X$ be an open set. For any $x \in U$, there exists $m \in \mathbb{N}$ such that $B(x, 2/m_x) \subseteq U$. On the other hand, since $x \in \overline{\{x_1, x_2, \ldots, x_n, \ldots\}}$, one can find a point $x_{n,x} \in \{x_1, x_2, \ldots, x_n, \ldots\}$ such that $x \in B(x_{n,x}, 1/m_x)$. Hence,

$$x \in B(x_{n,x}, 1/m_x) \subset U,$$

and this implies that U is the union of a subcollection of $\{B(x_n, 1/m)\}_{n,m\geq 1}$.

2. Assume that there exists a countable collection $\{\mathcal{O}_n\}_{n\geq 1}$ of open subsets of X such that any open subset of X is the union of a subcollection of $\{\mathcal{O}_n\}_{n\geq 1}$. For every $n \geq 1$, we pick $x_n \in \mathcal{O}_n$. The set $\{x_1, x_2, \ldots, x_n, \ldots\}$ is dense in X. □

2.4 Convergence, Cauchy sequence, and completeness

Given a metric space (X, d) and a sequence $\{x_n\}_{n\geq 1}$ in X

Definition 2.12 *The sequence $\{x_n\}_{n\geq 1}$ converges to $x \in X$ if*

$$\lim_{n\to\infty} d(x_n, x) = 0,$$

i.e., for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$x_n \in B(x_n, \varepsilon) \qquad \textit{for all } n \geq N_\varepsilon.$$

In this case, we say that x is a limit of $\{x_n\}_{n\geq 1}$ and denote by $x = \lim_{n\to\infty} x_n$.

Example. Given $X = (-1, 1]$ and $d(x, y) = |x - y|$ for all $x, y \in X$, let $\{x_n\}_{n\geq 1}$ be such that $x_n = -1 + \frac{1}{n}$. One has that $\{x_n\}_{n\geq 1}$ converges to 1 in $\mathbb{R}$ but does not converge in X since $-1 \notin X$.

Properties of convergence sequences. *Let $\{x_n\}_{n\geq 1}$ be a sequence in (X, d) which converges to x. Then the following holds:*

(a) The limit of $\{x_n\}_{n\geq 1}$ is unique.

(b) $\{x_n\}_{n\geq 1}$ is a bounded sequence, i.e., there exists $a \in X$ and $M > 0$ such that

$$x_n \in B(a, M) \qquad \textit{for all } n \geq 1.$$

(c) If $\{y_n\}_{n\geq 1}$ converges to y in (X,d) then

$$\lim_{n\to\infty} d(x_n,y_n) \ = \ d(x,y)\,.$$

Proof. (1). Assume that $\{x_n\}_{n\geq 1}$ also converges to y. We need to show that $x=y$. By the triangle inequality, we estimate

$$0 \ \leq \ d(x,y) \ \leq \ d(x,x_n)+d(x_n,y) \qquad \text{for all } n\geq 1\,.$$

Taking $n\to\infty$, we obtain that

$$0 \ \leq \ d(x,y) \ \leq \ \limsup_{n\to\infty}[d(x,x_n)+d(x_n,y)] \ = \ 0$$

and it yields $d(x,y)=0$.

(2). For $\varepsilon=1$, there exists $N_1\in\mathbb{N}$ such that

$$x_n \ \in \ B(x,1) \qquad \text{for all } n\geq N_1\,.$$

Choosing $M:= \ 1+\displaystyle\sum_{i=1}^{N_1} d(x,1)$, we have

$$x_n \ \in \ B(x,M) \qquad\qquad \text{for all } n\geq 1\,.$$

(**3.**) Observe that

$$0 \ \leq \ |d(x_n,y_n)-d(x,y)| \ \leq \ d(x_n,x)+d(y_n,y)\,.$$

Taking $n\to\infty$, we obtain that

$$0 \ \leq \ \limsup_{n\to\infty} \ |d(x_n,y_n)-d(x,y)| \ \leq \ \limsup_{n\to\infty} \ [d(x_n,x)+d(y_n,y)| \ = \ 0$$

and it yields

$$\lim_{n\to\infty} d(x_n,y_n) \ = \ d(x,y)\,.$$

The proof is complete. □

Proposition 2.12.1 *Let $E\subset X$. Then the following statements are equivalent*

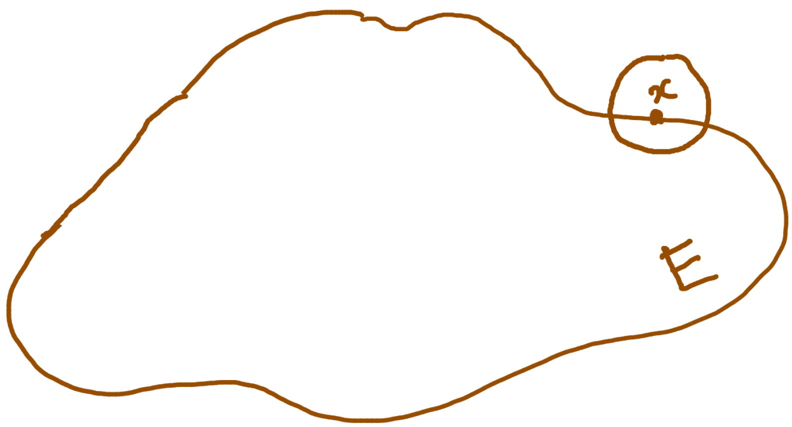
(i). $x\in\overline{E}$;

(ii) $B(x,r)\cap E\neq\varnothing$ for all $r>0$;

(iii) There exists a sequence $\{x_n\}_{n\geq 1}\subset E$ such that $\lim_{n\to\infty} x_n = x$.

Proof. $[(i) \implies (ii)]$ Recalling that

$$\overline{E} \;=\; X\backslash(X\backslash E)^o\,.$$



If $x \in \overline{E}$ then $x \notin (X\backslash E)^o$. By the definition, we have

$$B(x,r) \cap E \;\neq\; \varnothing \qquad \text{for all } r > 0$$

$[(ii) \implies (iii)]$ For every $n \geq 1$, one has

$$B(x, 1/n) \cap E \;\neq\; \varnothing\,.$$

Pick any $x_n \in B(x, 1/n) \cap E$ for all $n \geq 1$, the sequence $\{x_n\}_{n\geq 1}$ converges to x.

$[(iii) \implies (i)]$ Assume that there exists a sequence $\{x_n\}_{n\geq 1} \subset E$ such that $\lim_{n\to\infty} x_n = x$. We need to show that $x \in \overline{E} = X\backslash(X\backslash E)^o$. Assume by a contradiction, there exists $r > 0$ such that $B(x,r) \subset X\backslash E$. This implies that

$$B(x,r) \cap E \;=\; \varnothing$$

and it yields a contradiction. $\square$

Definition 2.13 (Cauchy sequence) *A sequence $\{x_n\}_{n\geq 1}$ is Cauchy if for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that*

$$d(x_n, x_m) \;<\; \varepsilon \qquad \textit{for all } n, m > N_\varepsilon\,.$$

The following lemma state a connection between Cauchy sequence and convergent sequence.

Lemma 2.14 *A convergent sequence is a Cauchy sequence.*

Proof. Let $\{x_n\}_n \geq$ be a convergent sequence in (X, d). Assume that

$$\lim_{n\to\infty} x_n \;=\; x \in X\,.$$

This implies that for every $\varepsilon' > 0$ there exists $N_{\varepsilon'}$ such that

$$d(x_n, x) \;\leq\; \varepsilon' \qquad \text{for all } n \geq N_{\varepsilon'}\,.$$

Using triangle inequality, we estimate

$$d(x_n, x_m) \;\leq\; d(x_n, x) + d(x, x_m) \;\leq\; 2\varepsilon' \qquad \text{for all } n, m \geq N_{\varepsilon'}$$

Choosing $\varepsilon = 2\varepsilon'$, one have that

$$d(x_n, x_m) \;\leq\; \varepsilon \qquad \text{for all } n, m \geq N_{\varepsilon}\,,$$

and it implies that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. □

Is a Cauchy sequence convergent?

In $\mathbb{R}$, $d(x, y) = |x - y|$, every Cauchy sequence converges to a limit in $\mathbb{R}$. However, it does not hold in general.

Example. Consider $X = (0, 1)$ and $d(x, y) = |x - y|$. One can see that the sequence $\{1/n\}_{n\geq}$ is a Cauchy sequence but it does not converges since $0 \notin X$. In this case, the metric space (X, d) is incomplete.

Definition 2.15 *The metric space (X, d) is complete if every Cauchy sequence is convergent. Otherwise, it is incomplete.*

Some examples.

(a) The metric space

$$X \;=\; \mathbb{R}, \qquad d(x, y) \;=\; |x - y|$$

is complete.

(b) The metric space

$$X \;=\; \mathbb{R}^n, \qquad d(x, y) \;=\; \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$$

is complete.

(c) Recalling that

$$\ell^p = \left\{ x = \{x_n\}_{n\geq 1} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

and

$$d_p(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \qquad \text{for all } x, y \in \ell^p,$$

one has that (ℓ^p, d_p) is complete.

(d) Let ℓ^∞ be the set of bounded sequence in $\mathbb{R}$. Denote by

$$d_\infty(x,y) = \sup_{n\geq 1} |x_n - y_n| \qquad x, y \in \ell^\infty .$$

Then (ℓ^∞, d_∞) is complete.

(e) Given two constant $-\infty < a < b < \infty$, denote by

$$C([a,b]) := \{f : [a,b] \to \mathbb{R} \text{ is continuous}\}$$

and

$$d(f,g) := \max_{t\in[a,b]} |f(t) - g(t)| \qquad \text{for all } f, g \in C([a,b]) .$$

The metric space $(C([a,b]), d)$ is complete.

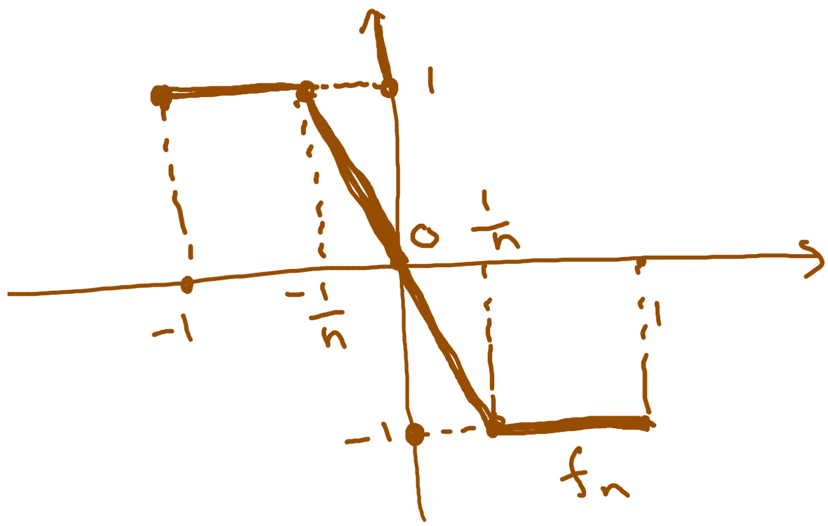
(f) However, if we consider $C([a,b])$ with the metric

$$d_1(f,g) = \int_a^b |f(t) - g(t)|\, dt,$$

the $(C([a,b]), d_1)$ is incomplete.

Let us give an example to show that the metric space $(C([a,b]), d_1)$ is incomplete. Without loss of generality, we assume that $a = -1, b = 1$. Consider the sequence of function $\{f_n\}_{n\geq} \in C([-1,1])$ such that

$$f_n(x) = \begin{cases} 1 & \text{if} \quad -1 \leq x \leq -\dfrac{1}{n}, \\ -nx & \text{if} \quad \dfrac{-1}{n} \leq x \leq \dfrac{1}{n}, \\ -1 & \text{if} \quad \dfrac{1}{n} \leq x \leq 1 . \end{cases}$$



A direct computation yields

$$d_1(f_n, f_m) \;=\; \frac{|m-n|}{mn} \qquad \text{for all } m, n \geq 1\,.$$

and it yields $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $C([-1,1])$.
However, f_n converges pointwise to

$$f(x) \;=\; \begin{cases} 1 & \text{if} \quad -1 \leq x < 0\,, \\ -1 & \text{if} \quad 0 < x \leq 1 \end{cases}$$

which is discontinuous. Hence, $(C([-1,1]), d_1)$ is incomplete. □

To conclude this subsection, let's show that

Proposition 2.15.1 *The metric space $(C([a,b]), d)$ is complete.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $C([a,b])$, i.e. for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$d(f_n, f_m) \;=\; \max_{t\in[a,b]} |f_n(t) - f_m(t)| \;<\; \varepsilon \qquad \text{for all } m, n \geq N_\varepsilon\,. \tag{2.14}$$

We need to find $f \in C([a,b])$ such that f_n converges to f in $(C([a,b]), d)$, i.e.,

$$\lim_{n\to\infty} d(f_n, f) \;=\; 0\,. \tag{2.15}$$

From (3.1), for every $t \in [a,b]$, it holds

$$|f_n(t) - f_m(t)| \ < \ \varepsilon \qquad\qquad \text{for all } m,n \geq N_\varepsilon\,.$$

Thus, the sequence $\{f_n(t)\}_{n\geq 1}$ is Cauchy in $\mathbb{R}$ for every $t \in [a,b]$. Denote by

$$f(t) \ := \ \lim_{n\to\infty} f_n(t) \qquad \text{for all } t \in [a,b]\,.$$

To complete the proof, we will show that

(i) f is continuous in $[a,b]$;

(ii) $\lim_{n\to\infty} \ d(f_m, f) = 0$.

1. To proof (i), we estimate

$$\begin{aligned}
|f(t) - f(s)| \ &\leq \ |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)| \\
&\leq \ |f(t) - f_m(t)| + |f_m(t) - f_n(t)| + |f_n(t) - f_n(s)| \\
&\qquad\qquad + |f_n(s) - f_m(s)| + |f_m(s) - f(s)| \\
&\leq \ 2d(f_m, f_n) + |f_n(t) - f_n(s)| + |f_m(t) - f(t)| + |f_m(s) - f(s)|\,.
\end{aligned}$$

For $\varepsilon > 0$, choose $N_\varepsilon > 0$ such that

$$d(f_m, f_n) \ < \ \frac{\varepsilon}{4} \qquad \text{for all } n,m \geq N_\varepsilon\,,$$

we have

$$|f(t - f(s)| \ \leq \ \frac{\varepsilon}{2} + |f_n(t) - f_n(s)| + |f_m(t) - f(t)| + |f_m(s) - f(s)|\,.$$

Taking $m \to \infty$, we obtain that

$$\begin{aligned}
|f(t) - f(s)| \ &\leq \ \frac{\varepsilon}{2} + |f_n(t) - f_n(s)| + \lim_{m\to\infty} [|f_m(t) - f(t)| + |f_m(s) - f(s)|] \\
&= \ \frac{\varepsilon}{2} + |f_n(t) - f_n(s)|\,.
\end{aligned}$$

Since f_n is continuous, there exists $\varepsilon > 0$ such that

$$|f_n(t) - f_n(s)| \ < \ \frac{\varepsilon}{2} \qquad \text{for all } s \in (t-\varepsilon, t+\varepsilon)$$

and this implies that

$$|f(t) - f(s)| \ < \ \varepsilon \qquad\qquad \text{for all } s \in (t-\varepsilon, t+\varepsilon)\,.$$

Hence, f is continuous on $[a,b]$.

2. We now show that

$$\lim_{n\to\infty} d(f_n, f) \;=\; 0\,.$$

We estimate

$$\begin{aligned} |f_n(t) - f(t)| \;&\leq\; |f_n(t) - f_m(t)| + |f_m(t) - f(t)| \\ &\leq\; d(f_n, f_m) + |f_m(t) - f(t)|\,. \end{aligned}$$

For $\varepsilon > 0$, choose $N_\varepsilon > 0$ such that

$$d(f_m, f_n) \;<\; \varepsilon \qquad \text{for all } n, m \geq N_\varepsilon\,,$$

we have

$$|f_n(t) - f(t)| \;<\; \varepsilon + |f_m(t) - f(t)|\,.$$

Taking $m \to \infty$, we obtain that

$$|f_n(t) - f(t)| \;<\; \varepsilon + \lim_{m\to\infty} |f_m(t) - f(t)| \;=\; \varepsilon \qquad \text{for all } n \geq N_\varepsilon\,.$$

Thus,

$$d(f_n, f) \;=\; \sup_{t\in[a.b]} |f_n(t) - f(t)| \;\leq\; \varepsilon \qquad n \geq N_\varepsilon\,,$$

and this implies that f_n converges to f in $(C([a,b]), d)$. Therefore, the metric space $(C([a,b]), d)$ is complete. □

2.5 Compact sets

Definition 2.16 *Given a metric space (X,d), the subset $K \subset X$ is compact if and only if for any open cover of K*

$$K \;\subseteq\; \bigcup_{\alpha\in\mathcal{I}} \mathcal{O}_\alpha, \qquad \mathcal{O}_\alpha \text{ open}\,,$$

there exists $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathcal{I}$ such that

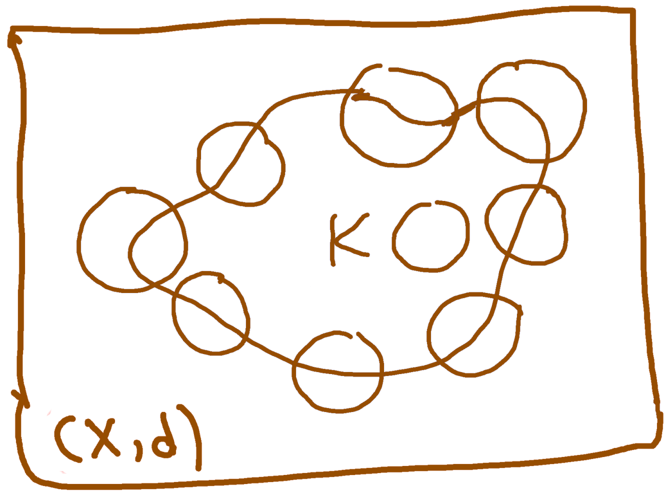
$$K \;\subseteq\; \bigcup_{i=1}^{N} \mathcal{O}_{\alpha_i}\,.$$

Examples. Consider $X = \mathbb{R}$ and $d(x,y) = |x-y|$. Then

- the set $[0,1)$ is not compact, and
- the set $[0,1]$ is compact.

Definition 2.17 *The set $K \subset X$ is called totally bounded if for every $\varepsilon > 0$ there exist a finite number of points $a_1, a_2, \ldots, a_N \in X$ such that*

$$K \subseteq \bigcup_{i=1}^{N} B(a_i, \varepsilon).$$



It is clear that if the set K is totally bounded then it is bounded. Indeed,

$$K \subseteq \bigcup_{i=1}^{N} B(a_i, \varepsilon) \quad \Longrightarrow \quad K \subseteq B(a_1, r_1) \quad \text{with} \quad r_1 = \varepsilon + \max_{i \in \{1,\ldots,N\}} d(a_1, a_i).$$

However, in general

$$K \text{ is bounded} \quad \not\Rightarrow \quad K \text{ is totally bounded}.$$

Example. Consider the metric space

$$X = \{x_1, x_2, \ldots, x_n, \ldots\}$$

and

$$d(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y. \end{cases}$$

The set $K = B(0, 3/2)$ is bounded. However, there does not exists a finite number of points $a_1, a_2, \ldots, a_N \in X$ such that

$$X = K \subseteq \bigcup_{i=1}^{N} B(a_i, 1/2) = \bigcup_{i=1}^{N} \{a_i\} = \{a_1, a_2, \ldots, a_N\}.$$

In this case, K is bounded but not totally bounded.

It is clear that if K is compact then K is totally bounded. Indeed, for every $\varepsilon > 0$, it holds

$$K \subseteq \bigcup_{x \in K} B(x, \varepsilon) .$$

Since K is compact, there exists a finite number of points $\{x_1, \ldots, x_N\} \subset K$ such that

$$K \subseteq \bigcup_{i=1}^{N} B(x_i, \varepsilon)$$

and K is totally bounded.

Theorem 2.18 *Given a metric space (X, d) and a subset $K \subset X$. The following statements are equivalent:*

(i) *K compact;*

(ii) *K is totally bounded and K is complete;*

(iii) *K is sequentially compact, i.e., for every sequence $\{x_n\}_{n \geq 1} \subseteq K$, there exists a subsequence $\{x_{n_k}\}_{n_k \geq 1} \subseteq \{x_n\}_{n \geq 1}$ which converges to $x \in K$.*

Proof. $[(ii) \implies (i)]$ Assume that K is not compact. Then there exists a collection of open subsets $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ such that

$$K \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda ,$$

but

$$K \nsubseteq \bigcup_{i=1}^{N} \mathcal{O}_{\lambda_i} \qquad \text{for all } N \in \mathbb{N}, \{\lambda_1, \lambda_2, \ldots, \Lambda_N\} \subset \Lambda.$$

Since K is totally bounded, one can construct a sequence of nonempty closed sets $\{F_i\}_{i \geq 1}$ satisfying the following properties:

(i) $F_{i+1} \subseteq F_i \subseteq K$ and $\lim_{i \to \infty} \operatorname{diam}(F_i) = 0$;

(ii) F_i can not be covered by a finite number of open sets of $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$.

Cantor intersection theorem implies that

$$\bigcap_{i=1}^{\infty} F_i \;=\; \{x\} \;\in\; K \;\subset\; \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda .$$

Thus, there exists $\lambda_0 \in \Lambda$ and $r_0 > 0$ such that

$$B(x, r_0) \;\subset\; \mathcal{O}_{\lambda_0} .$$

On the other hand, since $\lim_{i\to\infty} \operatorname{diam}(F_i) = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$F_{n_0} \subseteq B(x, r_0) \subseteq \mathcal{O}_{\lambda_0}$$

and it yields a contradiction.

To complete this part, let's construct the sequence of close set $\{F_i\}_{i\geq 1}$. Since K is totally bounded, it holds

$$K \subseteq \bigcup_{i=1}^{n_1} B(a_i, 1/2) \qquad \text{for all } a_i \in X\,.$$

There exists $i_1 \in \{1, \dots, n_1\}$ such that the set $B(a_{i_1}, 1/2) \cap K$ is non-empty and can not be covered by a finite number of open sets of $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$. In particular, the closed set

$$F_1 \doteq \overline{B(a_{i_1}, 1/2) \cap K} \neq \varnothing$$

is totally bounded with $\operatorname{diam}(F_1) \leq 1$ and can not be covered by a finite number of open sets of $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$.

Similarly, one can find a ball $B(a_{i_2}, 1/4)$ such that $B(a_{i_2}, 1/4) \cap F_1$ is non-empty and can not be covered by a finite number of open sets of $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$. In particular, the closed set

$$F_2 \doteq \overline{B(a_{i_2}, 1/4) \cap F_1} \neq \varnothing$$

is totally bounded with $\operatorname{diam}(F_2) \leq 1/2$ and can not be covered by a finite number of open sets of $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$. Continuing in this way we can construct a sequence of sets $\{F_1\}_{i\geq 1}$ which satisfies (i) and (ii).

[$(iii) \implies (ii)$] It is easy to show that K is complete. Now, assume that K is not totally bounded. Then there exists $\delta > 0$ such that K can not be covered by a finite number of ball with radius δ. In this case, we will construct a sequence $\{x_n\}_{n\geq 1}$ which does not admit any convergent subsequence. Take any $x_1 \in K$, we pick

$$x_2 \in K\backslash B(x_1, \delta), \qquad x_3 \in K\backslash[B(x_1, \delta) \cup B(x_2, \delta)]\,,$$

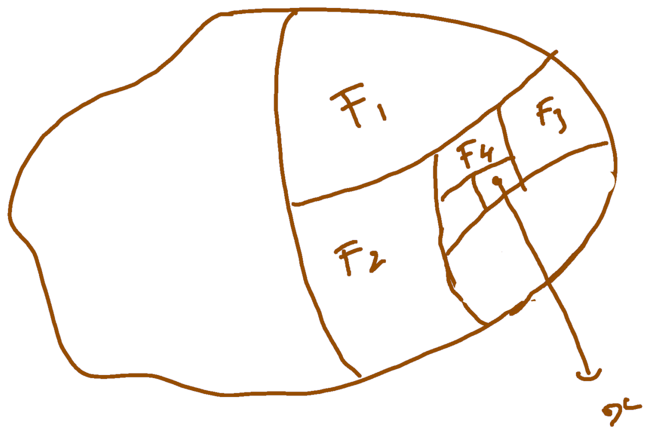
and

$$x_{n+1} \in K\backslash[B(x_1, \delta) \cup B(x_2, \delta) \cdots \cup B(x_n, \delta)]\,.$$

With this construction, we have that the sequence $\{x_n\}_{n\geq 1} \subset K$ satisfies

$$d(x_n, x_m) > \delta \qquad \text{for all } n \neq m\,.$$

Thus, the sequence $\{x_n\}_{n\geq 1}$ does not admit any convergent subsequence.



[$(i) \implies (iii)$] Assume that K is compact. Given any sequence $\{x_n\}_{n\geq 1} \subseteq K$, we need to construct a subsequence $\{x_{n_k}\}_{n_k\geq 1} \subseteq \{x_n\}_{n\geq 1}$ which converges to $x \in K$. Consider the closed set

$$F_k = \overline{\{x_k, x_{k+1}, \dots\}},$$

we first claim that

$$\bigcap_{k=1}^{\infty} F_k \neq \varnothing.$$

Let $\mathcal{O}_k = X \backslash F_k$ be open sets in X. Assume by a contradiction that $\bigcap_{k=1}^{\infty} F_k = \varnothing$, then De Morgan's identities

$$\bigcup_{k=1}^{\infty} \mathcal{O}_k = \bigcup_{k=1}^{\infty} [X \backslash F_k] = X \backslash \left[\bigcap_{k=1}^{\infty} F_k\right] = X.$$

In particular, $K \subseteq \bigcup_{k=1}^{\infty} \mathcal{O}_k$ and thus K is covered by finite number of open sets in $\{\mathcal{O}_k\}_{k\geq 1}$, i.e.,

$$K \subseteq \bigcup_{i=1}^{N} \mathcal{O}_{k_i} = X \backslash \left[\bigcap_{i=1}^{N} F_{k_i}\right] \qquad \text{for some } k_i \geq 1.$$

This yields a contradiction.

Now, take a point $\bar{x} \in \bigcap_{k=1}^{\infty} F_k$, we will construct a subsequence $\{x_{n_k}\}_{k\geq 1} \subset \{x_n\}_{n\geq 1}$ such that $\lim_{k\to\infty} x_{n_k} = \bar{x}$. For every $\ell \geq 1$, it holds

$$B(\bar{x}, 1/\ell) \bigcap \{x_\ell, x_{\ell+1}, \dots\} \neq \varnothing .$$

By induction, for any $k \geq 1$, we pick a point $x_{n_{k+1}} \in B\left(\bar{x}, \frac{1}{k+1}\right) \bigcap \{x_{n_k}, x_{n_k+1}, \dots\}$. It is clear that $\{x_{n_k}\}_{k\geq 1}$ is a subsequence of $\{x_n\}_{n\geq 1}$ and

$$0 \leq \limsup_{k\to 1} d(x_{n_k}, \bar{x}) \leq \limsup_{k\to\infty} \frac{1}{k} = 0 .$$

Thus, the subsequence $\{x_{n_k}\}_{k\geq 1} \subset \{x_n\}_{n\geq 1}$ converges to $\bar{x}$. □

As a consequence, we have the following corollary

Corollary 2.19 *Given a set $K \subset \mathbb{R}^n$, the followings are equivalent:*

(i) K is compact;

(ii) K is closed and bounded;

(iii) K is sequentially compact.

Notice that the equivalence of (i) and (ii) is known as the Heine-Borel theorem and that of (i) and (iii) the Bolzano-Weierstrass theorem.

Corollary 2.20 *A compact metric space (X, d) is separable.*

Proof. Since X is compact, X is totally bounded. Thus, for every natural number n, we have

$$X = \bigcup_{i=1}^{M_n} B(x_{n,i}, 1/n).$$

We can see that the set $S = \bigcup_{n\geq 1} \{x_{n,1}, \dots, x_{n,M_n}\}$ is countable and dense in X. □

Let us give some applications of theorem 2.18.

Proposition 2.20.1 *Given two metric spaces (X, d) and (Y, σ), let $f : X \to Y$ be a continuous map. Then, $f(A)$ is compact in (Y, σ) for all A compact in (X, d).*

Proof. From theorem 2.18, we only need to show that the set $f(A)$ is sequentially compact in (Y,σ), i.e., for every sequence $\{f(x_n)\}_{n\geq 1} \subset f(A)$, find a subset sequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $\{f(x_{n_k})\}_{k\geq 1}$ converges $\bar{y} \in f(A)$ in (Y,σ). Since A is compact, there exists $\{x_{n_k}\}_{k\geq 1} \subset \{x_n\}_{n\geq 1}$ such that $\{x_{n_k}\}_{k\geq 1}$ converges to $\bar{x} \in A$. By the continuity of f, it holds that

$$\lim_{k\to\infty} f(x_{n_k}) \;=\; f(\bar{x}) \;\in\; f(A)\,.$$

The proof is complete. □

Proposition 2.20.2 *Assume that (X,d) is a compact metric space. Then any continuous map $f : X \to Y$ is uniformly continuous, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sigma(f(x), f(y)) \;<\; \varepsilon \qquad \textit{for all } d(x,y) < \delta\,.$$

Proof. The map f is continuous on X, i.e., for every $\varepsilon > 0$ and $x \in X$ there exists $r_x > 0$ such that

$$\sigma(f(x), f(y)) \;\leq\; \frac{\varepsilon}{2} \qquad \text{for all } y \in B(x, r_x)\,.$$

By the triangle inequality, we get

$$\sigma(f(y), f(z)) \;\leq\; \varepsilon \qquad \text{for all } y,z \in B(x, r_x)\,.$$

Set $\mathcal{O}_x := B(x, r_x)$, we have that $X = \bigcup_{x\in X} \mathcal{O}_x$.

Lebesgue covering lemma. *Assume that the metric space (X,d) is compact and $X = \bigcup\limits_{x\in X} \mathcal{O}_x$. Then there exists a constant $\delta > 0$ such that for every $x \in X$, it holds*

$$B(x,\delta) \;\subset\; \mathcal{O}_z \qquad \text{for some } z \in X\,.$$

Thus, for any $x, y \in X$ such that $d(x,y) < \delta$, there exists $z \in X$ such that

$$x, y \;\in\; B(x,\delta) \;\subset\; \mathcal{O}_z \qquad \text{for some } z \in X\,.$$

In particular, this implies that

$$\sigma(f(y), f(x)) \;\leq\; \varepsilon$$

and this complete the proof. □

Following the same idea in Proposition 2.15.1, one can show that

Proposition 2.20.3 *Assume that (X,d) is a compact metric space. Then the metric space $(C(X), d_\infty)$ with*

$$C(X) \;=\; \{f : X \to \mathbb{R} \textit{ is continuous}\}$$

and

$$d_\infty(f,g) \;=\; \max_{x\in X} d(f(x), g(x)) \qquad \textit{for all } f,g \in C(X)$$

is complete.

2.6 Basic theorems

In this subsection, we will introduce some basic theorems.

Theorem 2.21 (Completion) *For a metric space (X, d), there exists a complete metric space $(\tilde{X}, \tilde{d})$ such that there exist a dense set $W \subset \tilde{X}$ and a bijective isometry $T : X \to W$, i.e.,*

$$T(X) \ = \ W \qquad \text{and} \qquad d(x,y) \ = \ \tilde{d}(\tilde{T}(x), \tilde{T}(y)) \qquad \textit{for all } x, y \in X\,.$$

1. The Banach contraction principle. Given two metric space (X, d) and (Y, σ), we say that the map $f : X \to Y$ is contractive if and only if there exists $0 < c < 1$ such that

$$\sigma(f(x), f(y)) \ \leq \ c \cdot d(x,y) \qquad \text{for all } x, y \in X\,.$$

The following holds:

Theorem 2.22 (Banach contraction principle) *Let (X, d) be a complete metric space and the map $T : X \to X$ is contractive. Then T has a unique fixed points $\bar{x}$, i.e.,*

$$T(\bar{x}) \ = \ \bar{x}\,.$$

Proof. Since T is contractive, we have

$$d(T(x), T(y)) \ \leq \ c \cdot d(x,y) \qquad \text{for all } x, y \in X$$

for some $0 < c < 1$. This implies that T has at most one fixed point. It remains to show that T has a fixed points.

1. Take any $a \in X$, the sequence $\{x_n\}_{n \geq 0}$ is constructed by

$$x_0 \ = \ a, \qquad x_{n+1} \ = \ T(x_n) \qquad \text{for all } n \geq 0\,.$$

Observe that

$$d(x_{n+1}, x_n) \ = \ d(T(x_n), T(x_{n-1})) \ \leq \ c \cdot d(x_n, x_{n-1}) \qquad \text{for all } n \geq 1\,.$$

By induction, we get

$$d(x_{n+1}, x_n) \ \leq \ c^n \cdot d(a, T(a)) \qquad \text{for all } n \geq 0\,.$$

2. We show that $\{x_n\}_{n \geq 0}$ is Cauchy in (X, d). For any $0 \leq n < m \in \mathbb{N}$, it holds

$$\begin{aligned} d(x_n, x_m) \ &\leq \ \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \ \leq \ d(a, T(a)) \cdot \sum_{k=n}^{m-1} c^k \\ &= \ d(a, T(a)) \cdot \frac{c^n(1 - c^{m-n})}{1 - c} \ \leq \ d(a, T(a)) \cdot \frac{c^n}{1 - c}\,. \end{aligned}$$

Since $0 < c < 1$, it holds that $\lim_{n\to\infty} d(a,T(a)) \cdot \frac{c^n}{1-c} = 0$. Thus, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence.

3. Since (X,d) is complete, the Cauchy sequence $\{x_n\}_{n\geq 0}$ converges to $\bar{x} \in X$. Notice that T is continuous on X and it implies

$$\bar{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\bar{x}).$$

Thus, $\bar{x}$ is a unique fixed point of T. □

Give given an open set $\mathcal{O} \in \mathbb{R}^2$, let $g : \mathcal{O} \to \mathbb{R}$ be a continuous function. Consider the ordinary differential equation

$$\begin{cases} x'(t) = f(t,x(t)) & \text{for all } t \in (a,b), \\ x(t_0) = x_0. \end{cases} \tag{2.16}$$

Then, $x(\cdot)$ is a Carathéodory solution of (2.16) if and only if $x(\cdot)$ is absolutely continuous and

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds \qquad \text{for all } t \in (a,b).$$

Theorem 2.23 (The Picard Local Existence Theorem) *Assume that*

$$|f(t,x_2) - f(t,x_1)| \leq M \cdot |x_2 - x_1| \qquad \textit{for all } (t,x_2),(t,x_1) \in \mathcal{O}.$$

Then, for every $(t_0,x_0) \in \mathcal{O}$, there exists an open interval I containing t_0 on which the ODE (2.16) has a unique Carathéodory solution.

Idea of the proof. There exists a small interval I and a closed subset $X_I \subset \mathcal{C}(I)$ such that the operator $T : X_I \to X_I$

$$T[y](t) = y_0 + \int_{t_0}^{t} f(s,y(s))ds \qquad \text{for all } t \in I,$$

is contractive. Thus, T achieves a unique fixed point x which is the unique solution to the ODE (2.16). □

2. The Arzelà-Ascoli theorem. Given a metric space (X,d), a collection $\mathcal{F}$ of real-valued functions $f : X \to \mathbb{R}$ is *equicontinuous* at $x \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \leq \varepsilon \qquad \text{for all } f \in \mathcal{F}, y \in B(x,\delta).$$

We say that $\mathcal{F}$ is equicontinuous on X if it is equicontinuous at every point $x \in X$.

Theorem 2.24 (Arzelà-Ascoli) *Given a compact metric space (X,d), let $\{f_n\}_{n\geq 1}$ be a uniformly bounded and equicontinuous sequence of real value functions on X, i.e.,*

- *There exists $M > 0$ such that*

$$\sup_{x\in X} |f_n(x)| \ \leq\ M \qquad \textit{for all } n \in \mathbb{N};$$

- *For every $x \in X$ and $\varepsilon > 0$, there exists $\delta_x > 0$ such that*

$$|f_n(x) - f_n(y)| \ \leq\ \varepsilon \qquad \textit{for all } y \in B(x,\delta_x), n \in \mathbb{N}.$$

Then there exists $\{f_{n_k}\}_{n_k\geq 1} \subset \{f_n\}_{n\geq 1}$ such that $\{f_{n_k}\}_{k\geq 1}$ converges uniformly to $f \in \mathcal{C}(X,\mathbb{R})$, i.e.,

$$\lim_{n_k\to\infty} \left[\sup_{x\in X} \left|f_{n_k}(x) - f(x)\right|\right] \ =\ 0\,.$$

Sketch of proof. 1. Since X is a compact metric space, it is separable. Thus, one can construct a subsequence $\{f_{n_k}\}_{n_k\geq 1} \subseteq \{f_n\}_{n\geq 1}$ such that f_{n_k} converges point-wise on all of X to a real function f on X, i.e.,

$$\lim_{n_k\to\infty} f_{n_k}(x) \ =\ f(x) \qquad \text{for all } x \in X\,.$$

2. We claim that f is continuous. Fix any $x \in X$, for every $\varepsilon > 0$, there exists $\delta_x > 0$ such that

$$|f_{n_k}(x) - f_{n_k}(y)| \ \leq\ \varepsilon \qquad \text{for all } n_k \geq 1, y \in B(x,\delta_x)\,.$$

The triangle inequality implies that

$$\begin{aligned} |f(x) - f(y)| \ &\leq\ |f_{n_k}(x) - f(x)| + |f_{n_k}(x) - f(n_k)(y)| + |f_{n_k}(y) - f(y)| \\ &\leq\ \varepsilon + |f_{n_k}(x) - f(x)| + |f_{n_k}(y) - f(y)| \end{aligned}$$

for all $n_k \geq 1$ and $y \in B(x,\delta_x)$. Taking n_k to ∞, we get

$$|f(x) - f(y)| \ \leq\ \varepsilon \qquad \text{for all } y \in B(x,\delta_x)$$

and it yields the continuity of f at x.

3. To complete the proof, we need to show that f_{n_k} converges uniformly to f in $\mathcal{C}(X)$. Hint: Equicontinuity of $\{f_{n_k}\}_{n_k\geq 1}$, and totally boundedness of X. □

Corollary 2.25 *Given a compact metric space (X,d), let $\mathcal{F}$ be a subset of $C(X)$. Then $\mathcal{F}$ is compact if $\mathcal{F}$ is closed, uniformly bounded and equicontinuous.*

To conclude this subsection, we will introduce the Baire category theorem.

Theorem 2.26 (Baire category) *Let (X, d) be a complete metric space. The following holds:*

(i) For any $\{\mathcal{O}_n\}_{n\geq 1}$ countable collections of open dense subset of X, the set

$$\mathcal{O} \;=\; \bigcap_{n=1}^{\infty} \mathcal{O}_n$$

is dense in X.

(ii) For any $\{F_n\}_{n\geq 1}$ countable collections of hollow subset of X, the set

$$F \;=\; \bigcup_{n=1}^{\infty} F_n$$

is hollow. (Notice that B is hollow in X if and only if $X\backslash B$ is dense in X)

Sketch of proof. Using De Morgan's identity, one can show that (i) and (ii) are equivalent. Thus, we only need to prove (i). Given $x_0 \in X$ and $r_0 > 0$, we need to show that

$$\left[\bigcap_{n=1}^{\infty} \mathcal{O}_n\right] \bigcap B(x_0, r) \;\neq\; \varnothing\,.$$

Since $\mathcal{O}_1$ is dense in X, one has that the open set $\mathcal{O}_1 \cap B(x_0, r_0)$ is nonempty. This implies that there exist $x_1 \in X$ and $0 < r_1 < 1$ such that

$$\overline{B}(x_1, r_1) \;\subseteq\; \mathcal{O}_1 \cap B(x_0, r)$$

Similarly, since $\mathcal{O}_2$ is dense in X, one has that the open set $B(x_1, r_1) \cap \mathcal{O}_2$ is nonempty. This implies that there exist $x_2 \in X$ and $0 < r_2 < 1/2$ such that

$$\overline{B}(x_2, r_2) \;\subseteq\; \mathcal{O}_2 \cap B(x_1, r_1).$$

By induction method, one obtains a decreasing sequence of closed balls $\overline{B}(x_n, r_n)$ such that

$$\lim_{n\to\infty} r_n \;=\; 0 \qquad \text{and} \qquad \overline{B}(x_n, r_n) \;\subseteq\; \mathcal{O}_n \qquad \text{for all } n \geq 1.$$

In particular, by the Cantor intersection theorem, we get

$$\bar{x} \;=\; \bigcap_{n\geq 1} \overline{B}(x_n, r_n) \;\subseteq\; \bigcap_{n=1}^{\infty} \mathcal{O}_n$$

and this yields

$$\bar{x} \;\in\; \left[\bigcap_{n=1}^{\infty} \mathcal{O}_n\right] \bigcap B(x_0, r).$$

The proof is complete. □

Corollary 2.27 *Let (X, d) be a complete metric space and $\{F_n\}_{n\geq 1}$ a countable collection of closed subsets of X. If $\bigcup_{n=1}^{\infty} F_n$ has nonempty interior, then there exists $n_0 \in \mathbb{N}$ such that F_{n_0} has also nonempty interior.*

Proof. Assume by a contradiction that F_n does not have empty interior for all $n \geq 1$. This implies that $\mathcal{O}_n := X \setminus F_n$ is open dense subset of X for every n. Using the Baire category theorem, we have that $X \setminus \left(\bigcup_{n=1}^{\infty} F_n \right) = \bigcap_{n=1}^{\infty} \mathcal{O}_n$ is dense in X and it yields a contradiction. □

Corollary 2.28 *Let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space (X, d) that is pointwise bounded, i.e., for every $x \in X$ there exists $M_x > 0$ such that*

$$|f(x)| \leq M_x \qquad \text{for all } f \in \mathcal{F}.$$

Then there is a nonempty open subset $\mathcal{O}$ of X on which $\mathcal{F}$ is uniformly bounded, i.e.,

$$\sup_{x\in\mathcal{O}} |f(x)| \leq M \qquad \text{for all } f \in \mathcal{F}$$

for some $M > 0$.

Proof. For every $n \geq 1$, consider the closed set

$$E_n = \{x \in X : |f(x)| \leq n \quad \text{for all } f \in \mathcal{F}\}$$

Since $\mathcal{F}$ is pointwise bounded, it holds that $X = \bigcup_{n\geq 1} E_n$. By the previous corollary, there exists a natural number n_0 such that E_{n_0} contain an open ball $B(x_0, r_0)$ and this yields

$$\sup_{x\in B(x_0,r_0)} |f(x)| \leq n_0 \qquad \text{for all } f \in \mathcal{F}.$$

The proof is complete. □

Corollary 2.29 *Given a complete metric space (X, d), let $(f_n)_{n\geq 1}$ be a sequence in $C(X)$ that converges pointwise to the real value function f. Then there exists a dense subset D of X for which $(f_n)_{n\geq 1}$ is equicontinuous and f is continuous at each point in D.*

Idea of the proof. For every $n, m \in \mathbb{N}$, define

$$E_{n,m} = \left\{ x \in X : \left| f_j(x) - f_k(x) \right| \leq \frac{1}{m} \qquad \text{for all } j, k \geq m \right\},$$

we have that $E_{n,m}$ is closed in X and thus

$$D \doteq X \backslash \left[\bigcup_{n,m\in\mathbb{N}} \partial E_{n,m} \right] \text{ is dense in } X.$$

One shows that $(f_n)_{n\geq 1}$ is equicontinuous and f is continuous at each point in D. □

3 Banach spaces and linear operators

3.1 Normed spaces

1. Vector spaces. X is a vector space on the field $\mathbb{R}$ if the following holds:

(a). **Addition** $+ : X \times X \to X$

- $x + y = y + x$ (commutative)
- $(x + y) + z = x + (y + z)$ (associate)
- There exists 0 and $-x$ such that

$$x + (-x) = 0 \qquad \text{and} \qquad x + 0 = x\,.$$

(b). **Multiplication by scalar**: $\cdot : \mathbb{R} \times X \to X$

- $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$ for all $\alpha, \beta \in \mathbb{R}, x \in X$
- $1 \cdot x = x$ for all $x \in X$

(c). **Linear property**

$$\alpha \cdot (x + y) = \alpha x + \alpha y \quad \text{and} \quad (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$$

for all $\alpha, \beta \in \mathbb{R}, x, y \in X$.

Some examples. Let's introduce here some basic vector spaces

(a). $\ell^2 = \left\{ x = \{x_n\}_{n\geq 1} \ : \ \sum_{n=1}^{\infty} |x_n|^2 < +\infty \right\}$ is a vector space over $\mathbb{R}$.

(b). $C([a,b]) = \{f : [a,b] \to \mathbb{R} \ : \ f \text{ is continuous}\}$ is a vector space over $\mathbb{R}$.

(c). $C_0([a,b]) = \{f \in C([a,b]) \ : \ f(a) = f(b) = 0\}$ is a vector space over $\mathbb{R}$.

(d). $C_c([a,b]) = \{f \in C_0([a,b]) \ : \ f \text{ has a compact support}\}$ is a vector space over $\mathbb{R}$.

The following theorem holds

Theorem 3.1 *Let X be a non-empty vector space then X has a Hamel basis, i.e., there exists a linear independent subset $B \subset X$ such that*

$$\text{span}(B) = \left\{ \sum_{i=1}^{n} \lambda_i \cdot e_i \ : \ \lambda_i \in \mathbb{R}, e_i \in B, n \geq 1 \right\} = X.$$

2. Normed spaces. Given a vector space X, let's consider a metric distance d on X which has the following properties:

(P1). **Invariant under a translation.**

$$d(x+z, y+z) = d(x,y).$$

(P2). **Positively homogeneous.**

$$d(\lambda x, \lambda y) = \lambda \cdot d(x,y).$$

(P3). **Convexity.** Every ball $B(a,r)$ is convex.

The invariant under translation implies that

$$d(x,y) = d(x-y, 0).$$

Thus, the metric $d(\cdot,\cdot)$ can be entirely determined by

$$x \mapsto \|x\| := d(x,0).$$

Here, $\|x\|$ is called norm of X.

Definition 3.2 (Normed spaces) *Let X be a vector space. A norm on X is a map $x \mapsto \|x\|$ such that*

(i). $\|x\| \geq 0$ and $\|x\| = 0$ if any only if $x = 0$;

(ii). $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in X$;

(iii). Triangle inequality.

$$\|x+y\| \leq \|x\| + \|y\| \qquad \text{for all } x, y \in X.$$

We call $(X, \|\cdot\|)$ is a normed space.

It is clear that

Lemma 3.3 (Distance defined by a norm) *Let $\|\cdot\|$ be a norm on X. Then*

$$d(x,y) := \|x-y\| \qquad \text{for all } x, y \in X$$

is a metric distance and d satisfies (P1)-(P3).

Proof. It is clear that

$$d(x,y) \;=\; 0 \qquad \text{and} \qquad x \;=\; y$$

and

$$d(x,y) \;=\; d(y,x) \qquad \text{for all } x,y \in X\,.$$

Let's prove the triangle inequality

$$\begin{aligned} d(x,y) + d(y,z) \;&=\; |x-y| + \|y-z\| \\ &\leq\; \|x-y+y-x\| \;=\; \|x-z\| \;=\; d(x,z) \end{aligned}$$

for all $x,y,z \in X$.

The properties (P1) and (P2) are trivial. Let's show that $B(a,r)$ is convex, i.e., for every $x,y \in B(a,r)$, it holds

$$tx + (1-t)y \;\in\; B(a,r) \qquad \text{for all } t \in (0,1).$$

Using the triangle inequality, we estimate

$$\|tx + (1-t)y\| \;\leq\; t \cdot \|x\| + (1-t) \cdot \|y\| \;<\; ta + (1-t)a \;=\; a$$

and it yields the above inclusion. □

Let's recall some basic notations and defitnions:

- **Open ball.**
$$B(a,r) \;=\; \{x \in X \mid \|x-a\| < r\};$$
- **Closed ball.**
$$\overline{B}(a,r) \;=\; \{x \in X \mid \|x-a\| \leq r\};$$
- **Sphere.**
$$S(a,r) \;=\; \{x \in X \mid \|x-a\| = r\}.$$

Sequences and series. A sequence $\{x_n\}_{n\geq 1} \subset X$ is

- bounded if there exits $M > 0$ such that $\{x_n\}_{n\geq 1} \subseteq B(0,M)$.
- Cauchy if for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that
$$\|x_n - x_m\| \;\leq\; \varepsilon \qquad \text{for all } n,m \geq N_\varepsilon.$$
- converges to $x \in X$ if for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that
$$\|x_n - x\| \;\leq\; \varepsilon \qquad \text{for all } n \geq N_\varepsilon.$$

Given a sequence $\{x_n\}_{n\geq 1}$, the series

$$\sum_{n=1}^{\infty} x_n \;=\; x_1 + x_2 + \dots$$

converges to $x \in X$ if and only if its partial sum $s_n = \displaystyle\sum_{i=1}^{n} x_i$ converges to x.

Definition 3.4 *A series* $\displaystyle\sum_{n=1}^{\infty} x_n$ *is absolutely convergent in* $(X, \|\cdot\|)$ *if and only if* $\displaystyle\sum_{n=1}^{\infty} \|x_n\|$ *converges in* $\mathbb{R}$.

Continuity. Given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the function $f : X \to Y$ is continuous at x if and only if for every $\varepsilon >$, there exists $\delta > 0$ such that

$$\|f(x) - f(y)\|_Y \;\leq\; \varepsilon \qquad \text{for all } \|x - y\|_X \;\leq\; \delta.$$

Equivalent norms. Given a vector space X, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if any only if there exists $\lambda \geq 1$ such that

$$\frac{1}{\lambda} \cdot \|x\|_1 \;\leq\; \|x\|_2 \;\leq\; \lambda \cdot \|x\|_1$$

for all $x \in X$.

3.2 Banach space

Definition 3.5 *A normed space* $(X, \|\cdot\|)$ *is called Banach if it is complete, i.e., every Cauchy sequence* $\{x_n\}_{n\geq 1}$ *converges to* $x \in X$.

Example 1. Consider the finite-dimensional space

$$\mathbb{R}^n \;=\; \{x = (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

with Euclidean norm

$$\|x\|_2 \;=\; \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

We have that $(\mathbb{R}^n, \|\cdot\|_2)$ is a Banach space.

Example 2. On $\mathbb{R}^n$, consider an alternative norm

$$\|x\|_p \;=\; \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}$$

for $p \geq 1$. Then $(\mathbb{R}^n, \|\cdot\|_p)$ is also a Banach space.

Example 3. Given a closed interval $[a, b]$, recalling that

$$C([a,b],\mathbb{R}) \;=\; \{f : [a.b] \to \mathbb{R} : f \text{ is continuous}\}$$

and

$$\|f\|_\infty \;=\; \max_{x\in[a,b]} |f(x)| \qquad \text{for all } f \in C([a,b],\mathbb{R}).$$

Then $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

Proof. Given a Cauchy sequence $\{f_n\}_{n\geq 1} \subset C([a,b],\mathbb{R})$, we need to show that $\{f_n\}_{n\geq 1}$ converges to $f \in C([a,b],\mathbb{R})$.

1. It is clear that $\{f_n\}_{n\geq 1}$ is bounded in $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$, i.e.,

$$\|f_n\|_\infty \;=\; \max_{x\in[a.b]} |f_n(x)| \;\leq\; M \qquad \text{for all } n \geq 1$$

for some constant M. Thus, $\{f_n\}_{n\geq 1}$ is uniformly bounded.

2. We claim that $\{f_n\}_{n\geq 1}$ is equicontinuous on $[a, b]$. Given $x \in [a, b]$, for any $\varepsilon > 0$, we need to find $\delta > 0$ such that

$$|f_n(x) - f_n(y)| \;\leq\; \varepsilon \qquad \text{for all } y \in B(x,\delta), n \geq 1. \tag{3.1}$$

Since $\{f_n\}_{n\geq 1}$ is Cauchy in $C([a,b],\mathbb{R}), \|\cdot\|_\infty)$, one can find $1 < n_0 \in \mathbb{N}$ such that

$$\|f_n - f_{n_0}\|_\infty \;\leq\; \frac{\varepsilon}{3} \qquad \text{for all } n \geq n_0.$$

On the other hand, by the continuity of f_i, for every $i \in \{1, 2, \dots, n_0\}$, there exists $\delta_i > 0$ such that

$$|f_i(y) - f_i(x)| \;\leq\; \frac{\varepsilon}{3} \qquad \text{for all } y \in B(x,\delta_i), i \in \{1,2,\dots,n_0\}.$$

We show that (3.1) holds for $\delta = \min_{i\in\{1,2,\dots,n_0\}} \delta_i$. Indeed, for every $n \geq n_0$ and $y \in B(x,\delta)$, it holds

$$|f_n(y) - f_n(x)| \;\leq\; |f_{n_0}(y) - f_{n_0}(x)| + 2\cdot\|f_n - f_{n_0}\|_\infty \;\leq\; \frac{\varepsilon}{3} + 2\cdot\frac{\varepsilon}{3} \;=\; \varepsilon\,.$$

Thus, $\{f_n\}_{n\geq 1}$ is equicontinuous on $[a, b]$

3. By the Arzalà-Ascoli theorem, there exists a subsequence $\{f_{n_k}\}_{k\geq 1} \subset \{f_n\}_{n\geq 1}$ which converges to f in $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$. Since $\{f_n\}_{n\geq 1}$ is Cauchy, it converges to f in $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$. □

Example 4. Fixed $p \geq 1$, recalling that

$$\ell^p \;=\; \left\{ x = \{x_n\}_{n\geq 1} \;\middle|\; \sum_{n=1}^{\infty} |x_n|^p < +\infty \right\},$$

and

$$\|x\|_p \;=\; \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}.$$

Then $(\ell^p, \|\cdot\|_p)$ is a Banach space.

Lemma 3.6 *Let $(X, \|\cdot\|)$ be a Banach space and $Y \subseteq X$ be a subspace of X. Then $(Y, \|\cdot\|)$ is a Banach space if any only if Y is closed in X.*

Proof. Assume that Y is closed in X. For Cauchy sequence $\{x_n\}_{n\geq 1} \subset Y \subseteq X$, it converges to x in X. Since Y is closed, one has that $x \in Y$. Thus, $\{x_n\}_{n\geq 1}$ converges to x in $(Y, \|\cdot\|)$.

Assume that $(Y, \|\cdot\|)$ is a Banach space. Let $\{x_n\}_{n\geq 1}$ be a sequence in Y which converges to $x \in X$. We have that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in Y. Thus, it converges to $y \in X$ and it yields $x = y$. □

Lemma 3.7 *Let $(X, \|\cdot\|)$ be a Banach space. If the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent then it converges.*

Proof. For every $n \geq 1$, consider the partial sum

$$s_n \;=\; x_1 + x_2 + \cdots + x_n.$$

We need to show that $\{s_n\}_{n\geq 1}$ is a Cauchy sequence in X. One has that

$$\|s_m - s_n\| \;\leq\; \sum_{k=n+1}^{m} \|x_k\| \qquad \text{for all } n < m\,.$$

Since the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, it holds

$$\sum_{k=n+1}^{m} \|x_k\| \;\leq\; \varepsilon \qquad \text{for all } n, m \geq N_\varepsilon$$

for some N_ε. This implies that

$$\|s_m - s_n\| \;\leq\; \varepsilon \qquad \text{for all } n, m \geq N_\varepsilon$$

and $\{s_n\}_{n\geq 1}$ is a Cauchy sequence in X. Thus, $\{s_n\}_{n\geq 1}$ converges in $(X, \|\cdot\|)$ □

3.3 Finite dimensional normed spaces

We say that X is a finite dimensional vector space if there exists $\{e_1, e_2, \ldots, e_n\}$ such that

$$X = \text{span}\,\{e_1, e_2, \ldots, e_N\} := \left\{ \sum_{i=1}^{n} \lambda_i \cdot e_i \;\middle|\; \lambda_i \in \mathbb{R} \right\}.$$

If $\{e_1, e_2, \ldots, e\}$ is linearly independent then $\dim(X) = n$.

Lemma 3.8 *A finite dimensional normed space $(X, \|\cdot\|)$ is complete.*

Proof. Assume that $\dim(X) = n$, we have

$$X = \left\{ \sum_{i=1}^{n} \lambda_i \cdot e_i \;\middle|\; \lambda_i \in \mathbb{R} \right\}$$

for some linearly independent set of vectors $\{e_1, e_2, \ldots, e_n\}$. Consider a Cauchy sequence $\{x_k\}_{k\geq 1}$ in $(X, \|\cdot\|)$. For every $k \geq 1$, one write

$$x_k = \sum_{i=1}^{n} \lambda_k^{(i)} \cdot e_i \,.$$

Linear combination lemma. *There exists a constant $c > 0$ such that*

$$\|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\| \geq c \cdot (|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|)\,.$$

We estimate

$$\|x_k - x_l\| = \left\| \sum_{i=1}^{n} \left(\lambda_k^{(i)} - \lambda_l^{(i)} \right) \cdot e_i \right\| \geq c \cdot \sum_{i=1}^{n} \left| \lambda_k^{(i)} - \lambda_l^{(i)} \right|.$$

In particular,

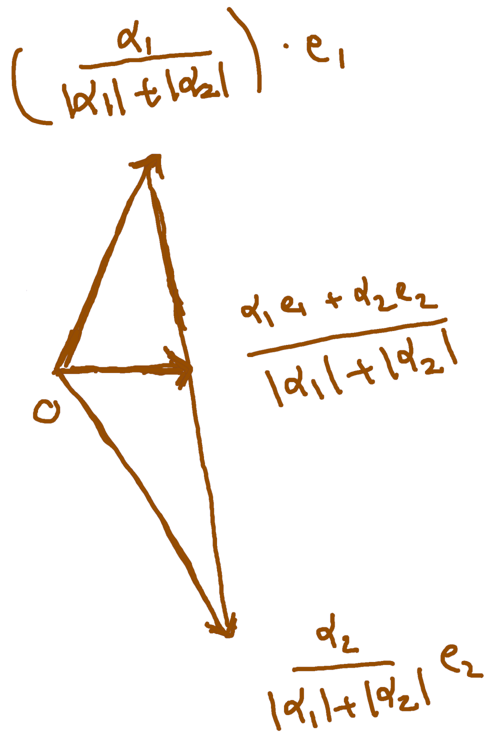
$$\left| \lambda_k^{(i)} - \lambda_l^{(i)} \right| \leq \frac{\|x_k - x_l\|}{c} \qquad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Thus, the sequence $\{\lambda_k^{(i)}\}_{k\geq 1}$ is Cauchy in $\mathbb{R}$ and

$$\lim_{k\to\infty} \lambda_k^{(i)} = \overline{\lambda}^{(i)}.$$

Finally, we show that $\{x_k\}_{k\geq 1}$ converges to $\bar{x} = \sum_{i=1}^{n} \overline{\lambda}^{(i)} \cdot e_i$. Indeed, using the triangle inequality, we estimate

$$\begin{aligned} \|x_k - \bar{x}\| &= \left\| \sum_{i=1}^{n} \left(\lambda_k^{(i)} - \bar{\lambda}^{(i)} \right) \cdot e_i \right\| \leq \sum_{i=1}^{n} \left| \lambda_k^{(i)} - \bar{\lambda}^{(i)} \right| \cdot \|e_i\| \\ &\leq \max\{\|e_1\|, \|e_2\|, \ldots, \|e_n\|\} \cdot \sum_{i=1}^{n} \left| \lambda_k^{(i)} - \bar{\lambda}^{(i)} \right|. \end{aligned}$$



Since $\{\lambda_k^{(i)}\}_{k\geq 1}$ converges to $\bar{\lambda}^{(i)}$ for all $i \in \{1, 2, \ldots, n\}$, the right hand side converges to 0 as k tends to $+\infty$ and this complete the proof. □

As a consequence of theorem 2.18, the following holds

Corollary 3.9 (Compactness) *Let $(X, \|\cdot\|)$ be a finite dimensional normed space. A subset $M \subseteq X$ is compact if and only if M is closed and bounded.*

The following theorem state that every norm in a finite dimensional vector space is equivalent.

Theorem 3.10 *Let X be a finite dimensional vector space. All norms in X are equivalent.*

Proof. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms in X. We need to find $\lambda > 1$ such that

$$\frac{1}{\lambda} \cdot \|x\|_1 \;\leq\; \|x\|_2 \;\leq\; \lambda \cdot \|x\|_1 \qquad \text{for all } x \in X. \tag{3.2}$$

1. Assume that $\dim(X) = n$. There exists n linear independent set of vectors $\{e_1, e_2, \ldots, e_n\}$ such that

$$X \;=\; \text{span}\{e_1, e_2, \ldots, e_n\}\,.$$

By the linear combination lemma, for any $k = 1, 2$, there exists $\lambda_k < 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i \cdot e_i \right\|_k \geq \lambda_k \cdot \sum_{i=1}^{n} |\alpha_i|.$$

2. For any $x \in X$, one can write

$$x = \sum_{i=1}^{n} \alpha_i \cdot e_i$$

for some $\alpha_i \in \mathbb{R}$. We estimate

$$\lambda_1 \cdot \sum_{i=1}^{n} |\alpha_i| \leq \|x\|_1 = \left\| \sum_{i=1}^{n} \alpha_i \cdot e_i \right\|_1 \leq \sum_{i=1}^{n} |\alpha_i| \cdot \|e_i\|_1 \leq M_1 \cdot \sum_{i=1}^{n} |\alpha_i|$$

and

$$\lambda_2 \cdot \sum_{i=1}^{n} |\alpha_i| \leq \|x\|_2 = \left\| \sum_{i=1}^{n} \alpha_i \cdot e_i \right\|_2 \leq \sum_{i=1}^{n} |\alpha_i| \cdot \|e_i\|_2 \leq M_2 \cdot \sum_{i=1}^{n} |\alpha_i|$$

where $M_k = \max_{i \in \{1,2,\ldots n\}} \|e_i\|_k$ for $k = 1, 2$. This implies that

$$\frac{\lambda_2}{M_1} \cdot \|x\|_1 \leq \|x\|_2 \leq \frac{M_2}{\lambda_1} \cdot \|x_1\|.$$

Set $\lambda := \max \left\{ \frac{M_1}{\lambda_2}, \frac{M_2}{\lambda_1} \right\}$, (3.2) holds. □

Question. *It is known that if $(X, \|\cdot\|)$ is a finite dimensional normed space then the closed unit ball*

$$\overline{B(0,1)} = \left\{ x \in X \;\middle|\; \|x\| \leq 1 \right\}$$

is compact. Is the reversed side still true?

Theorem 3.11 *Let $(X, \|\cdot\|)$ be a normed space. If $\overline{B(0,1)}$ is compact then X is finite dimensional.*

Proof. 1. Assume that $\overline{B(0,1)}$ is compact. There exists a set of N vectors $\{p_1, p_2, \ldots, p_N\}$ such that

$$\overline{B(0,1)} = \bigcup_{i=1}^{N} B(p_i, 1/2).$$

The vector subspace

$$V = \text{span}\,\{p_1, p_2, \ldots . p_N\}$$

is finite dimensional and closed in X.

2. To complete the proof, we will show that

$$V \;=\; X\,.$$

Assume by a contradiction, there exists $x_0 \in X$ such that $x_0 \notin V$. Denote by

$$r \;=\; d_V(x_0) \;=\; \inf_{y\in V}\; \|x_0 - y\|$$

Since V is a finite dimensional space, one can show that r is positive and there exists $y_0 \in V$ such that

$$\|x_0 - y_0\| \;=\; d_V(x_0) \;=\; r\,.$$

Set

$$z_0 \;=\; \frac{x_0 - y_0}{\|x_0 - y_0\|} \;\in\; \overline{B(0,1)}\,.$$

There exists $k \in \{1, 2, \ldots, N\}$ such that

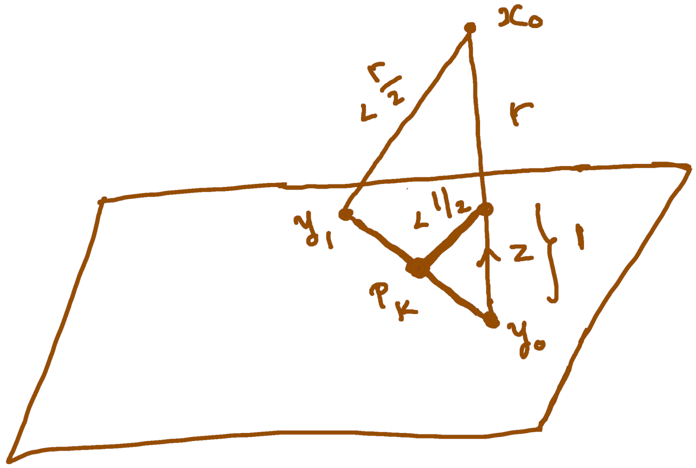
$$z_0 \;\in\; B(p_k, 1/2) \qquad \Longrightarrow \qquad \|z_0 - p_k\| \;<\; \frac{1}{2}\,.$$

Observe that

$$x_0 \;=\; y_0 + r\cdot z_0 \;=\; (y_0 + r\cdot p_k) + r\cdot(z - p_k)$$

Thus, the vector

$$y_1 \;=\; y_0 + r\cdot p_k \;\in\; V\,,$$



satisfies

$$\|x_0 - y_1\| \;=\; r\cdot\|z - p_k\| \;<\; \frac{r}{2}\,.$$

This implies that

$$r = d_V(x_0) < \frac{r}{2}.$$

and it yields a contradiction. $\square$

3.4 Linear bounded operators

Given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, consider a map

$$T : D(T) \subseteq X \longrightarrow R(T) \subseteq Y.$$

Definition 3.12 *We say that T is a linear operator if*

(i). $D(T)$ is a subspace of X

(ii). For any $x, y \in D(T)$ and $\alpha, \beta \in \mathbb{R}$, it holds

$$T(\alpha \cdot x + \beta \cdot y) = \alpha \cdot T(x) + \beta \cdot T(y).$$

Notice that If T is linear then

(a). $0 \in D(T)$ and $T(0) = 0$;

(b). $R(T)$ is a subspace of Y;

(c). For any $\alpha_i \in \mathbb{R}$ and $x_i \in X$, it holds

$$T\left(\sum_{i=1}^{n} \alpha_i \cdot x_i\right) = \sum_{i=1}^{n} \alpha_i \cdot T(x_i).$$

Some examples.

Example 1. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}$. Given a unit vector $v \in \mathbb{R}^n$, the map $T_v : X \to Y$, denoted by

$$T_v(x) = \langle x, v\rangle \qquad \text{for all } x \in R^n.$$

is a linear operator.

Riesz representation theorem in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}$ be a linear operator. Show that there exists a unique $v \in \mathbb{R}^n$ such that

$$T(x) = \langle v, x\rangle = \sum_{i=1}^{n} v_i x_i$$

for all $x \in \mathbb{R}^n$.

Example 2. (Integration) The map $T : C([a,b]) \to \mathbb{R}$, denoted by

$$T(f) \; = \; \int_a^b \; f(t) \; dt \qquad \text{for all } f \in C([a,b])$$

is a linear operator.

Example 3. (Differentiation) The map $T : D(T) \subseteq C([a,b]) \to R(T) \subseteq C([a,b])$, denoted by

$$T(f) \; = \; f' \qquad \text{for all } f \in C([a,b])$$

is a linear operator.

Example 4. (Projection) Let $(\mathbb{R}^n, \| \cdot \|)$ be an Euclidean space and Y be a vector subspace of Y. The projection $P_Y : \mathbb{R}^n \to Y$, denoted by

$$\|x - P_Y(x)\| \; = \; \min \; \{\|x - y\| \mid y \in Y\} \qquad \text{for all } x \in X,$$

is a linear operator.

Proof. Assume that

$$Y \; = \; \text{span} \; \{e_1, e_2, \ldots, e_m\} \qquad \text{and} \qquad X \; = \; \text{span} \; \{e_1, e_2, \ldots, e_n\}$$

where $\|e_i\| = 1$ and $e_i \perp e_j$ for all $i \neq j$. For any

$$x \; = \; \sum_{i=1}^{n} \alpha_i \cdot e_i \; \in \; \mathbb{R}^n \, ,$$

it holds

$$P_Y(x) \; = \; \sum_{i=1}^{m} \alpha_i \cdot e_i \, .$$

Indeed, for any

$$y \; = \; \sum_{i=1}^{m} \; \lambda_i \cdot e_i \; \in \; Y$$

We compute that

$$\begin{aligned} \|x - y\|^2 \; &= \; \left\| \sum_{i=1}^{m} (\lambda_i - \alpha_i) \cdot e_i + \sum_{j=m+1}^{m} \; \alpha_j \cdot e_j \right\|^2 \\ &= \; \sum_{i=1}^{m} \; |\lambda_i - \alpha_i|^2 + \sum_{j=m+1}^{n} \; |\alpha_j|^2 \end{aligned}$$

$$\geq \sum_{j=m+1}^{n} |\alpha_j|^2$$

This implies that

$$P_Y(x) = \sum_{i=1}^{m} \alpha_i \cdot e_i = \underset{y \in Y}{\text{argmin}} \|x - y\|^2.$$

Thus, P_Y is a linear operator. $\square$

Null space. The kernel (null space) of T is denoted by

$$\mathcal{N}(T) = T^{-1}\{0\} = \left\{x \in \mathcal{D}(T) \;\middle|\; T(x) = 0\right\}.$$

Since T is linear, the set $\mathcal{N}(T)$ is a vector space.

Lemma 3.13 *Let $T : \mathcal{D}(T) \to \mathcal{R}(T)$ be linear. Then T is invertible if only if $\mathcal{N}(T) = 0$. Moreover, $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ is linear.*

Proof. 1. Since T is surjective, one only needs to show that T is injective, i.e.,

$$T(x) \neq T(y) \qquad \text{for all } x \neq y.$$

Equivalently,

$$T(z) \neq 0 \qquad \text{for all } z \neq 0 \qquad \Longleftrightarrow \mathcal{N}(T) = T^{-}(0) = 0.$$

2. T^{-1} is linear. Indeed,

$$T^{-1}(\alpha_1 \cdot y_1 + \alpha_2 \cdot y_2) = x$$

implies that

$$T(x) = \alpha_1 \cdot y_1 + \alpha_2 \cdot y_2.$$

Set $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$, we have

$$y_1 = T(x_1) \qquad \text{and} \qquad y_2 = T(x_2).$$

This implies that

$$T(\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2) = T(x)$$

and it yields

$$x = \alpha_1 \cdot x_1 + \alpha_2 \cdot x_2.$$

Equivalently,

$$T^{-1}(\alpha_1 \cdot y_1 + \alpha_2 \cdot y_2) = \alpha_1 \cdot T^{-1}(y_1) + \alpha_2 \cdot T^{-1}(y_2)$$

and this complete the proof. $\square$

Definition 3.14 (Bounded linear operators) *Let $T : D(T) = X \to Y$ be linear operator. We say that T is bounded if and only if*

$$\|T\|_\infty := \sup_{\|x\|\le 1} \|T(x)\| < +\infty.$$

Some examples.

Example 1. (Matrices as linear operators) Given $n \times m$ matrix A, the map $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$T(x) = [A \cdot x^T]^T$$

is a linear bounded operator.

Example 2. For any $p \ge 1$, recalling that

$$\ell^p := \left\{ x = (x_n)_{n\ge 1} : \sum_{n=1}^{\infty} |x_n|^p < +\infty \right\}.$$

Given an arbitrary sequence $\lambda = (\lambda_n)_{n\ge 1}$, define $T : X \to X$ such that

$$T(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots).$$

Two cases may occur

(i). If the sequence $(\lambda_n)_{n\ge 1}$ is bounded then T is bounded and

$$\|T\|_\infty = \sup_{\|x\|\le 1} \|T(x)\| = \sup_{k\ge 1} |\lambda_k|.$$

(ii) If the sequence $(\lambda_n)_{n\ge 1}$ is unbounded then T is unbounded.

Example 3. Consider the normed space

$$X = C((0,1), \mathbb{R}) = \{f : (0,1) \to \mathbb{R} : f \text{ is continuous and bounded}\}$$

and

$$\|f\| = \sup_{x\in(0,1)} |f(t)|$$

The differential operator $\Lambda(f) = f'$ is linear but not bounded. Indeed, let

$$f_k(x) = \sin(k\pi x) \qquad x \in (0,1).$$

We have

$$\Lambda(f)(x) = f_k'(x) = k\pi \cdot \cos(k\pi x) \qquad x \in (0,1).$$

A direct computation yields

$$\|f_k\| \;=\; 1 \qquad \text{and} \qquad \|\Lambda(f_k)\| \;=\; k\pi\,.$$

Thus,

$$\sup_{\|f\|\le 1} \|\Lambda(f)\| \;=\; +\infty.$$

□

Lemma 3.15 *Let $T : X \to Y$ be a linear bounded operator. Then*

$$\|T(x)\|_Y \;\le\; \|T\|_\infty \cdot \|x\|_X \qquad \textit{for all } x \in X\,,$$

and

$$\|T\|_\infty \;=\; \sup_{\|x\|_X=1} \|T(x)\|_Y \;=\; \sup_{x\in X\setminus\{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

Theorem 3.16 (Linear operators in finite dimensional spaces) *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be normed spaces with $\dim(X) = n$. Then a linear operator $T : X \to Y$ is bounded.*

Proof. Assume that

$$X \;=\; \text{span}\,\{e_1, e_2, \dots, e_n\}$$

where $\{e_1, e_2, ..., e_n\}$ is linearly independent,

For any $x \in X$, it holds

$$x \;=\; \sum_{i=1}^{n} \alpha_i \cdot e_i \qquad \alpha_i \in \mathbb{R}.$$

Since T is linear, we estimate

$$\|T(x)\|_Y \;=\; \left\| \sum_{i=1}^{n} \alpha_i^x \cdot T(e_i) \right\|_Y \;\le\; M \cdot \sum_{i=1}^{n} |\alpha_i^x|$$

where $M = \max_{i\in\overline{1,n}} \|T(e_i)\|$.

On the other hand, by the linear combination lemma, there exists a constant λ such such that

$$\left\| \sum_{i=1}^{n} \alpha_i \cdot e_i \right\|_X \;\ge\; \lambda \cdot \sum_{i=1}^{n} |\alpha_i| \qquad \text{for all } \alpha_i \in \mathbb{R}.$$

This implies that

$$\|T(x)\|_Y \;\le\; \frac{M}{\lambda} \cdot \|x\|_X \qquad \text{for all } x \in X.$$

Thus, T is bounded. □

Theorem 3.17 (Continuity and boundedness) *Let $T : X \to Y$ be a linear operator. Then T is continuous at 0 if and only if T is bounded.*

Proof. 1. Assume that T is bounded. We have

$$\|T(x)\| \;\leq\; \|T\|_\infty \cdot \|x\| \qquad \text{for all } x \in X.$$

The linearity of T implies that

$$\|T(x) - T(y)\| \;=\; \|T(x-y)\| \;\leq\; \|T\|_\infty \cdot \|x - y\| \qquad \text{for all } x, y \in X.$$

Thus, T is continuous.

2. Assume that T is continuous at 0. There exists $\delta > 0$ such that

$$\|T(x)\| \;\leq\; 1 \qquad \text{for all } x \in \overline{B}(0, \delta).$$

Since $\overline{B}(0,\delta) = \delta \cdot \overline{B}(0,1)$, we obtain that

$$\|T(x)\| \;\leq\; \frac{1}{\delta} \qquad \text{for all } x \in \overline{B}(0,1).$$

Thus,

$$\|T\|_\infty \;=\; \sup_{x \in \overline{B}(0,1)} \|T(x)\| \;\leq\; \frac{1}{\delta}$$

and T is bounded. □

Corollary 3.18 *Let $T : X \to Y$ be a linear operator. Then the followings hold*

- *If T is continuous at x_0 then T is continuous on X.*
- *If T is a bounded operator then the null set $\mathcal{N}(T) = T^{-1}(\{0\})$ is a closed vector space in X.*

Given two normed space $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, denote by

$$B(X,Y) \;=\; \{T : X \to Y \mid T \text{is a bounded linear operator}\}$$

and

$$\|T\|_\infty \;=\; \sup_{\|x\| \leq 1} \|T(x)\|.$$

It is easy to show that $\big(B(X,Y), \|\cdot\|_\infty\big)$ is normed vector space.

Theorem 3.19 (Completeness) *Assume that $(Y, \|\cdot\|_Y)$ is a Banach space. Then $\big(B(X,Y), \|\cdot\|_\infty\big)$ is a Banach space.*

Proof. Let $\{T_n\}_{n\geq 1}$ be a Cauchy sequence in $\big(B(X,Y), \|\cdot\|_\infty\big)$. We need to find $T \in B(X,Y)$ such that

$$\lim_{n\to\infty} \|T_n - T\|_\infty \;=\; 0.$$

1. For any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\|T_n - T_m\|_\infty \;=\; \sup_{\|x\|=1} \|T_n(x) - T_m(x)\| \;\leq\; \varepsilon \qquad \text{for all } n, m > N_\varepsilon.$$

In particular, for every $x \in S(0,1)$, the sequence $\{T_n(x)\}_{n\geq 1}$ is a Cauchy sequence in $(Y, \|\cdot\|_Y)$. Since $(Y, \|\cdot\|_Y)$ is complete, it holds

$$\lim_{n\to\infty} T_n(x) \;=\; y := T(x).$$

The map T is define on $S(0,1)$. We extend the map T to X by

$$T(0) \;=\; 0 \qquad \text{and} \qquad T(x) \;=\; \|x\| \cdot T\left(\frac{x}{\|x\|}\right) \qquad \text{for all } x \in X\backslash\{0\}.$$

It is clear that for all $x \in X$

$$\lim_{n\to\infty} T_n(x) \;=\; \|x\| \cdot \lim_{n\to\infty} T_n\left(\frac{x}{\|x\|}\right) \;=\; \|x\| \cdot T\left(\frac{x}{\|x\|}\right) \;=\; T(x). \tag{3.3}$$

2. We claim that $T \in B(X,Y)$.

- T is linear. Indeed, for any $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, it holds

$$\begin{aligned} T(\alpha_1 x_1 + \alpha_2 x_2) &= \lim_{n\to\infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) \;=\; \lim_{n\to\infty} [\alpha_1 \cdot T_n(x_1) + \alpha_2 \cdot T(x_2)] \\ &= \alpha_1 \cdot \lim_{n\to\infty} T(x_1) + \alpha_2 \cdot \lim_{n\to\infty} T(x_2) \;=\; \alpha_1 \cdot T(x_1) + \alpha_2 \cdot T(x_2). \end{aligned}$$

- T is bounded. Indeed, since $\{T_n\}_{n\geq 1}$ is a Cauchy sequence in $\big(B(X,Y), \|\cdot\|_\infty\big)$, it is bounded, i.e. there exists $M > 0$ such that

$$\|T_n\|_\infty \;\leq\; M \qquad \text{for all } n \geq 1$$

  Thus, for any $x \in X$, it holds

$$\|T(x)\| \;=\; \lim_{n\to\infty} \|T_n(x)\| \;\leq\; M \cdot \|x\|$$

  and it yields $\|T\|_\infty \leq M$.

3. To complete the proof, we show that T_n converges to T in $\big(B(X,Y), \|\cdot\|_\infty\big)$, i.e.

$$\lim_{n\to\infty} \|T_n - T\|_\infty \;=\; 0\,.$$

For every $\varepsilon > 0$, find $N_\varepsilon > 0$ such that

$$\sup_{\|x\|=1} \big\|T_n(x) - T(x)\big\| \;\leq\; \varepsilon \qquad \text{for all } n \geq N_\varepsilon.$$

Since $\{T_n\}_{n\geq 1}$ is Cauchy in $\big(B(X,Y), \|\cdot\|_\infty\big)$, there exists $N_\varepsilon > 0$ such that

$$\|T_n - T_m\|_\infty \;<\; \varepsilon \qquad \text{for all } n, m \geq N_\varepsilon.$$

Thus, for every $x \in S(0,1)$, it holds

$$\begin{aligned} \|T_n(x) - T(x)\| &\leq \|T_n(x) - T_m(x)\| + \|T_m(x) - T(x)\| \\ &\leq \|T_n - T_m\|_\infty + \|T_m(x) - T(x)\| \\ &\leq \varepsilon + \|T_m(x) - T(x)\| \qquad \text{for all } n, m \geq N_\varepsilon. \end{aligned}$$

Taking m to $+\infty$, we get

$$\|T_n(x) - T(x)\| \;\leq\; \varepsilon \qquad \text{for all } x \in S(0,1), n \geq N_\varepsilon$$

and it yields

$$\|T_n - T\|_\infty \;\leq\; \varepsilon \qquad \text{for all } n \geq N_\varepsilon.$$

□

Corollary 3.20 *Given $(X, \|\cdot\|)$ a normed space, the normed space $(B(X,\mathbb{R}), \|\cdot\|_\infty)$ is Banach.*

3.5 Fundamental theorems

1. The uniform boundedness principle. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces with $(X, \|\cdot\|_X)$ complete. Denote by

$$B(X,Y) \;=\; \big\{T : X \to Y \;\big|\; T \text{ is a linear bounded operator}\big\}.$$

the set of all linear bounded operators.

Theorem 3.21 (Banach-Steinhaus) *Let $\mathcal{F} \subset B(X,Y)$ be any subset of bounded linear operators. Assume that for any $x \in X$, there exists $M_x > 0$ such that*

$$\sup_{T\in\mathcal{F}} \|T(x)\|_Y \;\leq\; M_x.$$

Then $\mathcal{F}$ is uniformly bounded in $B(X,Y)$, i.e., there exists a constant $M > 0$ such that

$$\sup_{T\in\mathcal{F}} \|T\|_\infty \;\leq\; M.$$

Proof. The proof is divided into several steps:

1. Observe that

$$\sup_{x\in B(x_0,r)} \|T(x)\|_Y \;\leq\; K \qquad \Longrightarrow \qquad \|T\|_\infty \;\leq\; \frac{\|T(x_0)\|_Y + K}{r}.$$

Hence, if there exists $n_0, r_0 > 0$ and $x_0 \in X$ such that

$$\|T(x)\|_Y \;\leq\; n_0 \qquad \text{for all } x \in B(x_0, r_0), T \in \mathcal{F}, \tag{3.4}$$

then

$$\sup_{T\in\mathcal{F}} \|T\|_\infty \;\leq\; M \;:=\; \sup_{T\in\mathcal{F}} \frac{\|T(x_0)\|_Y + K_0}{r_0} \;\leq\; \frac{M_{x_0} + n_0}{r_0}.$$

For any $n \in \mathbb{N}$, consider the set

$$S_n \;:=\; \left\{x \in X \;\middle|\; \|T(x)\|_Y > n \quad \text{for some } T \in \mathcal{F}\right\}.$$

If there exists $n_0 \in \mathbb{N}$ such that S_{n_0} is not dense in X then there exists $x_0 \in X$ and $r_0 > 0$ such that

$$B(x_0, r_0) \bigcap S_{n_0} \;=\; \varnothing$$

and this yields (3.4).

2. Assume that S_n is dense for every $n \in \mathbb{N}$. Observe that the set

$$\begin{aligned} S_n \;&=\; \left\{x \in T^{-1}(Y\backslash \bar{B}(0,n)) \;\text{ for some } T \in \mathcal{F}\right\} \\ &=\; \bigcup_{T\in\mathcal{F}} T^{-1}(Y\backslash \bar{B}(0,n)) \end{aligned}$$

is open. Using Baire Category theorem, the set

$$S \;=\; \bigcap_{n=1}^{\infty} S_n \text{ is dense in } X.$$

In particular, S is non-empty. Let $\bar{x} \in S = \bigcap_{n=1}^{\infty} S_n$. We have that for every $n \geq 1$, there exists $T_n \in \mathcal{F}$ such that $\|T_n(x)\|_Y > n$. This implies that

$$\sup_{T\in\mathcal{F}} \|T(\bar{x})\|_Y \;=\; \sup_{n\geq 1} \|T_n(\bar{x})\|_Y \;=\; +\infty$$

and it yields a contradiction. □

Corollary 3.22 (Continuity of the point-wise limit) *Let $\Lambda_n \subset B(X,Y)$ be such that*

$$\lim_{n\to\infty} \Lambda_n(x) \;=\; \Lambda(x) \qquad \textit{for all } x \in X.$$

Then Λ is a bounded linear operator and

$$\|\Lambda\|_\infty \;\leq\; \sup_{n\in\mathbb{N}} \|\Lambda_n\|_\infty \;<\; \infty.$$

2. The open mapping theorem. Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two normed spaces.

Definition 3.23 *We say that the function $f : X \to Y$ is open if and only if*

$$f(U) \textit{ is open in } Y \textit{ for every open } U \textit{ in } X.$$

It is easy to show that f is open if and only if for every $x \in X$ and $r > 0$, it holds

$$f(B_X(x,r)) \;\supseteq\; B_Y(f(x),\delta)$$

for some $\delta > 0$.

Theorem 3.24 (Open mapping theorem) *Assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|)$ are Banach spaces. Let $\Lambda : X \to Y$ be linear, bounded, and surjective ($\Lambda(X) = Y$). Then the map Λ is open*

Proof. By the linear property of Λ,

$$\Lambda(B_X(x,r)) \;=\; \Lambda(x) + r\cdot\Lambda(B_X(0,1)),$$

we need to show that there exists $\delta > 0$ such that

$$B_Y(0,\delta) \;\subseteq\; \Lambda(B_X(0,1)).$$

1. Let's first show that

$$B_Y(0,\delta_1) \;\subseteq\; \overline{\Lambda(B_X(0,1))}$$

for some $\delta_1 > 0$. Observe that

$$B_Y(0,\delta_1) \;=\; \frac{1}{2}\cdot B_Y(y_0,\delta_1) + \frac{1}{2}\cdot B_Y(-y_0,\delta_0)$$

for every $y_0 \in Y$. Thus, if we can show that

$$B_Y(y_0,\delta_0) \;\subseteq\; \overline{\Lambda(B_X(0,1))}$$

then by the symmetry of $\overline{\Lambda(B_X(0,1))}$, we have

$$B_Y(-y_0,\delta_0) \;\subseteq\; \overline{\Lambda(B_X(0,1))}$$

and the convexity of $\overline{\Lambda(B_X(0,1))}$ yields

$$B_Y(0,\delta_1) \;=\; \frac{1}{2}\cdot B_Y(y_0,\delta_1)+\frac{1}{2}\cdot B_Y(-y_0,\delta_0) \;\subseteq\; \overline{\Lambda(B_X(0,1))}.$$

Therefore, one only needs to show that $\overline{\Lambda(B_X(0,1))}$ has a nonempty interior.

2. Assume by a contradiction that $\overline{\Lambda(B_X(0,1))}$ has empty interior. It is clear that

$$\overline{\Lambda(B_X(0,n))} \;=\; n\cdot\overline{\Lambda(B_X(0,1))}$$

has also empty interior for every $n\geq 1$. Thus, the set

$$F_n \;=\; Y\backslash\overline{\Lambda(B_X(0,n))}$$

is open and dense in Y. By Barie category theory, the set

$$F \;=\; \bigcap_{n=1}^{\infty} F_n$$

is dense in X. In particular, F is nonempty. On the other hand, since $\Lambda(X)=Y$, we have

$$F \;=\; \bigcap_{n=1}^{\infty} F_n \;=\; Y\backslash\left[\bigcup_{n=1}^{\infty}\Lambda(B(0,n))\right] \;=\; Y\backslash\Lambda(X)=\varnothing$$

and it yields a contradiction.

3. We already showed that $\overline{\Lambda(B_X(0,1))}$ contains an open ball $B_Y(0,\delta_1)$ for some $\delta_1>0$. To conclude the proof, let's show that

$$B_Y(0,\delta_1/2) \;\subset\; \Lambda(\overline{B_X(0,1)}).$$

Given any $\bar{y}\in B(0,\delta_1/2)$, we want to find a sequence $s_n\in B(0,1)$ such that

$$\lim_{n\to\infty}\Lambda(s_n) \;=\; \bar{y}.$$

How to construct $\{s_n\}_{n\geq 1}$ The idea is to find

$$s_n \;=\; \sum_{i=1}^{n} x_i$$

such that

- $\|x_i\|\leq 2^{-i}$ for all $i\geq 1$,
- $\|\bar{y}-\Lambda(s_n)\|\leq 2^{-(n+1)}\delta_1$ for all $n\geq 1$.

In this case, one can see that $\{s_n\}_{n\geq 1}$ is a Cauchy sequence and thus converges to $\bar{s} \in \overline{B}(0,1)$. Thus,

$$\bar{y} \ = \ \lim_{n\to\infty} \Lambda(s_n) \ = \ \Lambda(\bar{s}) \ \in \ \Lambda(\overline{B}_X(0,1)).$$

4. To complete the proof, let us construct $\{x_n\}_{n\geq 1}$. By the linear property of Λ, we have that

$$B_Y(0, 2^{-n}\delta_1) \ \subseteq \ \Lambda(\overline{B}_X(0, 2^{-n})) \qquad \text{for all } n \geq 1.$$

For $n = 1$, we have

$$\bar{y} \ \in \ B_Y(0, \delta_1/2) \ \subseteq \ \Lambda(\overline{B}_X(0, 2^{-1}).$$

Pick $x_1 \in \overline{B}(0, 2^{-1})$ such that

$$\bar{y}_1 := \bar{y} - \Lambda(x_1) \ \in \ B_Y(0, 2^{-2}\delta_1).$$

For $n = 2$, we have

$$\bar{y}_1 \ \in \ B_Y(0, 2^{-2}\delta_1) \ \subseteq \ \Lambda(\overline{B}_X(0, 2^{-2}\delta_1)).$$

Pick $x_2 \in \overline{B}_X(0, 2^{-2})$ such that

$$\bar{y}_2 := \bar{y}_1 - \Lambda(x_2) \ \in \ B_Y(0, 2^{-3}\delta_1)$$

and this yields

$$\|\bar{y} - \Lambda(x_1) - \Lambda(x_2)\| \ = \ \|y_3\| \ \leq \ 2^{-3} \cdot \delta_1.$$

Assume that we can construct $\{x_1, x_2, \ldots, x_n\}$ such that

$$\bar{y}_{n+1} \ = \ \bar{y} - \Lambda\Big(\sum_{i=1}^{n} x_n\Big) \ \in \ B_Y(0, 2^{-(n+1)}\delta_1) \qquad \text{and} \qquad \|x_i\| \leq 2^{-i} \quad \text{for all } i \in \overline{1,n}.$$

We have

$$\bar{y}_{n+1} \ = \ \in \ B_Y(0, 2^{-(n+1)}\delta_1) \ \subseteq \ \Lambda(\overline{B}_X(0, 2^{-(n+1)})).$$

and there exists $x_{n+1} \in \overline{B_X(0, 2^{-(n+1)})}$ such that

$$\bar{y}_{n+1} \ = \ \bar{y} - \sum_{i=1}^{n+1} \Lambda(x_i) \ \in \ B_Y(0, 2^{-(n+2)}\delta_1).$$

The proof is complete. □

Corollary 3.25 *If X, Y are Banach and $\Lambda : X \to Y$ is continuous, linear, and bijective. Then $\Lambda^{-1} : Y \to X$ is a linear bounded operator.*

2. The closed graph theorem. Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ Banach spaces, the product space

$$X \times Y \ = \ \{(x,y) \bigm| x \in X, y \in Y\}$$

with

$$\|(x,y)\| \ = \ \|x\| + \|y\|$$

is also a Banach space.

Definition 3.26 *Let $\Lambda : \mathrm{Dom}(\Lambda) \subseteq X \to Y$ be a linear operator. We say that Λ is closed is the graph of Λ*

$$\mathrm{Graph}(\Lambda) \ = \ \{(x, T(x)) \mid x \in \mathrm{Dom}(T)\}$$

is closed in $X \times Y$.

From the definition, one can see that Λ is closed if any only if for every $\{x_n\}_{n\geq 1} \subset \mathrm{Dom}(\Lambda)$, if

$$\lim_{n\to\infty} x_n \ = \ x \qquad \text{and} \qquad \lim_{n\to\infty} T(x_n) \ = \ y$$

then

$$x \in \mathrm{Dom}(\Lambda) \qquad \text{and} \qquad T(x) \ = \ y.$$

It is clear that if Λ is continuous then Λ is closed.

Theorem 3.27 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. If $T : X \to Y$ is a closed linear operator with $\mathrm{Dom}(T) = X$ then T is bounded.*

Proof. Since T is a closed linear operator, the graph of T

$$\Gamma := \mathrm{Graph}(T) \ = \ \{(x, T(x)) : x \in X\}$$

is a closed vector subspace in $X \times Y$. Since $X \times Y$ is a Banach space, $(\Gamma, \|\cdot\|)$ is also a Banach space. Consider the following projections

$$\pi_X : \Gamma \to X \qquad \text{and} \qquad \pi_Y : \Gamma \to Y$$

defined by

$$\pi_X(x, T(x)) \ = \ x \qquad \text{and} \qquad \pi_Y(x, T(x)) \ = \ T(x) \qquad \text{for all } x \in X.$$

One can see that both π_X and π_Y are linear bounded operators. Moreover,

$$T(x) \ = \ \pi_Y \circ \pi_X^{-1}(x) \qquad \text{for all } x \in X.$$

Thus, T is continuous if the linear map

$$\pi_X^{-1} : \Gamma \to X$$

is continuous. Since $(\Gamma, \|\cdot\|)$ and $(X, \|\cdot\|_X)$ are Banach spaces and

$$\pi_X : X \to \Gamma$$

is linear, bounded, and surjective, the corollary 3.25 implies that $\pi^{-1} : \Gamma \to X$ is continuous. □

3.6 Hilbert spaces

In the Euclidean space $\mathbb{R}^n$, the inner product

$$\langle x, y\rangle \;=\; \sum_{i=1}^{n} x_i y_i$$

is useful in many ways:

(i). Define the Euclidian norm;

(ii) Determine the perpendicular spaces and projections. Moreover, given a set $\{v_1, v_2, \ldots, v_n\}$ of mutually orthogonal vectors with $\|v_i\| = 1$, for every $x \in \mathbb{R}^n$, it holds

$$x \;=\; \sum_{i=1}^{n} \alpha_i v_i \qquad \text{with} \qquad \alpha_i \;=\; \langle x, v_i\rangle \qquad \text{for all } i \in \overline{1, n}.$$

(iii) (Riesz representation formula) For any linear operator $\varphi : \mathbb{R}^n \to \mathbb{R}$, there exists a unique $v \in \mathbb{R}^n$ such that

$$\varphi(x) \;=\; \langle v, x\rangle \qquad \text{for all } x \in \mathbb{R}^n;$$

(iv) Symmetric operators $A : \mathbb{R}^n \to \mathbb{R}^n$, i.e.,

$$\langle Ax, y\rangle \;=\; \langle x, Ay\rangle \qquad \text{for all } x, y \in \mathbb{R}^n;$$

(v) Positive operator $A : \mathbb{R}^n \to \mathbb{R}^n$, i.e.,

$$\langle Ax, x\rangle \;\geq\; 0 \qquad \text{for all } x \in \mathbb{R}^n.$$

Goal: We would like to extend these properties to infinite-dimensional normed vector spaces.

3.6.1 Spaces with inner product.

Definition 3.28 *Let H be a vector space over $\mathbb{R}$. An inner product on H is a symmetric bilinear map $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ such that the following holds*

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if any only if $x = 0$;

(ii) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in H$;

(iii) For all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{R}$, it holds

$$\langle \alpha x + \beta y, z\rangle \;=\; \alpha \cdot \langle x, z\rangle + \beta \cdot \langle y, z\rangle.$$

Examples:

(a). *The ℓ^2 space*

$$\ell^2 \ = \ \left\{ x = \{x_n\}_{n\geq 1} \mid \sum_{n=1}^{\infty} x_n^2 < +\infty \right\}.$$

with

$$\langle x, y\rangle \ = \ \sum_{n=1}^{\infty} x_n y_n \qquad \text{for all } x, y \in \ell^2.$$

(b). *Given $a < b$, recalling that*

$$C([a,b]) \ = \ \{f : [a,b] \to \mathbb{R} \mid f \textit{ is continuous}\},$$

we define an inner product on $C([a,b])$

$$\langle f, g\rangle \ = \ \int_a^b f(x)g(x)dx \qquad \text{for all } f, g \in C([a,b]).$$

Now let us denote by

$$\|x\| \ = \ \sqrt{\langle x, x\rangle} \qquad \text{for all } x \in H.$$

It is clear that

$$\|tx\| \ = \ |t| \cdot \|x\|, \qquad \|x\| \ \geq \ 0 \qquad \text{and} \qquad \|x\| \ = \ 0 \ \textit{ if and only if } \ x \ = \ 0.$$

Moreover, we also have

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 \ &= \ \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \ 2\langle x, x\rangle + 2\langle y, y\rangle \ = \ 2 \cdot \left(\|x\|^2 + \|y\|^2\right). \end{aligned}$$

To verify that $(H, \|\cdot\|)$ is a normed space, we need to prove the basic triangle inequality

$$\|x+y\| \ \leq \ \|x\| + \|y\| \qquad \text{for all } x, y \in H.$$

Theorem 3.29 (Basic inequalities) *Given the vector space H and its inner product $\langle \cdot, \cdot\rangle$. The followings hold*

(i) Cauchy-Schwarz's inequality

$$|\langle x, y\rangle| \ \leq \ \|x\| \cdot \|y\|.$$

(ii) Minkowski's inequality

$$\|x+y\| \ \leq \ \|x\| + \|y\|.$$

Proof. Given $x, y \in X$, it holds

$$\langle x + \lambda y, x + \lambda y \rangle \ \geq \ 0 \qquad \text{for all } \lambda \in \mathbb{R}.$$

Equivalently,

$$\|y\|^2 \cdot \lambda^2 + 2 \cdot \langle x, y \rangle \cdot \lambda + \|x\|^2 \qquad \text{for all } \lambda \in \mathbb{R}$$

and this implies that

$$\langle x, y \rangle^2 - \|x\|^2 \cdot \|y\|^2 \ \leq \ 0.$$

The proof of (i) is complete.

(ii). Using the Cauchy-Schwarz's inequality, we estimate

$$\begin{aligned} \|x + y\|^2 \ &= \ \langle x + y, x + y \rangle \ = \ \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\ &\leq \ \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \ = \ [\|x\| + \|y\|]^2 \end{aligned}$$

and it yields (ii). □

Definition 3.30 *The vector space H with inner product $\langle \cdot, \cdot \rangle$ is called a Hilbert space if it is complete.*

We are going to study deeper into the structure of Hilbert space. Given $S \subseteq H$, the following set

$$\text{span}(S) \ = \ \left\{ \sum_{i=1}^{n} c_i x_i \mid c_i \in \mathbb{R}, x_i \in S, n \geq 1 \right\}$$

is a subspace of H. *We say that S is total if*

$$\overline{\text{span}(S)} \ = \ H.$$

For any $x, y \in H$, we say that x and y are orthogonal, denote by $x \perp y$, if

$$\langle x, y \rangle \ = \ 0.$$

The following set

$$S^{\perp} \ := \ \{ y \in H \mid y \perp x \qquad \text{for all } x \in S \}$$

is called the orthogonal space of S.

Theorem 3.31 (Perpendicular projection) *Let H be a Hilbert space and V be a closed vector subspace of H. The following holds*

(i) $V^{\perp}$ is a complement of V in H, i.e.,

$$H \ = \ V \oplus V^{\perp}.$$

(ii) If $x = y + z$ for $y \in V$ and $z \in V^{\perp}$ then y is a unique point in V such that

$$\|x - y\| \;=\; d_V(x) \;=\; \min\{\|x - w\| \mid w \in V\},$$

and z is a unique point in $V^{\perp}$ such that

$$\|x - z\| \;=\; d_{V^{\perp}}(x) \;=\; \min\{\|x - w\| \mid w \in V\}.$$

Define

$$P_V(x) \;=\; y \qquad \text{and} \qquad P_{V^{\perp}} \;:=\; z.$$

(iii) The maps $P_V(\cdot) : H \to V$ and $P_{V^{\perp}} : X \to V^{\perp}$ are linear bounded operators with norms ≤ 1.

Answer. The proof is divided into several steps:

1. Let $x \in X$. Then there exists a unique $y \in V$ such that

$$\|x - y\| \;=\; d_V(x) \;=\; \inf_{v \in V} \|x - y\|.$$

Existence. By the definition, there exists a sequence $\{y_n\}_{n \geq 1}$ in V such that

$$\lim_{n \to \infty} \|x - y_n\| \;=\; d_V(x).$$

We show that $\{y_n\}_{n \geq 1}$ is a Cauchy sequence. Indeed, by the convexity of V, we estimate

$$\begin{aligned} \|y_m - y_n\|^2 \;&=\; 2\|x - y_n\|^2 + 2\|x - y_n\|^2 - 4\left\|x - \frac{y_n + y_m}{2}\right\|^2 \\ &\geq\; 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d_V^2(x) \end{aligned}$$

Since the right hand side tends to 0 as $n, m \to \infty$, one has that $\{y_n\}_{n \geq 1}$ is a Cauchy sequence. By the closeness of V, the sequence $\{y_n\}_{n \geq 1}$ converges to $y \in V$.

Uniqueness. Assume that there are $\bar{y}_1, \bar{y}_2 \in V$ such that

$$\|x - \bar{y}_1\| \;=\; \|x - \bar{y}_2\| \;=\; d_V(x).$$

This implies that

$$\begin{aligned} \|\bar{y}_1 - \bar{y}_2\|^2 \;&=\; 2\|x - \bar{y}_1\|^2 + 2\|x - \bar{y}_1\|^2 - 4\left\|x - \frac{\bar{y}_1 + \bar{y}_2}{2}\right\|^2 \\ &\leq\; 2d_V(x) + 2d_V(s) - 4d_V(x) \;=\; 0 \end{aligned}$$

and it yields $\bar{y}_1 = \bar{y}_2$.

Lemma 3.32 *For K closed and convex set of H and $x \in X$, there exists a unique y such that*
$$\|x - y\| \;=\; d_V(x).$$

2. Let $P_V : H \to V$ be such that
$$\|x - P_V(x)\| \;=\; d_V(x) \qquad \text{for all } x \in H.$$

For any $x \in H$, we set
$$P_{V^\perp}(x) \;:=\; x - P_V(x).$$

We show that
$$P_{V^\perp}(x) \;\in\; V^\perp.$$

Indeed, for any $v \in V$, let's consider the smooth function $f_v : \mathbb{R} \to \mathbb{R}$ such that
$$f_v(t) \;=\; \|x - (P_V(x) + tv)\|^2$$

It is clear that $f_v(0) = d_V(x)$ and $f_v(t) \geq d_V^2(x)$. This implies that
$$\frac{d}{dt} f_v(t) \;=\; 0$$

and it yields
$$\langle x - P_V(x), v\rangle \;=\; 0.$$

Therefore, the map $P_V^\perp : H \to V^\perp$ satisfies
$$x \;=\; P_V(x) + P_{V^\perp}(x) \qquad \text{for all } x \in H.$$

One can easily show that $V \cap V' = 0$ and it yields
$$H \;=\; V \oplus V^\perp.$$

To complete (i) and (ii), we show that
$$\|x - P_{V^\perp(x)}\| \;=\; d_{V^\perp}(x) \qquad \text{for all } x \in H. \tag{3.5}$$

By the definition of $P_{V^\perp}$, we have that
$$\langle x - P_{V^\perp}(x), w\rangle \;=\; \langle P_V(x), w\rangle \;=\; 0 \qquad \text{for all } w \in V^\perp.$$

This implies that
$$\|x - P_V{}^\perp(x)\|^2 \;=\; \|x - z\|^2 - \|z - P_{V^\perp}(x)\|^2 \;\leq\; \|x - z\|^2 \qquad \text{for all } z \in V^\perp$$

and it yields (3.5).

4. We show that the map $P_V : H \to V$ is linear. Indeed, if $y_i = P_V(x_i)$ for $i = 1, 2$ then for any given $\alpha_i \in \mathbb{R}$, it holds

$$\alpha_1 y_1 + \alpha_2 y_2 \in V.$$

Since $\alpha_i(x_i - y_i) \in V^\perp$ for all $i = 1, 2$, it holds that

$$(\alpha_1 x_1 + \alpha_2 x_2) - (\alpha_1 y_1 + \alpha_2 y_2) \in V^\perp$$

and it yields

$$P_V(\alpha x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 P_V(x_1) + \alpha_2 P_V(x_2).$$

Thus, $P_V : H \to V$ and $P_{V^\perp} : H \to V^\perp$ are linear. Finally,

$$\|P_V(x)\|^2 + \|P_{V^\perp}(x)\|^2 = \|x\|^2$$

one has that

$$\|P_V(x)\|, \|P_{V^\perp}(x)\| \le \|x\|.$$

Therefore, P_V and $P_{V^\perp}$ are bounded with norm ≤ 1. □

Corollary 3.33 *If V is a closed subspace of the Hilbert space H then*

$$\left(V^\perp\right)^\perp = V.$$

Proof. For a given $x \in V$, by the definition of $V^\perp$ it holds

$$x \perp v \qquad \text{for all } v \in V^\perp.$$

This implies that $x \in \left(V^\perp\right)^\perp$. Thus,

$$V \subseteq \left(V^\perp\right)^\perp.$$

To complete the proof, we show that

$$x \in V \qquad \text{for all } x \in \left(V^\perp\right)^\perp.$$

For a given $x \in \left(V^\perp\right)^\perp$, it holds

$$x \perp v \qquad \text{for all } v \in V^\perp.$$

By the projection theorem, we have

$$P_{V^\perp}(x) = x - P_V(x) \in V^\perp.$$

Thus,

$$\left\|P_{V^\perp}(x)\right\|^2 = \langle x, P_{V^\perp}(x)\rangle - \langle P_V(x), P_{V^\perp}(x)\rangle = 0$$

and this implies that

$$P_{V^\perp}(x) = 0 \qquad \Longrightarrow \qquad x \in V.$$

□

Definition 3.34 (Orthonormal sets) *We say that $E \subset H$ is orthonormal if*

$$\langle v, w\rangle = 0 \qquad \text{and} \qquad \|v\| = \|w\| = 1 \qquad \textit{for all } v \neq w \in E.$$

A natural question is that for any given a set

$$S = \{v_1, v_2, \ldots, v_n, \ldots\} \subset H,$$

can one construct an orthonormal set

$$E = \{e_1, e_2, \ldots, e_n, \ldots\}$$

such that $\text{span}(S) = \text{span}(E)$?

Gram-Schmdit orthogonalization. The set E is constructed by induction. Indeed, we first set

$$e_1 = \frac{v_1}{\|v_1\|} \qquad \text{so that} \qquad \|e_1\| = 1, \qquad V_1 := \text{span}(e_1) = \text{span}(v).$$

Assume that $e_1, e_2, \ldots, e_n$ have been constructed such that

$$\|e_i\| = 1, \qquad \langle e_i, e_j\rangle \qquad \text{for all } i \neq j \in \{1, 2, \ldots, n\},$$

and

$$V_n := \text{span}\{e_1, e_2, \ldots, e_n\} = \text{span}\{v_1, v_2, \ldots, v_n\}.$$

The unit vector e_{n+1} is defined by

$$e_{n+1} = \frac{v_{n+1} - P_V(v_{n+1})}{\|v_{n+1} - P_V(v_{n+1})\|}.$$

It is clear that

$$\|e_{n+1}\| = 1 \qquad \text{and} \qquad e_{n+1} \perp V_n.$$

This implies that $\{e_1, e_2, \ldots, e_{n+1}\}$ is orthonormal and

$$\text{span}\{e_1, e_2, \ldots, e_{n+1}\} = \text{span}\{v_1, v_2, \ldots, v_{n+1}\}.$$

□

Remark. *The general formula of e_n is*

$$e_n = \frac{v_n - \sum_{k=1}^{n-1}\langle v_n, e_k\rangle \cdot e_k}{\left\| v_n - \sum_{k=1}^{n-1}\langle v_n, e_k\rangle \cdot e_k \right\|} \qquad \textit{for all } n \geq 1.$$

Let $E = \{e_1, e_2, \ldots, e_n\}$ be orthonormal. Then for any $x \in \text{span } E$, it holds

$$x = \sum_{i=1}^{n} \langle x, e_i \rangle \cdot e_i.$$

Can we extend this to the infinite sum?

Theorem 3.35 *Let $S = \{e_1, e_2, \ldots, e_n, \ldots\}$ be a orthonormal set in a Hilbert space H. Assume that S is complete, i.e.,*

$$H = \overline{\text{span}\{e_1, e_2, \ldots, e_n, \ldots\}}.$$

Then, for every $x \in H$, it holds

$$x = \sum_{n=1}^{\infty} \alpha_n \cdot e_n \qquad \text{with} \qquad \alpha_n = \langle x, e_n \rangle,$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} \alpha_n^2. \qquad [Parseval' s identity]$$

Proof. 1. For every $n \geq 1$, we define a closed vector subspace of H

$$V_n = \text{span } \{e_1, e_2, \ldots, e_n\}.$$

By the projection theorem, for any given $x \in H$, it holds

$$P_{V_n}(x) = \sum_{i=1}^{n} \alpha_i \cdot e_i \qquad \text{with} \qquad \alpha_i = \langle x, e_i \rangle \qquad \text{for all } i \in \{1, 2, \ldots, n\},$$

and

$$\|P_{V_n}(x)\| = \sum_{i=1}^{n} \alpha_i^2 \leq \|x\|^2.$$

In particular, the sequence $(P_{V_n}(x))$ is a Cauchy sequence and thus converges to $\bar{s} \in H$.

2. To complete the proof, we will show that $\bar{s} = x$. For every $k \geq 1$, one has

$$\langle x - \bar{s}, e_k \rangle = \lim_{n \to \infty} \left\langle x - P_{V_n}(x), e_k \right\rangle = 0$$

and this implies that

$$x - \bar{s} \perp S \qquad \Longrightarrow \qquad x - \bar{s} \perp \overline{\text{span}(S)} = H.$$

Thus, $x - \bar{s} = 0$ and this complete the proof. □

Fourier series. Given $\ell > 0$, denote by

$$\mathbf{L}^2(-\ell,\ell) \;=\; \left\{ f : (-\ell,\ell) \to \mathbb{R} \;\Big|\; \int_{-\ell}^{\ell} |f(x)|^2 \, dx \right\}.$$

The following holds:

Lemma 3.36 *The trigonometric set*

$$\mathcal{F} \;=\; \left\{ \frac{1}{2\sqrt{\ell}}, \frac{1}{\sqrt{\ell}} \cdot \sin\left(\frac{m\pi x}{\ell}\right), \frac{1}{\sqrt{\ell}} \cos\left(\frac{m\pi x}{\ell}\right) \;\Big|\; m = 1, 2, ... \right\}$$

is a orthonormal complete in $\mathbf{L}^2(-\ell,\ell)$.

From the above Theorem, it holds that for any function $f \in \mathbf{L}^2(-\ell,\ell)$, its Fourier series is

$$f \;\simeq\; \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cdot \cos\frac{m\pi x}{\ell} + b_m \cdot \sin\frac{m\pi x}{\ell} \right)$$

where a_m and b_m are Fourier coefficients and computed by

$$a_n \;=\; \frac{1}{\ell} \cdot \int_{-\ell}^{\ell} f(x) \cdot \cos\frac{m\pi x}{\ell} \, dx$$

and

$$b_n \;=\; \frac{1}{\ell} \cdot \int_{-\ell}^{\ell} f(x) \cdot \sin\frac{m\pi x}{\ell} \, dx$$

for all $m = 0, 1, 2, ...$

3.6.2 Linear functionals on Hilbert spaces

Given a Hilbert space H, the dual space of H is denoted by

$$H^* \;=\; \{ f : H \to \mathbb{R} \mid f \text{ is linear continuous operator} \}.$$

For any $x \in H$, let $f_x : H \to \mathbb{R}$ be such that

$$f_x(y) \;=\; \langle x, y \rangle \qquad \text{for all } y \in H.$$

It is easy to see that f_x is linear and bounded with

$$\|f_x\|_\infty \;=\; \|x\|.$$

In particular, this implies that f_x is in H^*.

Question: Is this true that $H^* = \{ f_x : x \in H \}$?

Theorem 3.37 (Riesz representation) *Let H be a Hilbert space and let $T : H \to \mathbb{R}$ be a linear bounded operator. Then there exists a unique $v_T \in H$ such that*

$$T(x) \;=\; \langle v_T, x\rangle \qquad \text{for all } x \in H.$$

Proof. Since T is linear and continuous, it holds that

$$V := \ker(T) \;=\; \{x : T(x) = 0\}$$

is a closed subspace of H. By the projection theorem, it holds

$$H \;=\; V \oplus V^{\perp}.$$

If $V = H$ then $T(x) = 0$ for every $x \in H$. In this case, one have that $v_T = 0$. Otherwise, there exists $x_0 \in H$ such that $T(x_0) \neq 0$. For every $x \in H$, we can write

$$x \;=\; \frac{T(x)}{T(x_0)} \cdot x_0 + x_V \qquad \text{with} \qquad x_V \;=\; \left(x - \frac{T(x)}{T(x_0)} \cdot x_0\right).$$

By the linear property of T, we have

$$T(x_V) \;=\; T\left(x - \frac{T(x)}{T(x_0)} \cdot x_0\right) \;=\; 0.$$

This implies that

$$x_V \;\in\; \mathrm{Ker}(T) \qquad \text{and} \qquad H = \mathrm{Ker}(T) + \mathrm{span}\{x_0\}.$$

Since $\mathrm{Ker}(T) \cap \mathrm{span}\{x_0\} = \{0\}$, we then have

$$H \;=\; \mathrm{Ker}(T) \oplus \mathrm{span}(x_0) \;=\; V \oplus \mathrm{span}(x_0).$$

Hence, $V^{\perp}$ is one dimensional vector space and

$$V^{\perp} \;=\; \mathrm{span}\{v_0\} \qquad \text{for some } \; 0 \neq v_0 \in H.$$

Let us now consider the vector

$$0 \;\neq\; v_T \;=\; \frac{T(v_0)}{\|v_0\|^2} \cdot v_0 \;\in\; V^{\perp}$$

such that

$$T(v_T) \;=\; \langle v_T, v_T\rangle \qquad \text{and} \qquad V^{\perp} \;=\; \mathrm{span}\{v_T\}.$$

We are going to show that

$$T(x) \;=\; \langle v_T, x\rangle \qquad \text{for all } x \in H.$$

For any $x \in H$, we can write

$$x \;=\; v_x + \alpha \cdot v_T$$

for $v_x \in V$ and $\alpha \in \mathbb{R}$. By the linear property of T, it holds

$$T(x) \;=\; T(v_x + \alpha \cdot v_T) \;=\; T(v_x) + \alpha \cdot T(v_T) \;=\; \alpha T(v_T) \;=\; \alpha\langle v_T, v_T\rangle \;=\; \langle x, v_T\rangle.$$

The uniqueness of v_T is almost straight forward. $\square$

Corollary 3.38 *For every Hilbert space H, it holds*

$$H^* \simeq H.$$

Proof. Consider the map $\Lambda : H \to H^*$ such that

$$\Lambda(x) = f_x \qquad \text{for all } x \in H.$$

One has that Λ is linear and bounded with $\|\Lambda(x)\| = \|x\|$. In particular, Λ is injective. On the other hand, from the above theorem, it holds

$$\Lambda(H) = H^*.$$

Thus, Λ is bijective and $\|\Lambda(x)\| = \|x\|$. □

From the above corollary, one has the following definition

Definition 3.39 (Weak convergence) *A sequence $\{x_n\}_{n\geq 1} \subset H$ weakly converges to $x \in H$, denote by $x_n \rightharpoonup x$, if*

$$\lim_{n\to\infty} \langle x_n, v\rangle = \langle x, v\rangle \qquad \text{for all } v \in H.$$

Example 1. Let H be a Hilbert space such that

$$H = \overline{\text{span}\{e_1, e_2, \dots, e_n, \dots\}}$$

where $\{e_1, e_2, \dots, e_n, \dots\}$ is an orthonormal set, i.e.,

$$\|e_i\| = 1 \qquad \text{and} \qquad \langle e_i, e_j\rangle = 0 \qquad \text{for all } i \neq j.$$

Then, the sequence $\{e_n\}_{n\geq 1}$ does not converge in H but converges weakly to 0.

Proof. It is clear that

$$\|e_n - e_m\| = 1 \qquad \text{for all } n \neq m.$$

Thus, $\{e_n\}_{n\geq 1}$ does not converge in H. However, for every $v \in H$, it holds that

$$v = \sum_{n=1}^{\infty} \alpha_i \cdot e_i.$$

Therefore

$$\lim_{n\to\infty} \langle e_n, v\rangle = \lim_{n\to\infty} \alpha_n = 0,$$

and $\{e_n\}_{n\geq 1}$ converges weakly to 0.

Basic properties: The followings hold (i). If $x_n \to x$ then $x_n \rightharpoonup x$.

(ii). If $x_n \rightharpoonup x$ then x_n converges to x.

$$\liminf_{n\to\infty} \ \|x_n\| \ \geq \ \|x\|.$$

(iii). If $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$ then

Proof. (ii). Assume that $x_n \rightharpoonup x$, we have that

$$\lim_{n\to\infty} \ \langle x_n, x\rangle \ = \ \langle x, x\rangle \ = \ \|x\|^2.$$

Since $|\langle x_n, x\rangle| \leq \|x_n\| \cdot \|x\|$, it holds

$$\liminf_{n\to\infty} \ \|x_n\| \cdot \|x\| \ \geq \ \|x\|^2$$

and it yields (ii).

(iii). Assume that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$. We have

$$\begin{aligned} \lim_{n\to\infty} \ \|x_n - x\|^2 \ &= \ \lim_{n\to\infty} \langle x_n - x, x_n - x\rangle \ = \ \lim_{n\to\infty} \left[\|x_n\|^2 + \|x\|^2 - 2\langle x_n, x\rangle\right] \\ &= \ \|x\|^2 + \|x\|^2 - 2\langle x, x\rangle \ = \ 0 \end{aligned}$$

and it yields (iii). □

Theorem 3.40 *Let H be a Hilbert space and let $\{x_n\}_{n\geq 1}$ be a sequence in the Hilbert space H. Then the followings hold:*

(i) If x_n converges weakly to x then $(x_n)_{n\geq 1}$ is bounded.

(ii) If $(x_n)_{n\geq 1}$ is bounded then there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges weakly to $x \in H$.

Proof. 1. Let us first prove (i). For every $n \geq 1$, we define a linear bounded operator $\varphi_n : H \to \mathbb{R}$ such that

$$\varphi_n(v) \ = \ \langle x_n, v\rangle \qquad \text{for all } v \in H.$$

Since $(x_n)_{n\geq 1}$ converges weakly to x then the sequence of linear bounded operators $(\varphi_n)_{n\geq 1}$ converges point-wise to the linear bounded operator $\varphi : H \to \mathbb{R}$ defined by

$$\varphi(v) \ = \ \langle x, v\rangle \qquad \text{for all } v \in H.$$

As a consequence of the uniform boundedness principle, we have that the sequence of linear bounded operators $(\varphi_n)_{n\geq 1}$ is bounded and thus

$$\sup_{n\geq 1} \|x_n\| \ = \ \sup_{n\geq 1} \|\varphi_n\|_\infty \ \leq \ M$$

for some constant $M > 0$.

2. Consider a closed and separable vector subspace $X = \overline{\text{span}\{x_1, x_2, \ldots, x_n, \ldots\}}$ of H. For every $n \geq 1$, let $\varphi_n : X \to \mathbb{R}$ be a bounded linear operator such that

$$\varphi_n(v) \;=\; \langle x_n, v\rangle \qquad \text{for all } v \in X.$$

Since $\{x_n\}_{n\geq 1}$ is bounded, one has that

$$\|\varphi_n\|_\infty \;=\; \sup_{\|v\|=1, v\in X} |\langle x_n, v\rangle| \;=\; \|x_n\| \;\leq\; M \qquad \text{for all } n \geq 1$$

for some $M > 0$. Using the Banach-Alaoglu's theorem, there exist a subsequence $(\varphi_{n_k\geq 1})_{k\geq 1}$ and $\varphi \in X^*$ such that

$$\lim_{k\to\infty} \varphi_{n_k}(v) \;=\; \varphi(v) \qquad \text{for all } v \in X.$$

By the Riesz representation theorem, there exists a unique $x \in X$ such that

$$\varphi(v) \;=\; \langle x, v\rangle \qquad \text{for all } v \in X$$

3. Finally, we show that $(x_{n_k})_{k\geq 1}$ converges weakly to x, i.e.,

$$\lim_{k\to\infty} \langle x_{n_k}, y\rangle \;=\; \langle x, y\rangle \qquad \text{for all } y \in H.$$

For any given $y \in H$, we have that

$$y \;=\; P_X(y) + P_{X^\perp}(y) \;\in\; X \oplus X^\perp.$$

Thus,

$$\begin{aligned}
\lim_{k\to\infty} \langle x_{n_k}, y\rangle &= \lim_{k\to\infty} \langle x_{n_k}, P_X(y)\rangle + \langle x_{n_k}, P_{X^\perp}(y)\rangle \\
&= \lim_{k\to\infty} \langle x_{n_k}, P_X(y)\rangle \;=\; \langle x, P_X(y)\rangle \;=\; \langle x, P_X(y) + P_{X^\perp}(y)\rangle \\
&= \langle x, y\rangle
\end{aligned}$$

and this complete the proof. □

We now show that a compact operator maps weakly convergent sequences into strongly convergent ones.

Definition 3.41 *Given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the map $f : X \to Y$ is compact if and only if*

$$\overline{f(K)} \text{ is compact for every bounded set } K.$$

The following holds:

Lemma 3.42 *Assume that $(Y, \|\cdot\|_Y)$ is complete. Let $\{T_n : X \to Y\}_{n\geq 1}$ be a sequence of linear bounded and compact operators such that*

$$\lim_{n\to\infty} \|T_n - T\|_\infty = 0.$$

Then T is compact.

Proof. It is clear that T is linear and bounded. Thus, the map T is compact if the set $\overline{T(B(0,1))}$ is compact. Since Y is complete and $\overline{T(B(0,1))}$ is closed in $(Y, \|\cdot\|_Y)$, it is sufficiently to prove that

$$\overline{T(B(0,1))} \text{ is totally bounded.}$$

Given $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\|T_n - T\|_\infty < \frac{\varepsilon}{2} \qquad \text{for all } n \geq N_\varepsilon$$

and it yields

$$\overline{T(B(0,1))} \subset \overline{T_n(B(0,1))} + B(0,\varepsilon/2) \qquad \text{for all } n \geq N_\varepsilon.$$

On the other hand, since T_n is compact, the set $\overline{T_n(B(0,1))}$ is compact . Thus

$$\overline{T_n(B(0,1))} \subset \bigcup_{i=1}^{M_n} B_Y(y_i^n, \varepsilon/2).$$

This implies that

$$\overline{T(B(0,1))} \subset \bigcup_{i=1}^{M_n} B_Y(y_i^n, \varepsilon/2) + B(0,\varepsilon/2) = \bigcup_{i=1}^{M_n} B_Y(y_i^n) \qquad \text{for all } n \geq N_\varepsilon$$

and it complete the proof. □

Proposition 3.42.1 *Given a Hilbert space H, let $\Lambda : H \to H$ be a linear bounded and compact operator. If x_n is weakly convergent to x then*

$$\lim_{n\to\infty} \Lambda(x_n) = \Lambda(x).$$

Proof. Consider a subsequence $(x_n)_{n\in J_1}$ of $(x_n)_{n\geq 1}$, we show that there exists subsequence $(x_n)_{n\in J_2}$ with $J_2 \subseteq J_1$ such that

$$\lim_{J_2 \ni n\to\infty} \|\Lambda(x_n) - \Lambda_n(x)\| = 0.$$

From Theorem 3.40, the sequence $(x_n)_{n\in J_1}$ is bounded by M, i.e., $\|x_n\| \leq M$ for all $x \in J_1$. Since Λ is a compact operator, it holds that there exists a subsequence $(x_n)_{n\in J_2}$ such that

$$\lim_{J_2 \ni n\to\infty} \Lambda(x_n) = \bar{y} \qquad \text{for some } \bar{y} \in H.$$

To complete the proof, we need to show that $\bar{y} = \Lambda(x)$. By the Riesz representation theorem, calling Λ^* the adjoint operator of Λ, i.e.,

$$\langle \Lambda^*(x), y \rangle \; = \; \langle x, \Lambda(y) \rangle \qquad \text{for all } x, y \in H.$$

One has that

$$\lim_{n\to\infty} \langle v, \Lambda(x_n) - \Lambda(x) \rangle \; = \; \lim_{n\to\infty} \langle \Lambda^*(v), x_n - x \rangle \; = \; 0 \qquad \text{for all } v \in H.$$

Thus, $(\Lambda(x_n))_{n\in I_2}$ weakly converges to $\Lambda(x)$ and thus $\Lambda(x) = y$. □

3.7 Positive definite operators

Consider the system of linear equations

$$Ax \; = \; b, \qquad A \in \mathbb{M}^{n\times n}, \qquad b \in \mathbb{R}^n.$$

One condition which guarantees the existence and uniqueness of solutions is that A be strictly positive definite, i.e., $\langle Ax, x \rangle > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. Indeed, in this case, the matrix A must have full rank and the unique solution is $x = A^{-1}b$. In this subsection, we aim to extend this result to an infinite-dimensional Hilbert space H.

Definition 3.43 *A linear operator $A : H \to H$ is strictly positive definition if there exists $\beta > 0$ such that*

$$\langle Au, u \rangle \; \geq \; \beta \cdot \|u\|^2 \qquad \textit{for all } u \in H.$$

It is clear that if A is linear and strictly positive then A is one-to-one and

$$\|Au\| \; \geq \; \beta \cdot \|u\| \qquad \text{for all } u \in H.$$

Therefore, in addition if A is surjective then its inverse $A^{-1} : H \to H$ is linear and bounded with

$$\left\|A^{-1}\right\|_\infty \; \leq \; \frac{1}{\beta}.$$

A natural question is whether a strictly position linear operator is surjective?

Theorem 3.44 *Given a Hilbert space, let $A : H \mapsto H$ be a bounded linear operator which is strictly positive definite. Then, A is bijective and its inverse $A^{-1} : H \to H$ is a linear bounded operator.*

Proof. It is known that A is one-to-one and

$$\|Au\| \; \geq \; \beta \cdot \|u\| \qquad \text{for all } u \in H. \tag{3.6}$$

1. We claim that Range(A) is a closed vector subspace. Indeed, given a sequence $(v_n)_{n\geq 1} \in$ Range(A) which converges to $v \in H$, we show that $v \in$ Range(A). By assumption, we have

$$v_n \ = \ Au_n \qquad \text{and} \qquad \lim_{n\to\infty} A(u_n) \ = \ v.$$

In particular, $(v_n)_{n\geq 1}$ is a Cauchy sequence and (3.6) implies that $(u_n)_{n\geq 1}$ is also a Cauchy sequence. Thus, $(u_n)_{n\geq 1}$ converges to $u \in H$ and the continuity of A yields $v = A(u) \in \text{Range}(A)$.

2. By Theorem 3.31, we can write

$$H \ = \ \text{Range}(A) \oplus [\text{Range}(A)]^{\perp}.$$

If $[\text{Range}(A)]^{\perp} \neq \emptyset$ then there exists a unique vector $w \in [\text{Range}(A)]^{\perp}$ and thus

$$\beta \cdot \|w^2\| \ \leq \ \langle Aw, w\rangle \ = \ 0$$

and this yields a contradiction. Hence, $[\text{Range}(A)]^{\perp} = \emptyset$ and $\text{Range}(A) = H$ which means that A is surjective. □

Theorem 3.45 (Lax-Milgram) *Given a Hilbert space, let $B : H \times H \to \mathbb{R}$ be a continuous bilinear functionals, i.e.,*

$$B(a_1u_1 + a_2u_2, b_1v_1 + b_2v_2) \ = \ \sum_{i,j\in\{1,2\}} a_ib_jB(u_i, v_j)$$

and

$$B(u, v) \ \leq \ C \cdot \|u\| \cdot \|v\| \qquad \textit{for all } u, v \in H$$

for some constant $C > 0$. In addition, assume that B is strictly positive definite, i.e., there exists $\beta > 0$ such that

$$B[u, u] \ \geq \ \beta \cdot \|u\|^2 \qquad \textit{for all } u \in H$$

Then, for every $f \in H$, there exists a unique $u_f \in H$ such that

$$B[u_f, v] \ = \ \langle f, v\rangle \qquad \textit{for all } v \in H.$$

Moreover,

$$\|u_f\| \ \leq \ \frac{\|f\|}{\beta} \qquad \textit{for all } f \in H.$$

Proof. Fixed $u \in H$, the map $v \mapsto B(u, v)$ is linear bounded operator on H. Thus, by Riesz representation theorem, there exists a unique vector $A(u) \in H$ such that

$$B(u, v) \ = \ \langle A(u), v\rangle \qquad \text{for all } v \in H.$$

It is clear that $A(u)$ a linear bounded operator with $\|A\|_\infty \leq C$ since

$$\|A(u)\| \;=\; \frac{\langle A(u), A(u)\rangle}{\|A(u)\|} \;=\; \frac{B[u, Au]}{\|A(u)\|} \;\leq\; C \cdot \|u\| \qquad \text{for all } u \neq 0.$$

Moreover,

$$\langle A(u), u\rangle \;=\; B(u, u) \;\geq\; \beta \cdot \|u\|^2 \qquad \text{for all } u \in H$$

proving that A is strictly positive definition. Applying the previous Theorem, we obtain that A is bijective and its inverse is a linear bounded operator with $\|A\|_\infty \leq \frac{1}{\beta}$. Thus, for every $f \in H$, the unique vector $u_f = A^-1(f)$ satisfies

$$B[u_f, v] \;=\; \langle A(A^{-1}(f)), v\rangle \;=\; \langle f, v\rangle \qquad \text{for all } v \in H$$

and the proof is complete. □

4 Duality and weak convergence in Banach space

4.1 Dual spaces

Let $(X, \|\cdot\|)$ be a normed space. The dual space of X is defined by

$$X^* \;:=\; \left\{f \in X^\sharp : f \text{ is continuous}\right\}$$

with $X^\sharp = \{f : X \to \mathbb{R} : f \text{ is linear}\}$. It is clear that both $X^\sharp$ and X^* are vector spaces.

Lemma 4.1 *Assume that* $\dim(X) = n < \infty$. *Then* $X^\sharp = X^*$ *and*

$$\dim(X^*) \;=\; \dim(X^\sharp) \;=\; n.$$

Proof. Since $\dim(X) = n < \infty$, every linear map $f : X \to \mathbb{R}$ is continuous and this implies that $X^\sharp = X^*$. Assume that

$$X \;=\; \text{span}\,\{e_1, e_2, \dots, e_n\}$$

where $\{e_1, e_2, \dots, e_n\} \subset X$ are linearly independent. For every $i \in \{1, 2, \dots, n\}$, we consider the linear function $f_i : X \to \mathbb{R}$ such that

$$f_i(e_i) \;=\; 1 \qquad \text{and} \qquad f_i(e_j) \;=\; 0 \qquad \text{for all } j \neq i.$$

The set $\{f_1, f_2, \dots, f_n\}$ is linearly independent. Thus, we now prove that

$$X^\sharp \;=\; \text{span}\,\{f_1, f_2, \dots, f_n\}.$$

For a given $f \in X^\sharp$, we set $\beta_i = f(e_i)$ for all $i \in \{1, 2, \ldots, n\}$. For any $x = \sum_{i=1}^{n} \alpha_i \cdot e_i$, it holds that

$$\begin{aligned} f(x) &= f\left(\sum_{i=1}^{n} \alpha_i \cdots e_i\right) = \sum_{i=1}^{n} \alpha_i \cdot f(e_i) \\ &= \sum_{i=1}^{n} \beta_i \cdot \alpha_i \cdot f_i(e_i) = \sum_{i=1}^{n} \beta_i \cdot f_i(\alpha_i e_i) = \sum_{i=1}^{n} \beta_i \cdot f_i(x). \end{aligned}$$

Thus,

$$f(x) = \sum_{i=1}^{n} \beta_i \cdot f_i(x) \qquad \text{for all } x \in X,$$

it yields $\dim(X^\sharp) = n$. □

Remark 4.2 *If* $\dim(X) = n$ *then the map* $T : X \to X^*$ *such that*

$$T(x) = \sum_{i=1}^{n} \alpha_i f_i \qquad \textit{for all } x = \sum_{i=1}^{n} \alpha_i e_i$$

is linear, bounded, and bijective.

Definition 4.3 (Isomorphism) *Given two normed space* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$, *we say that* X *is isomorphic with* Y, *denote by* $X \simeq Y$, *if any only if there exists a bijective isometric linear operator* $T : X \to Y$, *i.e.,*

$$\|T(x)\|_Y = \|x\|_X \qquad \textit{for all } x \in X.$$

Some basic examples.

Examples 1. Given $n \in \mathbb{Z}^+$, it holds

$$[\mathbb{R}^n]^* \simeq \mathbb{R}^n.$$

Proof. Assume that

$$\mathbb{R}^n = \{e_1, e_2, \ldots .e_n\}.$$

Consider the linear function $f_i : X \to \mathbb{R}$ such that

$$f_i(e_i) = 1 \qquad \text{and} \qquad f_i(e_j) = 0 \qquad \text{for all } j \neq i.$$

Let $T : \mathbb{R}^n \to [\mathbb{R}^n]^*$ be such that

$$T[x] = \sum_{i=1}^{n} \alpha_i \cdot f_i \qquad \text{for all } x = \sum_{i=1}^{n} \alpha_i \cdot e_i.$$

One easily see that T is bijective and linear operator. It remains to show that T is isometric. For any given $x = \sum_{i=1}^{n} \alpha_i \cdot e_i$, we estimate

$$\begin{aligned}
\|T[x]\|_\infty &= \left\|\sum_{i=1}^{n} \alpha_i \cdot f_i\right\|_\infty = \sup_{\|z\|=1} \left\|\sum_{i=1}^{n} \alpha_i \cdot f_i(z)\right\| \\
&\qquad\qquad (z = \sum_{i=1}^{n} \beta_i \cdot e_i) \\
&= \sup_{\sum_{i=1}^{n} \beta_i^2 = 1} |\alpha_i \cdot \beta_i| \le \sup_{\sum_{i=1}^{n} \beta_i^2 = 1} \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2} \cdot \left(\sum_{i=1}^{n} |\beta_i|^2\right)^{1/2} \\
&\le \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2} = \|x\|.
\end{aligned}$$

On the other hand, by choosing $\beta_i = \dfrac{\alpha_i}{\sum_{i=1}^{n} \alpha_i^2}$ we have

$$\|T[x](z)\| = |\alpha_i \cdot \beta_i| = 1 \qquad \text{with} \qquad z = \sum_{i=1}^{n} \beta_i \cdot e_i \in \overline{B}(0,1)$$

and this implies that

$$\|T[x]\|_\infty = \|x\| \qquad \text{for all } x \in \mathbb{R}^n.$$

The proof is complete. □

Example 2. Recalling that

$$\ell^\infty = \left\{x = \{x_n\}_{n\ge1} \mid \sup_{n\ge1} |x_n| < \infty\right\}, \qquad \|x\|_\infty = \sup_{n\ge1} |x_n|$$

and

$$\ell^1 = \left\{x = \{x_n\}_{n\ge1} \mid \sum_{n=1}^{\infty} |x_n| < \infty\right\}, \qquad \|x\|_\infty = \sum_{n=1}^{\infty} |x_n|,$$

we have that

$$(\ell^1)^* \simeq \ell^\infty.$$

Proof. Let $f_i : \ell^1 \to \mathbb{R}$ be such that

$$f_i(e_i) = 1 \qquad \text{and} \qquad f_i(e_j) = 0 \qquad \text{for all } j \ne i.$$

One has that $f_i \in (\ell^1)^*$ and

$$f_i(x) = \beta_i \qquad \text{for} \qquad x = \sum_{j=1}^{\infty} \beta_j \cdot e_j.$$

Consider $T : \ell^\infty \to (\ell^1)^*$ such that

$$T[z] \;=\; \sum_{i=1}^{\infty} \alpha_i \cdot f_i \qquad \text{for} \qquad z \;=\; \sum_{i=1}^{\infty} \alpha_i \cdot e_i \;\in\; \ell^\infty.$$

For any $x = \sum_{j=1}^{\infty} \beta_j \cdot e_j \in \ell^1$, we estimate

$$\begin{aligned} |T[z](x)| \;&=\; \left| \sum_{i=1}^{\infty} \alpha_i \cdot f_i(x) \right| \;=\; \left| \sum_{i=1}^{\infty} \alpha_i \beta_i \right| \\ &\leq\; \left(\sup_{i \geq 1} |\alpha_i| \right) \cdot \sum_{i=1}^{\infty} |\beta_i| \;=\; \|z\|_\infty \cdot \|x\|_1. \end{aligned}$$

On the other hand, for any $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{Z}^+$ such that

$$\|z\|_\infty \;\leq\; \alpha_{i_\varepsilon} + \varepsilon.$$

By choosing $x = e_{i_\varepsilon}$, we have

$$|T[z](x)| \;=\; |\alpha_{i_\varepsilon}| \;\geq\; \|z\|_\infty - \varepsilon$$

and this implies

$$\|T[z]\|_\infty \;\geq\; |T[z](x)| \;\geq\; \|z\|_\infty - \varepsilon.$$

Therefore,

$$\|T[z]\|_\infty \;=\; \|z\|_\infty \qquad \text{for all } z \in \ell^\infty.$$

□

Example 3. For any $p \in (1, +\infty)$, let q be its conjugate, i.e., $1/p + 1/q = 1$. Then the dual space of ℓ^p is isomorphic with ℓ^q.

4.2 Direct sum

Definition 4.4 *Let M and N be vector subspaces of X. We say that the direct sum*

$$M \oplus N \;=\; X$$

if any only if

$$M \cap N = \{0\} \quad \text{and} \quad M + N \;=\; \{m + n \mid m \in M, n \in N\} \;=\; X.$$

In this case, we say that N is the complement of M in X.

Notice that the complement of M in X is not unique. Indeed, let's consider $X = \mathbb{R}^3$ and $M = \mathbb{R}^2 \times \{0\}$. One can see that both

$$N_1 \;=\; \{0\} \times \mathbb{R} \qquad \text{and} \qquad N_2 \;=\; \mathrm{span}\{1,1,1\}$$

are the complements of M.

The following lemma holds.

Lemma 4.5 *(i) For any $f \in X^\sharp$ such that*

$$f(x_0) \;\neq\; 0 \quad \text{for some } x_0 \in X.$$

Then $\ker(f)$ *is of co-dimension* 1 *of* X *and*

$$X \;=\; \ker(f) \oplus \mathrm{span}\{x_0\}.$$

(ii) Let $V \subseteq X$ be a subspace of co-dimension 1 *of X. Then there exists $f \in X^\sharp$ such that*

$$\ker(f) \;=\; V.$$

Proof. (i) By the linear property of f, it holds

$$f\left(x - \frac{f(x)}{f(x_0)} \cdot x_0\right) \;=\; f(x) - \frac{f(x)}{f(x_0)} \cdot f(x_0) \;=\; 0 \qquad \text{for all } x \in X.$$

Thus,

$$x - \frac{f(x)}{f(x_0)} \cdot x_0 \;\in\; \ker(f)$$

and it yields

$$x \;=\; \left(x - \frac{f(x)}{f(x_0)} \cdot x_0\right) + \frac{f(x)}{f(x_0)} \cdot x_0 \;\in\; \ker(f) \cap \mathrm{span}(x_0) \qquad \text{for all } x \in X.$$

On the other hand, it is clear that

$$\ker(f) \cap \mathrm{span}\{x_0\} \;=\; \{0\}.$$

and it yields (i).

(ii) Since V a subspace of co-dimension 1 of X, there exist $0 \neq x_0 \in X$ such that

$$V \oplus \mathrm{span}(x_0) \;=\; X.$$

Let $f : X \to \mathbb{R}$ be such that $f(x_0) = 1$ and

$$f(v + \lambda x_0) \;=\; \lambda \qquad \text{for all } v \in V, \lambda \in \mathbb{R}.$$

One has that $f \in X^\sharp$ and $\ker(f) = V$. $\square$

Before going to state the linear extension result, let us first introduce the following lemma.

Lemma 4.6 *Let Y be a subspace of X . Then there exists a subspace Z of X such that*

$$Z \oplus Y \;=\; X. \tag{4.1}$$

Proof. Consider the collection of subspaces of X

$$\mathcal{F} \;=\; \{W \textit{ subspace of } X \textit{ such that } W \cap Y = \{0\}\}$$

and the relation

$$W_1 \;\preceq\; W_2 \qquad \text{if and only if} \qquad W_1 \;\subseteq\; W_2.$$

It is clear that $\mathcal{F}$ is a partially ordered set. Let us now show that every totally ordered subset of $\mathcal{F}$ has an upper bound. Given a totally ordered subset $\mathcal{F}_1$ of $\mathcal{F}$, i.e., for any $W_1, W_2 \in \mathcal{F}_1$, it holds

$$W_1 \;\preceq\; W_2 \qquad \text{or} \qquad W_2 \;\preceq\; W_1.$$

Denote by

$$\widetilde{W} \;=\; \bigcup_{W\in\mathcal{F}_1} W.$$

It is clear that $\widetilde{W} \cap Y = \{0\}$. Let's show that $\widetilde{W}$ is a subspace of X. Indeed, for any $x, y \in \widetilde{W}$, there exist $W_x, W_y \in \mathcal{F}_1$ such that

$$x \;\in\; W_x \qquad \text{and} \qquad y \;\in\; W_y.$$

Without loss of generality, we assume that $W_x \subseteq W_y$. In this case, we have that $x, y \in W_y$ and thus

$$\alpha\cdot x + \beta\cdot y \;\in\; W_y \;\subseteq\; \widetilde{W} \qquad \text{for all } \alpha, \beta \in \mathbb{R}.$$

Therefore, $\widetilde{W}$ is a subspace of X and is an upper bound of $\mathcal{F}_1$.

By the Zorn's lemma, $\mathcal{F}$ has a maximal element Z. To complete this step, we show that (4.1) holds. By the definition, one has that

$$Z \cap Y \;=\; \varnothing.$$

Let's show that

$$Z + Y \;=\; X.$$

Assume by a contradiction, there exists $x \in X$ such that $x \notin Z + Y$. In particular, $x \notin Z$ and $x \notin Y$. Denote by

$$Z \;\subsetneq\; Z_1 \;:=\; Z + \text{span}\{x\}$$

It is clear that

$$Z_1 \cap Y = \{0\}.$$

Indeed, if there exists $0 \neq x_0 \in Z_1 \cap Y$ then $\lambda x + z = x_0 \in Y$ for some $z \in Z$ and $\lambda \neq 0$. This implies

$$x = \frac{1}{\lambda} \cdot x_0 - \frac{1}{\lambda} \cdot z \in Z + Y$$

and it yields a contradiction. Thus, (4.1) holds. □

Corollary 4.7 (Linear extension) *Let Y be a subspace of X and let $T : Y \to \mathbb{R}$ be a linear function. Then there exists a extension $\widetilde{T}$ of T defined on X, i.e.,*

$$\widetilde{T} : X \to \mathbb{R} \text{ is linear} \quad \text{and} \quad \widetilde{T}(y) = T(y) \qquad \text{for all } y \in Y.$$

Proof. Let Z be a subspace of X such that

$$Z \oplus Y = X.$$

A linear extension $\widetilde{T} : X \to \mathbb{R}$ of T is defined by

$$\widetilde{T}(x) = T(x_y) \qquad \text{for all } x = x_y + x_z \in Y + Z$$

and the proof is complete. □

Remark 4.8 *If $Y \oplus V = X$ then for every $x \in X$, there exist unique $x_Y \in Y$ and $x_V \in V$ such that*

$$x = x_Y + x_V.$$

Thus, the projection maps $\pi_Y : X \to Y$ and $\pi_V : X \to V$, defined by

$$\pi_Y(x) = x_Y \qquad \text{and} \qquad \pi_V(x) = x_V,$$

are well-defined and linear.

Question. Are π_Y and π_V bounded? No in general.

Lemma 4.9 *Assume that Y is closed and V is a finite dimensional vector space. Then the maps π_Y and π_V are bounded.*

Proof. Since $\pi_Y(x) + \pi_V(x) = x$, the map π_V is bounded if π_Y is bounded. Thus, it is sufficient to show that π_Y is bounded. Assume by a contradiction, there exists $\{x_n\}_{n \geq 1} \subset X$ such that

$$\|x_n\| = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \|\pi_Y(x_n)\| = +\infty.$$

Denote by

$$y_n \;=\; \frac{\pi_Y(x_n)}{\|\pi_Y(x_n)\|} \;\in\; Y \qquad \text{and} \qquad v_n \;=\; \frac{\pi_V(x_n)}{\|\pi_Y(x_n)\|} \;\in\; V,$$

we have that

$$\|y_n\| \;=\; 1, \qquad \lim_{n\to\infty} y_n + v_n \;=\; 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|v_n\| \;=\; 1.$$

Since V is a finite dimensional vector space, there exists a subsequence $\{v_{n_k}\}_{k\geq 1} \subseteq \{v_n\}_{n\geq 1}$ such that $v_{n_k} \to \bar{v} \in V$. This implies that

$$\lim_{n_k\to\infty} y_{n_k} \;=\; -\,\bar{v} \;\in\; Y \qquad \text{and} \qquad \|\bar{v}\| \;=\; 1.$$

Thus,

$$0 \;\neq\; \bar{v} \;\in\; Y \cap V$$

and it yields a contradiction. □

4.2.1 Hahn-Banach extension theorem

Definition 4.10 *Let $(X, \|\cdot\|)$ be a normed space. The function $p : X \to \mathbb{R}$ is called sub-linear functional if for all $t > 0$, $x, y \in X$, it holds*

$$p(tx) \;=\; tp(x) \qquad \text{and} \qquad p(x+y) \;\leq\; p(x) + p(y). \tag{4.2}$$

If p is sub-linear functional then

$$p(0) \;=\; 0, \qquad -p(x) \;\leq\; p(-x)$$

and p is convex, i.e.,

$$p(\theta x + (1-\theta)y) \;\leq\; \theta p(x) + (1-\theta)p(y) \qquad \text{for all } \theta \in [0,1].$$

<u>Some examples:</u>

(a) $\|\cdot\|$ norm is a sublinear function.

(b) Let K be a bounded, open, convex set with $0 \in K$. The function

$$p(x) \;=\; \inf\{\lambda \geq 0 \mid x \in \lambda K\}$$

is sublinear functional.

Theorem 4.11 (Hahn-Banach extension theorem) *Let X be a normed space and let $p : X \to \mathbb{R}$ be a sublinear function. Given a subspace $V \subseteq X$, let $f : V \to \mathbb{R}$ be linear such that*

$$f(x) \ \leq \ p(x) \qquad \textit{for all } x \in V. \tag{4.3}$$

Then there exists a linear extension of $F : X \to \mathbb{R}$ such that

$$F(x) \ = \ f(x) \qquad \textit{for all } x \in V$$

and

$$-p(-x) \ \leq \ F(x) \ \leq \ p(x) \qquad \textit{for all } x \in X.$$

Proof. 1. Notice that if $F(x) \leq p(x)$ then

$$F(x) \ = \ -F(-x) \ \geq \ -p(-x).$$

Thus, if $X = V$ then

$$f(x) \ = \ F(x) \ \geq \ -p(-x) \qquad \text{for all } x \in X.$$

2. Otherwise, there exists $x_0 \in X \backslash V$. Denote by

$$V_0 \ = \ \{x + tx_0 \mid x \in V, t \in \mathbb{R}\},$$

we want to extend f to V_0 linearly. In this case, it holds

$$f(x + tx_0) \ = \ f(x) + tf(x_0) \qquad \text{for all } x \in V, t \in \mathbb{R}.$$

Thus, one needs to find the value $\beta = f(x_0)$ such that

$$f(x) + t\beta \ \leq \ p(x + tx_0) \qquad \text{for all } x \in V, t \in \mathbb{R}. \tag{4.4}$$

For $t > 0$, (4.4) holds if

$$\begin{aligned} \beta \ &\leq \ \frac{1}{t} \cdot \inf_{x\in V} [p(x + tx_0) - f(x)] \ = \ \inf_{x\in V} \left[p\left(\frac{x}{t} + x_0\right) - f\left(\frac{x}{t}\right)\right] \\ &= \ \inf_{y\in V} \ [p(y + x_0) - f(y)] \end{aligned}$$

For $t < 0$, (4.4) holds if

$$\begin{aligned} \beta \ &\geq \ -\frac{1}{t} \cdot \sup_{x\in V} [f(x) - p(x + tx_0)] \ = \ \sup_{x\in V} \left[f\left(\frac{-x}{t}\right) - p\left(\frac{-x}{t} - x_0\right)\right] \\ &= \ \sup_{z\in V} f(z) - p(z - x_0). \end{aligned}$$

Observe from (4.3) that

$$p(y + x_0) - f(y) \ \leq \ f(z) - p(z - x_0) \qquad \text{for all } y, z \in V,$$

we can choose

$$\beta \;=\; \sup_{z\in V}\, f(z) - p(z - x_0).$$

3. To extend f to X, let's introduce the collection

$$\mathcal{F} \;=\; \{(W,\Phi) \;\;|\;\; V \subseteq W \text{ subspace}, \Phi : W \to \mathbb{R} \text{ linear} \\ \text{such that } \Phi(x) \leq p(x) \text{ for all } x \in W\}.$$

and the relation

$$(W_1,\Phi_1) \;\preceq\; (W_2,\Phi_2) \quad \text{if and only if} \quad W_1 \subseteq W_2, \Phi_1(x) = \Phi_2(x) \text{ for all } x \in W_1.$$

It is clear that $(\mathcal{F},\preceq)$ is partially ordered. Moreover, every totally bounded ordered subset of $\mathcal{F}$ has an upper bound. By Zorn's lemma, $\mathcal{F}$ has a maximal element $(V_{\max}, F)$. To complete the proof, we show that

$$V_{\max} \;=\; X.$$

Assume by a contradiction, there exists $0 \neq x_0 \in X\backslash V_{\max}$. From the step 2, the linear function F can be extended to a strictly larger space and it yields a contradiction. □

Applications

1. A natural application to the case

$$p(x) \;=\; \|x\| \qquad \text{for all } x \in X.$$

The following holds:

Theorem 4.12 (Extension theorem for bounded linear operator) *Let $V \subseteq X$ be a subspace and let $f : V \to \mathbb{R}$ be a bounded linear operator. Then there exists $F : X \to \mathbb{R}$ bounded linear operator with $\|F\|_\infty = \|f\|_\infty$ such that*

$$F(x) \;=\; f(x) \qquad \textit{for all } x \in V.$$

Proof. Consider $p : X \to \mathbb{R}$ such that

$$p(x) \;=\; \|f\|_\infty \cdot \|x\| \qquad \text{for all } x \in X.$$

We have that p is sublinear functional and

$$f(x) \;\leq\; p(x) \qquad \text{for all } x \in V.$$

Using the Hahn Banach theorem, there exists $F : X \to \mathbb{R}$ linear such that

$$f(x) \;=\; F(x) \qquad \text{for all } x \in V$$

and

$$-p(-x) \;\leq\; F(x) \;\leq\; p(x) \qquad \text{for all } x \in X.$$

This implies that

$$-\|f\|_\infty \cdot \|x\| \;\leq\; F(x) \;\leq\; \|f\|_\infty \cdot \|x\| \qquad \text{for all } x \in X$$

and it yields

$$\|F\|_\infty \;\leq\; \|f\|_\infty.$$

Since $F(x) = f(x)$ for all $x \in V$, one then has that $\|F\|_\infty = \|f\|_\infty$. □

Corollary 4.13 *Let X be a normed space. For any $x \neq y \in X$, there exists $\Phi : X \to \mathbb{R}$ continuous and linear such that*

$$\Phi(x) \;\neq\; \Phi(y).$$

Proof. Consider the 2-dimensional vector space

$$V \;=\; \text{span}\{x, y\} \;=\; \{sx + ty \mid s, t \in \mathbb{R}\}.$$

Consider the linear function $f : V \to \mathbb{R}$ such that

$$f(sx + ty) \;=\; s \cdot \|x\| + t \cdot \|y\| \qquad \text{for all } s, t \in \mathbb{R}.$$

One has that f is bounded and

$$f(x) \;=\; \|x\| \;\neq\; -\|y\| \;=\; f(y).$$

By the extension theorem, there exists linear and continuous map $\Phi : X \to \mathbb{R}$ such that $\Phi(z) = f(z)$ for all $z \in V$, and it yields

$$\Phi(x) \;\neq\; \Phi(y)$$

and this complete the proof. □

2. Separation of convex sets. Given two disjoint convex nonempty sets $A, B \subset X$, our goal to find a bounded linear operator $\phi : X \to \mathbb{R}$ such that

$$\phi(A) \cap \phi(B) \;=\; \emptyset.$$

We prove the following theorem.

Theorem 4.14 *Let $(X, \|\cdot\|)$ be a normed space. For any two disjoint convex nonempty sets $A, B \subset X$, the followings hold:*

(i) If A is open then there exist $\phi \in X^*$ and a constant $c \in \mathbb{R}$ such that

$$\phi(a) \;<\; c \;\leq\; \phi(b) \qquad \textit{for all } a \in A, b \in B.$$

(ii) If A is compact and B is closed, then there exist $\phi \in X^*$ and $c_1, c_2 \in \mathbb{R}$ such that

$$\phi(a) \ \le \ c_1 \ < \ c_2 \ \le \ \phi(b) \qquad \textit{for all } a \in A, b \in B.$$

Proof. (i). Assume that A is open. The proof of (i) is divided in to several step:

1. Pick any two points a_0, b_0, we consider the following set

$$\Omega \ = \ A_0 - B_0 \qquad \text{with} \qquad A_0 \ = \ A - a_0, \quad B_0 \ = \ B - b_0.$$

It is clear that Ω is open and convex and

$$0 \ \in \ \Omega \qquad \text{but} \qquad b_0 - a_0 := x_0 \ \notin \ \Omega.$$

2. Consider the functional

$$p(x) \ = \ \inf\{\lambda \ge 0 : x \in \lambda\Omega\}.$$

Since Ω contains a small ball $B(0, \rho)$ and $x_0 \notin \Omega$, it holds

$$p(x_0) \ \ge \ 1 \qquad \text{and} \qquad p(x) \ \le \ \frac{1}{\rho} \cdot \|x\| \qquad \text{for all } x \in X.$$

Moreover, by the convexity of Ω, the function ρ satisfies

$$p(tx) \ = \ t \cdot p(x), \qquad p(x+y) \ \le \ p(x) + p(y) \qquad \text{for all } t \ge 0, x, y \in X.$$

3. Let $f : V := \text{span}\{x_0\} \to \mathbb{R}$ be linear such that

$$f(tx_0) \ = \ t \qquad \text{for all } t \in \mathbb{R}.$$

One has

$$f(x_0) \ = \ 1 \qquad \text{and} \qquad f(tx_0) \ = \ t \ \le \ p(tx_0).$$

By the Hahb-Banach extension theorem, there exists a linear functional $\phi : X \to \mathbb{R}$ such that

$$-p(-x) \ \le \ \phi(x) \ \le \ p(x) \qquad \text{for all } x \in X.$$

In particular, this implies that ϕ is bounded and

$$\|\phi\|_\infty \ \le \ \frac{1}{\rho}.$$

4. For any $a \in A$ and $b \in B$, we have

$$\phi(a) - \phi(b) + 1 \ = \ \phi(a - b + x_0) \ \le \ p(a - b + x_0) \ < \ 1$$

since $a - b + x_0 \in \Omega$ and Ω open. Thus,

$$\phi(a) \ < \ \phi(b) \qquad \text{for all } a \in A, b \in B.$$

In particular, the constant $c := \sup_{a\in A}\phi(a) < \infty$ and

$$\phi(a) \ < \ c \ \le \ \phi(b).$$

(ii). Assume that A is compact and B is closed. We have

$$d(A,B) \ = \ \inf\{|a-b| : a\in A, b\in b\} \ \doteq \ \rho \ > \ 0.$$

and the open set

$$A_\rho \ = \ \{x\in X : d(x,A) < \rho\}$$

had an empty intersection with B. Thus, there exists $\phi\in X^*$ and $c_2>0$ such that

$$\phi(a) \ < \ c_2 \ \le \ \phi(b) \qquad \text{for all } a\in A, b\in B.$$

Finally, since A is compact, we set $c_1 := \max_{a\in A}\phi(a)$ and get

$$\phi(a) \ \le \ c_1 \ < \ c_2 \ \le \ \phi(b) \qquad \text{for all } a\in A, b\in B.$$

The proof is complete. □

4.3 Weak convergence in Banach space

Given $(X,\|\cdot\|)$ a Banach space, its dual space is

$$X^* \ = \ \{\varphi : X\to\mathbb{R} \mid \varphi \textit{ is bounded linear operator}\}$$

with

$$\|\varphi\|_\infty \ = \ \sup_{\|x\|=1} |\varphi(x)|.$$

One has that $(X^*,\|\cdot\|_\infty)$ is also a Banach space.

Definition 4.15 (Weak convergence) *A sequence $\{x_n\}_{n\ge1}$ converges weakly to x, denote by $x_n \rightharpoonup x$, if*

$$\lim_{n\to\infty}\varphi(x_n) \ = \ \varphi(x) \qquad \textit{for all } \varphi\in X^*.$$

From the definition, it is clear that if x_n converges to x then x_n converges weakly to x.

Lemma 4.16 (uniqueness of weak limit) *If $x_n\rightharpoonup x$ and $x_n\rightharpoonup y$ then $x=y$.*

Answer. Assume that $x\neq y$. Then there exists $\phi : X\to\mathbb{R}$ bounded linear operator such that $\phi(x)\neq\phi(y)$. Since $x_n\rightharpoonup$ and $x_n\rightharpoonup y$, one has

$$\phi(x) \ = \ \lim_{n\to\infty}\phi(x_n) \ = \ \phi(y)$$

and it yields a contradiction. □

Weak star convergence. Consider the dual of X^*

$$(X^*)^* \ : \ \{\varphi^* : X^* \to \mathbb{R} \mid \varphi^* \textit{ is bounded linear operator}\}.$$

We first show that

Lemma 4.17 *For any given normed space $(X, \|\cdot\|)$, it holds*

$$X \simeq (X^*)^*$$

Proof. Consider $T : X \to X^*$ such that

$$T[x](\varphi) \ = \ \varphi(x) \qquad \text{for all } x \in X.$$

The followings hold:

- $T[x]$ is linear;
- For every $\varphi \in X^*$, one has

  $$|T[x](\varphi)| \ = \ |\varphi(x)| \ \leq \ \|x\| \cdot \|\varphi\|_\infty$$

  and this implies

  $$\|T[x]\|_\infty \ = \ \sup_{\|\varphi\|_\infty \leq 1} |T[x](\varphi)| \ \leq \ \|x\| \qquad \text{for all } x \in X.$$

  On the other hand, consider the linear function $\phi : V = \mathrm{span}\{x\} \to \mathbb{R}$ such that

  $$\phi(tx) \ = \ t \cdot \|x\| \qquad \text{for all } t \in \mathbb{R}.$$

  Using the Hahn Banach, there exists an extension $\Phi \in X^*$ such that

  $$\|\Phi\|_\infty \ = \ 1 \qquad \text{and} \qquad \Phi(x) \ = \ \|x\|.$$

  Thus,

  $$\|T[x]\|_\infty \ \geq \ \|T[x](\Phi)\| \ = \ \Phi(x) \ = \ \|x\|.$$

  and this yields $\|T[x]\|_\infty = \|x\|$.

The proof is complete. □

Definition 4.18 *We say that* $\{\varphi_n\}_{n\geq 1} \subseteq X^*$ *weak-star converges to* $\varphi \in X^*$*, denote by* $\varphi_n \overset{*}{\rightharpoonup} \varphi$*, if*

$$\lim_{n\to\infty} \varphi_n(x) \ = \ \varphi(x) \qquad \text{for all } x \in X.$$

We have the following theorem.

Theorem 4.19 (Banach Alaoglu) *Let* X *be a separable Banach space. Then, for every* $\{\varphi_n\}_{n\geq 1} \subseteq X^*$ *such that*

$$\|\varphi_n\|_\infty \ \leq \ M \ < \ +\infty \qquad \text{for all } n \geq 1,$$

there exists $\{\varphi_{n_k}\}_{k\geq 1} \subset \{\varphi_n\}_{n\geq 1}$ *such that* $\varphi_{n_k} \overset{*}{\rightharpoonup} \varphi \in X^*$*.*

Sketch of the proof. 1. Since X is separable, i.e.,

$$X \ = \ \text{closure}\ \{x_1, x_2, \cdots, x_n, \dots\},$$

One can show that that $\varphi_{n_k} \overset{*}{\rightharpoonup} \varphi$ if and only if

$$\lim_{k\to\infty} \varphi_{n_k}(x_i) \ = \ \varphi(x_i) \qquad \text{for all } i \geq 1.$$

2. Relying on this observation, we only need to construct the subsequence $(\varphi_{n_k})_{k\geq 1}$ of $(\varphi_n)_{n\geq 1}$ as follows:

- $\{\varphi_n(x_1)\}_{n\geq 1}$ is bounded in $\mathbb{R}$, there exists $\mathcal{I}_1 \subset \mathbb{N}$ such that

$$\lim_{\mathcal{I}_1 \ni \to \infty} \varphi_n(x_1) =: \varphi(x_1).$$

- By induction for every $k \geq 1$, there exists $\mathcal{I}_{k+1} \subset \mathcal{I}_k$ such that

$$\lim_{\mathcal{I}_{k+1} \ni \to \infty} \varphi_n(x_{k+1}) =: \varphi(x_{k+1}).$$

Choose $\{n_k\}_{k\geq 1}$ such that

$$n_k \ \in \ \mathcal{I}_k \qquad \text{for all } k \geq 1,$$

we then have

$$\lim_{k\to\infty} \varphi_{n_k}(x_i) \ = \ \varphi(x_i) \qquad \text{for all } i \geq 1.$$

Notice that the function φ is defined on $S = \{x_1, x_2, \dots, x_n, \dots\}$. For every $x_i, x_j \in S$, one has

$$|\varphi(x_i) - \varphi(x_j)| \ = \ \lim_{k\to\infty} |\varphi_{n_k}(x_i) - \varphi_{n_k}(x_j)| \ \leq \ M \cdot |x_i - x_j|.$$

In particular, the function φ is Lipschitz on S with a Lipschitz constant M. Since S is dense in X, one can extend φ to X by

$$\varphi(x) \;=\; \lim_{n\to\infty} \varphi(z_n)$$

for some $\{z_n\}_{n\geq 1} \subset S$ converging to x.

3. To complete the proof, we need to prove the following problem:

HW problem: *Given a Banach space X , let $\varphi \in X^*$ and $\{\varphi_n\}_{n\geq 1} \subset X^*$ be such that*

$$\|\varphi_n\|_\infty \;=\; \sup_{\|x\|\leq 1} \|\varphi_n(x)\| \;\leq\; M$$

for some constant $M > 0$. Assume that there exists a dense $S = \{x_1, x_2, ..., x_k, ...\}$ in X such that for every $y_k \in S$ it holds

$$\lim_{n\to\infty} \;\varphi_n(x_k) \;=\; \varphi(x_k)\,.$$

Show that φ_n is weakly star convergent to φ in X^* and $\|\varphi\|_\infty \leq M$. □

4.4 Adjoint operators

Let X be a Banach space. Its dual is the space X^* of all bounded linear functional $x^* : X \to \mathbb{R}$ such that

$$\|x^*\| \;=\; \sup_{\|x\|\leq 1} |x^*(x)| \;<\; +\infty.$$

For a convenience, we will use the notation

$$x^*(x) \;=\; \langle x^*, x\rangle \qquad \text{for all } x \in X, x^* \in X^*.$$

Notice that

$$\sup_{\|x^*\|=1} \langle x^*, x\rangle \;=\; \|x\| \qquad \text{for all } x \in X.$$

By using the uniform boundedness principle, we show that

Lemma 4.20 *Any sequence $x_n \in X$ which weakly converges to $x \in X$ is bounded.*

Proof. For every x_n, let $\varphi_n \in (X^*)^*$ be such that

$$\varphi_n(y^*) \;=\; \langle y^*, x_n\rangle \qquad \text{for all } y^* \in X^*.$$

As $n \to \infty$, we have the point-wise convergence

$$\lim_{n\to\infty} \varphi_n(y^*) \;=\; \lim_{n\to\infty} \langle y^*, x_n\rangle \;=\; \langle y^*, x\rangle \qquad \text{for all } y^* \in X^*.$$

In particular, this implies that

$$\sup_{n\geq 1} |\varphi_n(y^*)| \ < \ +\infty \qquad \text{for all } y^* \in X^*.$$

By the uniform boundedness principle, we conclude that sequence $(\varphi_n)_{n\geq 1}$ is uniformly bounded and thus

$$\sup_{n\geq 1} \|x_n\| \ = \ \sup_{n\geq 1} \|\varphi_n\|_\infty \ \leq \ M$$

for some constant $M \geq 0$. □

Given a Banach space Y, let $\Lambda : X \to Y$ be a bounded linear operator. For every $y^* \in Y^*$, the composed map $x^* : X \to \mathbb{R}$ defined as

$$x^*(x) \ = \ y^*(\Lambda(x)) \qquad \text{for all } x \in X$$

is bounded linear functional, i.e., $x^* := y^* \circ \Lambda \in X^*$.

Definition 4.21 *The map $\Lambda^* : Y^* \to X^*$ such that $\Lambda^* y^* = y^* \circ \Lambda$ is called the adjoint of Λ.*

From the above definition, one can see that

$$\langle \Lambda^* y^*, x\rangle \ = \ \langle y^*, \Lambda x\rangle \qquad \text{for all } x \in X, y^* \in Y^*.$$

Given subset $V \subseteq X$ and $W \subseteq X^*$, their orthogonal sets are defined as

$$V^\perp \ = \ \{x^* \in X^* : \langle x^*, v\rangle = 0 \quad \text{for all } v \in V\}$$

and

$$W^\perp \ = \ \{x \in X : \langle w^*, x\rangle = 0 \quad \text{for all } w^* \in W\}.$$

Lemma 4.22 *Given a bounded linear operator $\Lambda : X \to Y$, let Λ^* be its adjoint operator. Then*

$$\|\Lambda\|_\infty = \|\Lambda^*\|_\infty, \quad \mathrm{Ker}(\Lambda) \ = \ [\mathrm{Range}(\Lambda^*)]^\perp \quad \text{and} \qquad \mathrm{Ker}(\Lambda^*) \ = \ [\mathrm{Range}(\Lambda)]^\perp.$$

Proof. By the definition of $\|\cdot\|$ and the orthogonal sets, we have

$$\begin{aligned}\|\Lambda\|_\infty \ &= \ \sup_{\|x\|\leq 1} \|\Lambda(x)\| \ = \ \sup\{\langle y^*, \Lambda(x)\rangle : \|x\| \leq 1, \|y^*\|_\infty \leq 1\} \\ &= \ \sup\{\langle \Lambda^* y^*, x\rangle : \|x\| \leq 1, \|y^*\|_\infty \leq 1\} \\ &= \ \sup_{\|y^*\|_\infty \leq 1} \|\Lambda^* y^*\|_\infty \ = \ \|\Lambda^*\|_\infty.\end{aligned}$$

and

$$\begin{aligned}\text{Ker}(\Lambda) &= \{x \in X : \Lambda(x) = 0\} \\ &= \{x \in X : \langle y^*, \Lambda(x)\rangle = 0 \quad \text{for all } y^* \in Y^*\} \\ &= \{x \in X : \langle \Lambda^* y^*, x\rangle = 0 \quad \text{for all } y^* \in Y^*\} \\ &= \left\{x \in X : x \in [\text{Range}(\Lambda^*)]^\perp\right\} \;=\; [\text{Range}(\Lambda^*)]^\perp.\end{aligned}$$

The last equality is quite similar. □

Theorem 4.23 *Let X and Y be Banach spaces. The bounded linear operator $\Lambda : X \to Y$ is compact if and only if its adjoint $\Lambda^* : Y^* \to X^*$ is compact.*

Proof. Assume that Λ is compact. For a given a bounded sequences $(y_n^*)_{n\geq 1}$, we need to show that there exists a subsequence $(y_{n_k}^*)_{k\geq 1} \subseteq (y_n^*)_{n\geq 1}$ such that $\Lambda^*(y_{n_k}^*)$ converges in X^* as $n_k \to \infty$. By assumption, the set

$$E := \overline{\Lambda(B_1)} \qquad \text{with} \qquad B_1 \;=\; \{x \in X : \|x\| \leq 1\}$$

is a compact set in Y. For every $n \geq 1$, we define the function $f_n : E \to \mathbb{R}$ as

$$f_n(z) \;=\; y_n^*(z) \qquad \text{for all } z \in E,$$

Since $(y_n^*)_{n\geq 1}$ is a bounded sequence in Y^*, there exists $M > 0$ such that

$$|f_n(z) - f_n(z')| \;=\; |y_n^*(z) - y_n^*(z')| \;\leq\; M \cdot \|z - z'\| \quad \text{for all } z, z' \in E$$

and

$$\sup_{z\in E} |f_n(z)| \;\leq\; M \cdot \sup_{z\in E} \|z\| \;\leq\; M \cdot \|\Lambda\|_\infty.$$

Thus, the sequence of functions $(f_n)_{n\geq 1}$ is equicontinuous and uniformly bounded. By Arzelà-Ascoli theorem, there exists a subsequence $(f_{n_k})_{k\geq 1}$ which converges to a function f uniformly on the compact set E. In particular, the subsequence $(f_{n_k})_{k\geq 1}$ is a Cauchy sequence in $\mathcal{C}(E, \mathbb{R})$, i.e., for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\sup_{z\in E} |f_{n_k}(z) - f_{n_j}(z)| \;\leq\; \varepsilon \qquad \text{for all } n_k, n_j \geq N_\varepsilon.$$

Thus, by the definition of Λ^*, we estimate

$$\begin{aligned}\|\Lambda^*(y_k^*) - \Lambda^*(y_j^*)\|_\infty &= \sup_{\|x\|\leq 1} \left|\langle \Lambda^*(y_k^*) - \Lambda^*(y_j^*), x\rangle\right| \\ &= \sup_{\|x\|\leq 1} \left|\langle y_k^* - y_j^*, \Lambda x\rangle\right| \\ &= \sup_{\|x\|\leq 1} |f_{n_k}(\Lambda(x)) - f_{n_j}(\Lambda(x))| \;\leq\; \varepsilon \qquad \text{for all } n_k, n_j \geq N_\varepsilon\end{aligned}$$

and this shows that the subsequence $\Lambda^*(y_k)$ is Cauchy in X^*, hence it converges to $x^* \in X^*$. Therefore, Λ^* is a compact operator.

The converse implication can be proved by the same argument. □

Theorem 4.24 *Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous map. Then the integral operators*

$$\Lambda[f](x) \; = \; \int_a^b K(x,y) f(y) dy$$

is a compact linear operator from $C([a,b])$ into it self.

Proof. It is clear that Λ is a linear bounded operator from $C([a,b])$ to $C([a,b])$. To achieve the compactness of Λ, consider a bounded sequence of continuous functions $f_n \in C([a,b])$, we need to show that $(\Lambda[f_n])_{n\geq 1}$ admits an uniformly convergent subsequence.

1. Observe that K is uniformly continuous and bounded in $[a,b] \times [a,b]$, i.e., $\max\limits_{(x,y)\in[a,b]\times[a,b]} K(x,y) \leq \kappa$ and for every $\varepsilon > 0$, there exists $\delta_{\varepsilon>0}$ such that

$$|K(x,y) - |K(x',y')| \; \leq \; \varepsilon \qquad \text{for all } |x-x'| + |y-y'| \leq \delta_\varepsilon.$$

We have that

$$\big|\Lambda[f_n](x)\big| \; \leq \; \int_a^b |K(x,y)| \cdot |f_n(y)| dy \; \leq \; \kappa \cdot (b-a) \cdot \sup_{n\geq} \|f_n\|_\infty$$

and this yields

$$\big\|\Lambda[f_n]\big\|_\infty \; \leq \; \kappa \cdot (b-a) \cdot \sup_{n\geq} \|f_n\|_\infty \qquad \text{for all } n \geq 1.$$

Thus, $\Lambda[f_n]$ is uniformly bounded.

2. Let $\varepsilon > 0$ be given, we have that

$$\big|\Lambda[f_n](x) - \Lambda[f_n](x')\big| \; \leq \; (b-a) \cdot \sup_{n\geq} \|f_n\|_\infty \cdot \varepsilon$$

for all $|x - x'| \leq \delta_\varepsilon$. Thus, $\Lambda[f_n]$ is equicontinuous. Therefore, one can apply the Ascoli's theorem to complete the proof. □

5 Compact operators on a Hilbert Space

It is well-known that for any given linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$, the followings hold:

(i) The subspace Ker(A) and $[\text{Range}(A)]^\perp$ have the same dimension. In particular, this implies that A is one-to-one if A is onto.

(ii). If A is symmetric then its eigenvalues are reals and $\mathbb{R}^n$ admits an orthonormal basis consisting of eigenvectors of A.

We aim to extend these results to an infinite-dimensional Hilbert space H.

Theorem 5.1 (Fredholm) *Let $K : H \to H$ be a compact linear operator. Then the followings hold:*

(i) $\mathrm{Ker}(I-K)$ *is a finite-dimensional subspace;*

(ii) $\mathrm{Range}(I-K) = \mathrm{Ker}(I-K^*)^\perp$ *is closed;*

(iii) $\mathrm{Ker}(I-K) = \{0\}$ *if and only if* $\mathrm{Range(I-K)} = H$*;*

(iv) $\dim(\mathrm{Ker}(I-K)) = \dim(\mathrm{Ker}(I-K^*))$.

Proof.

1. Assume that $\mathrm{Ker}(I-K)$ is an infinite-dimensional subspace. By Gram-Schmdit process, one can construct an orthonormal sequence $(e_n)_{n\geq 1}$ in $\mathrm{Ker}(I-K)$. In this case, we have

$$e_n \;=\; K(e_n) \qquad \text{for all } n \geq 1$$

and

$$\|K(e_n) - K(e_m)\| \;=\; \|e_n - e_m\| \;=\; \sqrt{2} \qquad \text{for all } n \neq m.$$

Thus, K is not a compact operator.

2. Toward the proof of (ii), we observe that there exists a constant $\beta > 0$ such that

$$\|u - Ku\| \;\geq\; \beta \cdot \|u\| \qquad \text{for all } u \in (\mathrm{Ker}(I) - K)^\perp$$

Indeed, assume by a contradiction, there exists a sequence $u_n \in \mathrm{Ker}(I-K)^\perp$ such that

$$\|u_n\| \;=\; 1 \qquad \text{and} \qquad \|u_n - K(u_n)\| \;\leq\; \frac{1}{n}.$$

By extracting a subsequence and relabelling, we can assume that $u_n \rightharpoonup u$. Since K is compact, we then have that

$$\lim_{n\to\infty} K(u_n) \;=\; K(u) \qquad \Longrightarrow \qquad \lim_{n\to\infty} u_n \;=\; K(u).$$

By the uniqueness of weak convergence, we get

$$\lim_{n\to\infty} u_n \;=\; u \;=\; K(u), \qquad \|u\| \;=\; 1,$$

and this yields a contradiction since $u \in \mathrm{Ker}(I-K) \cap \mathrm{Ker}(I-K)^\perp$.

Consider a sequence $(v_n)_{n\geq 1} \subseteq \mathrm{Range}(I-K)$ which converges to v. To show that $\mathrm{Range}(I-K)$ is closed, we will find u such that $u - K(u) = v$. Since $v_n \in \mathrm{Range}(I-K)$, it holds

$$v_n \;=\; u_n - K(u_n) \qquad \text{for all } n \geq 1.$$

Set $V := \text{Ker}(I - K)$ a closed vector subspace of H. For every n, we have

$$z_n \ := \ u_n - \pi_V(u_n) \ \in \ \big[\text{Ker}(I-K)\big]^{\perp}, \qquad v_n \ = \ z_n - K(z_n).$$

From the above estimate, we have

$$\|v_m - v_n\| \ \geq \ \beta \cdot \|z_m - z_n\| \qquad \text{for all } m, n \geq 1$$

and this implies that $(z_n)_{n\geq 1}$ is a Cauchy sequence. Set $u := \lim_{n\to\infty} z_n$, we get

$$u - K(u) \ = \ \lim_{z\to\infty} (z_n - K(z_n)) \ = \ \lim_{n\to\infty} v_n \ = \ v.$$

Since both $\text{Range}(I-K)$ and $\text{Ker}(I-K^*)^{\perp}$ are closed, (ii) holds if any only if

$$[\text{Range}(I-K)]^{\perp} \ = \ \text{Ker}(I-K^*).$$

We have

$$\begin{aligned}
\text{Ker}(I-K^*) \ &= \ \{x \in H : (I-K^*)(x) = 0\} \\
&= \ \{x \in H : \langle y, (I-K^*)(x)\rangle \ \text{ for all } y \in H = 0\} \\
&= \ \{x \in H : \langle (I-K)(y), x\rangle \ \text{ for all } y \in H = 0\} \ = \ [\text{Range}(I-K)]^{\perp}.
\end{aligned}$$

3. Assume that $\text{Ker}(I-K) = 0$. If Range(I – K) $\neq H$ then we consider

$$H_n \ = \ (I-K)^n(H) \qquad \text{for all } n \geq 1.$$

By induction and the injective properties of $I - K$, we have that H_n is a closed subspace of H and

$$H \ \supset \ H_1 \ \supset \ H_2 \ \supset \ \cdots$$

Thus, for each $n \geq 1$, we can choose a unit vector $e_n \in H_n \cap H_{n+1}^{\perp}$. In this case, for every $m < n$, we have

$$K(e_m) - K(e_n) \ = \ e_m + z_m \qquad \text{with} \qquad z_m \ = \ (I-K)(e_n - e_m) - e_n \ \in \ H_{m+1}$$

and this implies that

$$\|K(e_m) - K(e_n)\| \ \geq \ \|e_m\| \ = \ 1.$$

In particular, K is not a compact operator and this yields a contradiction.

4. Assume that $\text{Range}(I-K) = H$ then by Lemma 4.22 have have

$$\text{Ker}(I-K^*) \ = \ [\text{Range}(I-K)]^{\perp} \ = \ H^{\perp} \ = \ \{0\}.$$

Since K^* is compact, by the previous step we have that $\text{Range}(I-K^*) = H$. By by Lemma 4.22, we get

$$\text{Ker}(I-K) \ = \ [\text{Range}(I-K^*)]^{\perp} \ = \ H^{\perp} \ = \ \{0\}$$

and this complete the proof of (iii).

5. To obtain (iv), we first show that

$$\dim(\mathrm{Ker}(I-K)) \;\geq\; \dim\big(\mathrm{Range}(I-K)^{\perp}\big).$$

Indeed, assume by a contradiction that

$$\dim(\mathrm{Ker}(I-K)) \;<\; \dim\big(\mathrm{Range}(I-K)^{\perp}\big). \tag{5.1}$$

Then there exists a linear map $A : \mathrm{Ker}(I-K) \to \mathrm{Range}(I-K)^{\perp}$ which is one-to-one but not onto. We extend A to a linear map $A : H \to \mathrm{Range}(I-K)^{\perp}$ such that

$$A(u) \;=\; 0 \qquad \text{for all } u \in [\mathrm{Ker}(I-K)]^{\perp}.$$

Observe that A is compact so is $K + A$. We show that $I - (K + A)$ is one-to-one. Indeed, for any $u \in H$, we write

$$u \;=\; u_1 + u_2, \qquad u_1 \in \mathrm{Ker}(I-K), \qquad u_2 \;=\; [\mathrm{Ker}(I-K)]^{\perp}.$$

Then

$$[I-(K+A)](u) \;=\; (I-K)(u_2) - A(u_1) \;\in\; \mathrm{Range}(I-K) \oplus \mathrm{Range}(I-K)^{\perp}.$$

From this observation, $u \in \mathrm{Ker}(I - (K + A))$ if only if

$$(I-K)(u_2) \;=\; A(u_1) \;=\; 0$$

and this yields $u_1 = u_2 = 0$. From (iii), it holds

$$\mathrm{Range}(I-(K+A)) \;=\; H.$$

Since A is not onto, there exists $v \in \mathrm{Range}(I-K)^{\perp}$ but $v \notin \mathrm{Range}(A)$. Thus, the equation

$$u - K(u) - A(u) \;=\; v$$

has no solution and this yield a contradiction.

6. Recalling that $\mathrm{Range}(I-K^*)^{\perp} = \mathrm{Ker}(I-K)$, we then get

$$\dim(\mathrm{Ker}(I-K^*)) \;\geq\; \mathrm{Range}(I-K^*)^{\perp} \;=\; \dim(\mathrm{Ker}(I-K)).$$

Interchanging the roles K and K^*, we obtain the opposite inequality. □

Remark 5.2 *Consider the linear equation*

$$u - K(u) \;=\; f \tag{5.2}$$

with K linear compact operator. Then *(i) If* $\mathrm{Ker}(I-K) = \{0\}$ *then for any* $f \in H$*, the linear equation has a unique solution.*

(ii) If $\mathrm{Ker}(I-H) \neq \{0\}$ *then the homogenous linear equation* $u - K(u) = 0$ *admits a nontrivial solution. In this case, (5.2) has solutions if any only if* $f \in \mathrm{Ker}(I-K^*)^{\perp}$*, i.e.,*

$$\langle f, u\rangle \ = \ 0 \qquad \textit{for all } u \in \mathrm{Ker}(I-K^*).$$

Definition 5.3 *Let* $\Lambda : H \to H$ *be a bounded linear operator. The resolvent and the spectrum of* Λ *are denoted by*

$$\rho(\Lambda) \ := \ \{\eta \in \mathbb{R} : \eta I - \Lambda \text{ is bijective}\} \qquad \text{and} \qquad \sigma(\Lambda) \ = \ \mathbb{R}\backslash\rho(\Lambda).$$

The point spectrum and the essential spectrum of Λ *is*

$$\sigma_p(\Lambda) \ = \ \{\eta \in \mathbb{R} : \eta I - \Lambda \text{ is not one to one}\} \qquad \text{and} \qquad \sigma_e(\Lambda) \ = \ \sigma(\Lambda)\backslash\sigma_p(\Lambda).$$

For every $\eta \in \sigma_p(\Lambda)$, there exists a nonzero vector $w \in H$ such that

$$\Lambda(w) \ = \ \eta w.$$

In this case, η is called an *eigenvalue* and w is an associated *eigenvector*.

Theorem 5.4 (Spectrum of a compact operator) *Let* H *be infinite-dimensional Hilbert space, and let* $K : H \to H$ *be a compact linear operator. Then*

(i) $\sigma(K) = \sigma_p(K) \cup \{0\}$.

(ii) Either $\sigma_p(K)$ *is finite or else* $\sigma_p(K) = \{\lambda_n : n \geq 1\}$ *such that* $\lim_{n\to\infty} \lambda_n = 0$.

Proof. 1. To prove (i), we first show that $0 \in \sigma(K)$. Assume by a contradiction that $0 \notin \sigma(K)$. Then the linear operator K is bijective. By the open mapping theorem, the linear map K^{-1} is continuous. Thus, $I = K^{-1} \circ K$ is compact operator and this yields a contradiction.

Now, given $\lambda \in \sigma(K)\backslash\{0\}$, we need to show that $\lambda \in \sigma_p(K)$. Assume by a contradiction that $\lambda \notin \sigma_p(K)$ then by the Fredholm alternative, one has

$$\mathrm{Ker}(\lambda I - K) \ = \ \{0\} \qquad \Longrightarrow \qquad \mathrm{Range}(\lambda I - K) \ = \ H.$$

Thus, $\lambda I - K$ is bijective and this yields a contradiction.

2. Assume that $\sigma_p(K)$ is not finite. Let $(\lambda_n)_{n\geq 1}$ be a sequence of distinct eigenvalue in $\sigma_p(K)$ such that $\lim\limits_{n\to\infty} \lambda_n = \lambda$. We claim that $\lim\limits_{n\to\infty} \lambda_n = 0$. Indeed, let w_n be an associated eigenvector

$$K(w_n) \ = \ \lambda_n \cdot w_n \qquad \text{for all } n \geq 1.$$

Set $H_n := \text{span}\{w_1, \cdots, w_n\}$. Since eigenvectors corresponding to distinct eigenvalue are linearly independent, we get that $H_n \subset H_{n+1}$. Moreover, for every $n \geq 1$, it holds

$$(K - \lambda_n I)(H_n) \ \subset \ H_{n-1}.$$

For each $n \geq 2$, we can choose a unit vector $e_n \in H_n \cap H_{n-1}^{\perp}$. For every $m < n$, we have

$$K(e_n) - K(e_m) \ = \ [(I - K)(e_m - e_n) - \lambda_m e_m] + \lambda_n e_n \ \in \ H_{n-1} + H_{n-1}^{\perp}$$

and this implies that

$$\|K(e_n) - K(e_m)\| \ = \ |\lambda_n| \cdot \|e_n\| \ \geq \ |\lambda|.$$

If $|\lambda| > 0$ then the operator K is not compact and this yields a contradiction. Thus, $\lambda = 0$ and $\lim_{n\to\infty} \lambda_n = 0$.

3. To conclude, we show that for any $r > 0$, the set $\sigma_p(K) \cap [r, \infty[$ is finite. Indeed, assume by a contradiction, one can fine a sequence of $(\lambda_n)_{n\geq 1} \subseteq \sigma_p(K)$ such that

$$r \ \leq \ \lambda_n \ \leq \ \|\Lambda\|_\infty \qquad \text{and} \qquad \lambda_n \neq \lambda_m.$$

In particular, $(\lambda_n)_{n\geq 1}$ admits a subsequence which is convergent and thus converges to 0. This yields a contradiction. Similarly, we can show that the set $\sigma_p(K) \cap]-\infty, -r]$ is finite. Therefore, the set $\sigma_p(K)$ is at most countable. □

Lemma 5.5 (Bounds on the spectrum of a symmetric operator) *Let $\Lambda : H \to H$ be a linear bounded operator on the Hilbert space H. Assume that Λ is selfadjoint, i.e.,*

$$\langle \Lambda(x), y \rangle \ = \ \langle x, \Lambda(y) \rangle \qquad \text{for all } x, y \in H.$$

Define

$$m \ := \ \inf_{\|u\|=1} \langle \Lambda(u), u \rangle, \qquad M \ := \ \sup_{\|u\|=1} \langle \Lambda(u), u \rangle$$

Then,

$$m, M \in \sigma(\Lambda) \subseteq [m, M] \qquad \text{and} \qquad \|\Lambda\|_\infty \ = \ \max\{-m, M\}.$$

Proof. 1. Let us first show that the spectrum $\sigma(\Lambda)$ is contained in $[m, M]$. For any $\eta > M$, we have that the linear operator $\eta I - \Lambda$ is strictly positive since

$$\langle (\eta I - \Lambda)(u), u \rangle \ \geq \ (\eta - M) \cdot \|u\|^2 \qquad \text{for all } u \in H.$$

By Lax-Milgram theorem, the linear operator $(\eta I - \Lambda)$ is bijective. In particular, the set $]M, \infty[$ is a subset of the resolvent set of Λ. Similarly, for any $\eta < M$, the linear operator $\Lambda - \eta I$ is strictly positive and bijective since

$$\langle (\Lambda - \eta I)(u), u \rangle \ \geq \ (m - \eta) \cdot \|u\|^2 \qquad \text{for all } u \in H.$$

Thus, the set $]-\infty, m[$ is a subset of the resolvent set of Λ and this yields

$$\sigma(\Lambda) \;=\; \mathbb{R}\backslash\rho(\Lambda) \;\subseteq\; [m, M].$$

2. Without lose of generality, we shall assume that $|m| \leq M$. Otherwise, one can replace Λ by $-\Lambda$. We will show that

$$\|\Lambda\|_\infty \;=\; \sup_{\|u\|\leq 1} \|\Lambda(u)\| \;=\; M.$$

It is sufficient to show that $\|\Lambda\|_\infty \leq M$. For every $u, v \in H$, we have

$$\begin{aligned}
4\langle \Lambda u, v\rangle \;&=\; \langle \Lambda(u+v), u+v\rangle - \langle \Lambda(u-v), u-v\rangle \\
&\leq\; M\|u+v\|^2 - m\|u-v\|^2 \;\leq\; M\cdot\left(\|u+v\|^2 + \|u-v\|^2\right) \\
&\leq\; 2M\cdot\left(\|u\|^2 + \|v\|^2\right).
\end{aligned}$$

Choosing $v = \dfrac{\|u\|}{\|\Lambda(u)\|}\cdot\Lambda(u)$, we get

$$\|\Lambda(u)\| \;\leq\; M\|u\| \qquad \text{for all } u \notin \mathrm{Ker}(\Lambda)$$

and this yields $\|\Lambda\|_\infty \leq M$.

3. Finally, we claim that $M \in \sigma(\Lambda)$. Indeed, let $(u_n)_{n\geq 1}$ be a sequence such that

$$\|u_n\| \;=\; 1 \qquad \text{and} \qquad \lim_{n\to\infty} \langle \Lambda(u_n), u_n\rangle \;=\; M.$$

We have

$$\|(M\cdot I - \Lambda)(u_n)\|^2 \;=\; M^2 + \|\Lambda(u_n)\|^2 - 2M\Lambda(u_n), u_n\rangle \;\leq\; 2M(1 - \Lambda(u_n), u_n\rangle)$$

and this implies that

$$\lim_{n\to\infty} \|(M\cdot I - \Lambda)(u_n)\| \;=\; 0.$$

Thus, the operator $M\cdot I - \Lambda$ can not have a bounded inverse and thus it is not bijective. □

Give a symmetric $n\times n$ matrix A, one can choose an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors of A. The following theorem shows that the results remains valid for compact symmetric operator.

Theorem 5.6 (Hilbert-Schimit) *Given a separable real Hilbert space H, let $K : H \to H$ be a compact and selfadjoint linear operator. Then there exists a countable orthonormal basis of H consisting of eigenvectors of H.*

Answer. 1. If H is a finite dimensional space then this is a classical result in linear algebra. Assume that $\dim(H) = \infty$. Let $\eta_0 = 0$ and $\{\eta_1, \eta_2, \dots\}$ be the set of all nonzero eigenvalues of K. Consider the eigenspaces

$$H_0 \ = \ \mathrm{Ker}(K) \qquad \text{and} \qquad H_n \ = \ \mathrm{Ker}(K - \eta_n I)$$

By the Fredholm theorem, we have that

$$\dim(H_n) \ < \ \infty \qquad \text{for all } n \geq 1.$$

On the other hand, for every $u \in H_m$ and $v \in H_n$ with $n \neq m$, it holds

$$\eta_m \langle u, v\rangle \ = \ \langle Ku, v\rangle \ = \ \langle u, Kv\rangle \ = \ \eta_n \langle u, v\rangle$$

and this yields $\langle u, v\rangle = 0$. Thus, the subspace H_m and H_n are orthogonal.

2. We claim that the vector subspace

$$\widetilde{H} \ = \ \left\{ \sum_{k=0}^{N} \alpha_k u_k : u_k \in H_k, \alpha_k \in \mathbb{R} \right\}.$$

is dense in H. Let us first show that

$$\widetilde{H}^{\perp} \ \subseteq \ \mathrm{Ker}(K) \ = \ H_0. \tag{5.3}$$

Notice that $K(\widetilde{H}) \subset \widetilde{H}$. For any $u \in \widetilde{H}^{\perp}$, one has

$$\langle K(u), v\rangle \ = \ \langle u, K(v)\rangle \ = \ 0 \qquad \text{for all } v \in \widetilde{H}$$

and this shows that $K(\widetilde{H})^{\perp} \subseteq \widetilde{H}^{\perp}$. Let $\widetilde{K}$ be the restriction of K to the subspace $\widetilde{H}^{\perp}$. Clearly, $\widetilde{K}$ is a compact and symmetric operator. Thus, the previous lemma yields

$$\|\widetilde{K}\|_{\infty} \ = \ \sup_{\|u\|=1, u \in \widetilde{H}^{\perp}} |\langle \widetilde{K}(u), u\rangle| \ = \ M.$$

If $M > 0$ then $\lambda = -M$ or $\lambda = M$ is in the spectrum of $\widetilde{K}$, i.e. there exists a unit vector $w \in \widetilde{H}^{\perp}$ such that

$$\widetilde{K}(w) \ = \ K(w) \ = \ \lambda \cdot w \qquad \Longrightarrow \qquad w \ \in \ \widetilde{H}.$$

This is contradiction. Thus, $\|\widetilde{K}\|_{\infty} = 0$ and this yields (5.3). In turn,

$$\widetilde{H}^{\perp} \ \subseteq \ H_0 \cap H_0^{\perp} \ = \ \{0\}.$$

and thus $\widetilde{H}$ is dense in H.

3. For each $k \geq 1$, H_n admits an orthonormal basis $\mathcal{B}_n = \{e_{n,1}, \dots, a_{n,N(n)}\}$. Since H is separable, the space H_0 admits a countable orthonormal basis $\mathcal{B}_0 = \{e_{0,1}, e_{0,2} \dots\}\}$. Hence $\mathcal{B} = \bigcup_{n=0}^{\infty}$ is an orthonormal basis of H. □

Remark 5.7 *Let* $\{\mathbf{w}_1, \mathbf{w}_2, \dots\}$ *be an orthonormal basis of a real Hilbert space* H, *consisting eigenvector of a linear, symmetric operator* K. *Let* $\lambda_1, \lambda_2, \dots$ *be the corresponding eigenvalues. If* $1 \notin \sigma(K)$ *then for any given* $f \in H$, *consider the equation*

$$u - K(u) = f$$

admits a unique solution u *such that*

$$u = \sum_{k=1}^{\infty} \frac{\langle f, \mathbf{w}_k \rangle}{1 - \lambda_k} \cdot \mathbf{w}_k.$$

References

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