Lecture note on Analysis II

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1 Why study measure theory?

1. **Passing limit under integral sign**: There are many reasons to pass limits under integrals. Some examples:

   (a) *Differentiation under the integral sign*: under what conditions can we write
   $$\frac{\partial}{\partial y}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx.$$
   Both integrals and derivatives are defined in terms of limits, so this is really a question of whether we can interchange limits.

   (b) Given a open set $\Omega \subseteq \mathbb{R}^d$ and a function $g : \partial\Omega \to \mathbb{R}$, consider the minimization problem
   $$\inf_f J(f) := \int_\Omega |\nabla f|^2\,dx$$
   subject to
   $$f \;=\; g \;\text{ on } \partial\Omega, \qquad f : \Omega \to \mathbb{R}.$$
   Basic question: *Is there a function f^* such that $J(f^*) = \inf J$?*

   The direct method is to consider a sequence of functions f_n so that
   $$\lim_{n\to\infty} J(f_n) \;=\; \inf_f J.$$
   Show that the sequence f_n has a limit f^* and that $\lim J(f_n) = J(f^*)$. One important step is understanding how we can pass a limit under an integral.

   (c) *Generalized solutions to differential equations:* Consider the problem of describing the electrical potential induced by a point charge. Heuristically (and scaling out physical constants), we are trying to solve
   $$\Delta f = \delta_0$$
   where f is the electrical potential and δ_0 is a Dirac mass at zero. However, how do we rigorously define the Dirac mass at zero? One can think of this as a limit of masses charged on a vanishing set, namely
   $$\delta_0 \;=\; \lim_{n\to\infty} C r^{-d}\chi_{B(0,r)}.$$

How do we accurately capture this "limiting process" when solving the partial differential equation? This often involves an integral formulation, and limits have to be passed under an integral. In other words, we often want to use approximating sequences of functions, and be able to pass limits under integrals (or into differential equations) rigorously.

2. **Probability**

   (a) How does one define Brownian motion? We need a rigorous way to define a probability measure over trajectories with the appropriate properties, and measure theory lets us do that.

   (b) How does one rigorously prove limit theorems? Approximating sequences, passing derivatives and limits under integrals.

   (c) How do we define conditional probabilities? How do we unify discrete and continuous probability? How do we give an axiomatic formulation of probability? Measure theory...

3. **Generalized Calculus and geometry**

   (a) How do we define the size of a set? Volume/area/path length? Fractals? Need a rigorous notion of size. For example, what is the area in $\mathbb{R}$ of the rational numbers? What is the right notion of area of the Sierpinski triangle, or Koch snowflake? This depends delicately on how we define our notions of length. For example in $\mathbb{R}^2$, we could define

$$\text{Area}_1(E) = \inf\left\{\sum_{i=1}^{m} area(R_n) : R_n \text{ is an open rectangle, } E \subset \bigcup_{n=1}^{m} R_n\right\}$$
$$\text{Area}_2(E) = \inf\left\{\sum_{i=1}^{\infty} area(R_n) : R_n \text{ is an open rectangle, } E \subset \bigcup_{n=1}^{\infty} R_n\right\}.$$

   Suppose that $E = \mathbb{Q}^2 \cap (0,1)^2$. One can show that

$$\text{Area}_1(E) \;=\; 1 \;\neq\; 0 \;=\; \text{Area}_2(E).$$

   (b) How do we define generalized derivatives? For example, consider the function $\chi_{(0,\infty)}$. What is the derivative of this function, and how should we think about it? For the function $f(x) = |x|$ how should we think about ∇f and Δf? Measure theory gives us a rigorous way to think about these objects.

This course will mostly be focused on the Lebesgue integration: namely defining a measure and integration on $\mathbb{R}^d$. Here is its outline:

- Review of Riemann integration (to highlight some shortcomings)

- Measures: abstract properties for measuring size of sets.
- Construction of measures: outer measures, Lebesgue measure, Carathéodory extension
- Measurable functions (analog of continuous functions)
- Lebesgue integration
- $\mathbf{L}^p$ spaces (deeper properties)
- Product measures
- A tour of Calculus topics for measure theory

2 A review on Riemann Integration

2.1 Riemann integrable functions

Given a bounded interval $[a, b]$, let $f : [a, b] \to \mathbb{R}$ be a bounded function. Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, i.e.,

$$a = x_0 < x_1 < \ldots x_n - 1 < x_n = b.$$

For any $i \in \overline{1, n}$, denote by

$$m_i := \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad \text{and} \qquad M_i := \sup_{x \in [x_{i-1}, x_i]} f(x).$$

Definition 2.1 (Darboux sums) *For any P partition of $[a, b]$, the lower and upper Darboux sums for f with respect to P are defined by*

$$L(f, P) := \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}), \qquad \text{[Lower Darboux sum]}$$

$$U(f, P) := \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}). \qquad \text{[Upper Darboux sum]}$$

Let $\mathcal{P}_{[a,b]}$ be a collection of all partitions of the interval $[a, b]$. From the above definition, $L(f, \cdot)$ and $U(f, \cdot)$ satisfies the following properties.

Basic properties

(i) For any $P \in \mathcal{P}_{[a,b]}$, it holds

$$m_f \cdot (b - a) \le L(f, P) \le U(f, P) \le M_f \cdot (b - a)$$

with $M_f := \sup_{[a,b]} f(x)$ and $m_f := \inf_{[a,b]} f(x)$.

(ii) $L(f,\cdot)$ is increasing and $U(f,\cdot)$ is decreasing w.r.t to $P \in \mathcal{P}_{[a,b]}$, i.e.,

$$L(f,P) \;\le\; L(f,P') \qquad \text{and} \qquad U(f,P') \;\le\; U(f,P) \qquad \text{for all } P \subseteq P'.$$

(iii) For any $P, P' \in \mathcal{P}_{[a,b]}$, it holds

$$L(f,P) \;\le\; U(f,P').$$

Definition 2.2 *Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Then*

- *The lower Riemann integral of f over $[a,b]$ is*

$$(R)\underline{\int_a^b} f dx \;:=\; \sup_{P \in \mathcal{P}_{[a,b]}} L(f,P).$$

- *The upper Riemann integral of f over $[a,b]$ is*

$$(R)\overline{\int_a^b} f dx \;:=\; \inf_{P \in \mathcal{P}_{[a,b]}} U(f,P).$$

From (i) and (ii), it is clear that

$$m_f \cdot (b-a) \;\le\; (R)\underline{\int_a^b} f dx \;\le\; (R)\overline{\int_a^b} f dx \;\le\; M_f \cdot (b-a). \tag{2.1}$$

Definition 2.3 (Riemann integration (Bernhard Riemann 1826-1866)) *The bounded function f is said Riemann integrable if*

$$(R)\underline{\int_a^b} f dx \;=\; (R)\overline{\int_a^b} f dx.$$

In this case, the Riemann integration of f over $[a,b]$ is denoted by

$$(R)\int_a^b f dx \;:=\; (R)\underline{\int_a^b} f dx \;=\; (R)\overline{\int_a^b} f dx.$$

From (2.1), one has

$$m_f \cdot (b-a) \;\le\; (R)\int_a^b f dx \;\le\; M_f \cdot (b-a)$$

provided that f is Riemann integrable.

How to check that f is Riemann integrable?

It is clear that

- If f is a constant function on $[a,b]$, i.e. $f(x) = C$ for all $x \in [a,b]$, then f is Riemann integrable and

$$(R)\int_a^b f dx \;=\; C \cdot (b-a).$$

- Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$, if

$$f(x) \;=\; c_i \qquad \text{for all } x \in (x_i, x_{i+1}), i \in \overline{0, n-1},$$

then f is Riemann integrable and

$$(R)\int_a^b f dx \;=\; \sum_{i=0}^{n-1} c_i \cdot (x_{i+1} - x_i).$$

In general, from (i)-(iii), we have

$$0 \;\leq\; (R)\overline{\int_a^b} f dx - (R)\underline{\int_a^b} f dx \;\leq\; \inf_{P \in \mathcal{P}_{[a,b]}} \left[U(f,P) - L(f,P)\right]. \tag{2.2}$$

Theorem 2.4 (Darboux's integrability condition) *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists $P_\varepsilon \in \mathcal{P}_{[a,b]}$ such that*

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) \;\leq\; \varepsilon. \tag{2.3}$$

Proof. 1. Assume that (2.3) holds for all $\varepsilon > 0$. This implies that

$$\inf_{P \in \mathcal{P}_{[a,b]}} U(f,P) - L(f,P) \;=\; 0.$$

From (2.2), we have

$$(R)\overline{\int_a^b} f dx \;=\; (R)\underline{\int_a^b} f dx$$

and it yields the Riemann integrability of f over $[a,b]$.

2. Assume that f is Riemann integrable, we have

$$\alpha =: (R)\int_a^b f dx \;=\; \sup_{P \in \mathcal{P}_{[a,b]}} L(f,P) \;=\; \inf_{P \in \mathcal{P}_{[a,b]}} U(f,P).$$

For any $\varepsilon > 0$, there exists $P_\varepsilon^L, P_\varepsilon^U \in \mathcal{P}_{[a,b]}$ such that

$$U\left(f, P_\varepsilon^L\right) - \frac{\varepsilon}{2} \;<\; \alpha \;<\; L\left(f, P_\varepsilon^L\right) + \frac{\varepsilon}{2}$$

Set $P_\varepsilon := P_\varepsilon^L \cup P_\varepsilon^U$. By the monotonicity of $L(f,\cdot)$ and $U(f,\cdot)$, we have

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) \ \leq \ U\left(f,P_\varepsilon^U\right) - L\left(f,P_\varepsilon^L\right) \ < \ \varepsilon.$$

Thus, (2.3) holds. □

As a consequence of Theorem 2.4, one obtains the following results.

Corollary 2.5 *If $f : [a,b] \to \mathbb{R}$ is continuous then f is Riemann integrable.*

Proof. Since f is continuous and $[a,b]$ is a compact set, it is bounded by some constant M and uniformly continuous. Thus, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \ < \ \frac{\varepsilon}{(b-a)} \qquad \text{for all } |x-y| \leq \delta.$$

Let $P_\varepsilon = \{x_0, x_1, \ldots, x_n\}$ be in $\mathcal{P}_{[a,b]}$ with n sufficiently large such that

$$|x_i - x_{i-1}| \ = \ \frac{b-a}{n} \ < \ \delta \qquad \text{for all } i \in \overline{1,n}.$$

We then have

$$M_i - m_i \ = \ \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \ < \ \frac{\varepsilon}{b-a}.$$

Thus,

$$\begin{aligned} U(f,P_\varepsilon) - L(f,P_\varepsilon) \ &= \ \sum_{i=1}^{n} (M_i - m_i)\cdot(x_i - x_{i-1}) \\ &\leq \ \frac{\varepsilon}{b-a}\cdot(b-a) \ = \ \varepsilon. \end{aligned}$$

The Darboux's integrability condition implies that f is integrable. □

Corollary 2.6 *If $f : [a,b] \to \mathbb{R}$ is monotone then f is Riemann integrable.*

Proof. Without loss of generality, assume that f is increasing. In this case, f is bounded and

$$f(a) \ \leq \ f(x) \ \leq \ f(b) \qquad x \in [a,b].$$

If $f(b) = f(a)$ then f is constant in [a,b] and thus it is Riemann integrable. Otherwise, given any $\varepsilon > 0$, let $P_\varepsilon = \{x_0, x_1, \ldots, x_n\}$ be in $\mathcal{P}_{[a,b]}$ be such that

$$x_i - x_{i-1} \ \leq \ \frac{\varepsilon}{f(b)-f(a)} \qquad \text{for all } i \in \overline{1,n}.$$

We compute

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) \ = \ \sum_{i=1}^{n} (M_i - m_i)\cdot(x_i - x_{i-1}) \ \leq \ \frac{\varepsilon}{f(b)-f(a)}\cdot\sum_{i=1}^{n}(M_i - m_i)$$

$$
\begin{aligned}
&= \frac{\varepsilon}{f(b)-f(a)} \cdot \sum_{i=1}^{n}[f(x_i)-f(x_{i-1})] \;=\; \frac{\varepsilon \cdot (f(x_n)-f(x_0))}{f(b)-f(a)} \\
&= \frac{\varepsilon}{f(b)-f(a)} \cdot (f(b)-f(a)) \;=\; \varepsilon,
\end{aligned}
$$

and the Darboux's integrability condition implies that f is integrable. □

Let us collect some basic properties of Riemann integrable functions.

Basic properties: *Give two Riemann integrable functions f and g on $[a, b]$, the following holds*

(i) f is integrable over $[c, d] \subset [a, b]$. Moreover,

$$
(R)\int_a^b f(x)dx \;=\; (R)\int_a^c f(x)dx + (R)\int_c^b f(x)dx.
$$

(ii) For any $\alpha, \beta \in \mathbb{R}$, it holds

$$
(R)\int_a^b \alpha f + \beta g \; dx \;=\; \alpha \cdot (R)\int_a^b f \; dx + \beta \cdot (R)\int_a^b g \; dx.
$$

Theorem 2.7 (Fundamental theorem in calculus) *Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Suppose that $F : [a, b] \to \mathbb{R}$ is continuous and*

$$
F'(x) \;=\; f(x) \qquad \text{for all } x \in (a, b).
$$

Then,

$$
F(b) - F(a) \;=\; (R)\int_a^b f(x)\; dx.
$$

Proof. Since f is Riemann integrable, we need to show that

$$
L(f, P) \;\leq\; F(b) - F(a) \;\leq\; U(f, P) \qquad \text{for all } P \in \mathcal{P}_{[a,b]}. \tag{2.4}
$$

Assume that $P = \{x_0, x_1, \ldots, x_n\}$. Using the mean value theorem, we compute

$$
\begin{aligned}
F(b) - F(a) &= \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \\
&= \sum_{i=1}^{m} F'(\xi_i) \cdot (x_i - x_{i-1}) \;=\; \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})
\end{aligned}
$$

for some $\xi_i \in (x_{i-1}, x_i)$. For all $i \in \overline{1, n}$, it holds

$$
m_i := \inf_{[x_{i-1}, x_i]} f(x) \;\leq\; f(\xi_i) \;\leq\; \sup_{[x_{i-1}, x_i]} f(x) := M_i.
$$

Thus,

$$\sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}) \leq F(b) - F(a) \;\leq\; \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1})$$

and it yields (2.4). □

Problem 1: Suppose that $f : [0,1] \to \mathbb{R}$ is Riemann integrable. Prove that

$$(R)\int_0^1 f(x)dx \;=\; \lim_{n\to\infty}\left(\frac{1}{n}\cdot\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\right).$$

Problem 2: Suppose that $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Prove that $|f|$ is Riemann integrable and

$$\left|(R)\int_a^b f(x)dx\right| \;\leq\; (R)\int_a^b |f(x)|dx.$$

2.2 Non-Riemann integrable functions

In this subsection, we prove some examples to show that the Riemann integration does not handle

- Functions with infinite discontinuities;
- Unbounded functions;
- Limits.

Example 2.1 (Dirichlet functions) *Let $f : [0,1] \to \mathbb{R}$ be such that*

$$f(x) \;=\; \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0,1] \\ \\ 0 & \text{if} \quad x \in [0,1] \backslash \mathbb{Q}. \end{cases}$$

It is clear that f is bounded in $[0,1]$ but discontinuous at every $x \in [0,1]$. Since $\mathbb{Q}$ is dense in $[0,1]$, one can show that

$$L(f,P) \;=\; 0 \qquad \text{and} \qquad U(f,P) \;=\; 1 \qquad \textit{for all } P \in \mathcal{P}_{[0,1]}.$$

This implies that

$$(R)\underline{\int_0^1} f dx \;=\; 0 \;\neq\; 1 \;=\; (R)\overline{\int_0^1} f dx$$

and f is not Riemann integrable.

Remark 2.8 *If $f : [a,b] \to \mathbb{R}$ is bounded and continuous at all but a finite points then f is Riemann integrable.*

Problem 3: *Is there a bounded and non-Riemann integrable function with countably infinite discontinuities?*

Example 2.2 *Consider the function $f : [0,1] \to \mathbb{R}$ such that*

$$f(0) \;=\; 0 \qquad \text{and} \qquad f(x) \;=\; \frac{1}{\sqrt{x}} \qquad \textit{for all } x \in (0,1].$$

It is clear that

$$U(f,P) \;=\; +\infty \qquad \textit{for all } P \in \mathcal{P}_{[0,1]}$$

and thus

$$R\overline{\int_0^1} f\ dx \;=\; +\infty.$$

However,

$$L(f,P) \;\le\; \lim_{a\to 0+}\int_a^1 f(x)dx \;=\; \lim_{a\to 0+}\int_a^1 \frac{1}{\sqrt{x}}dx \;=\; 2$$

and this implies that

$$R\underline{\int_0^1} f\ dx \;\le\; 2.$$

Thus, Riemann integration does not work with unbounded functions.

Example 2.3 *Assume that*

$$\mathbb{Q}\cap[0,1] \;=\; \{q_1, q_2, \ldots, q_n, \ldots\}.$$

For any $n \in \mathbb{Z}^+$, consider the function $f_n : [0,1] \to \mathbb{R}$ such that

$$f_n(x) \;=\; \begin{cases} 1 & \text{if} \quad x \in \{q_1, q_2, \ldots, q_n\} \\ \\ 0 & \text{if} \quad x \notin \{q_1, q_2, \ldots, q_n\}. \end{cases}$$

Since f_n is bounded and discontinuous at finite points, it is Riemann integrable. Moreover, one can see that

$$(R)\int_0^1 f_n\ dx \;=\; 0.$$

On the other hand, f_n converges point-wise to the function f defined in the example 1 which is non-Riemann integrable. Thus, Riemann integration does not work well with pointwise limits.

Problem 4: Suppose that $f_n : [0,1] \to \mathbb{R}$ are bounded, Riemann integrable functions, and that f_n converges uniformly to a function f. Then show that f is Riemann integrable, and that the integral of f_n converges to the integral of f.

We need to construct a theory which can remedy the problems illustrated in the last few examples. More precisely, the notion of the length of an interval to a larger collection of subsets of $\mathbb{R}$. This will lead us to measures.

3 Measures

3.1 σ-algebras and measurable sets

3.1.1 Sets

Let X be a non-empty set. Denote by

$$\mathcal{P}(X) \ = \ 2^X \ = \ \{A : A \subseteq X\}.$$

For any given $A \in \mathcal{P}(X)$, the complement of A is

$$A^c \ = \ X \backslash A \ = \ \{x \in X : x \notin A\}.$$

It is clear that

$$A \backslash B \ = \ A \cap B^c \qquad \text{for all } A, B \in \mathcal{P}(X).$$

De Morgan identity. *Let $\{A_n\}_{n\geq 1}$ be a sequence of sets in $\mathcal{P}(X)$. The followings hold*

$$\left(\bigcup_{i=1}^{n} A_n\right)^c \ = \ \bigcap_{n=1}^{\infty} A_n^c \qquad \text{and} \qquad \left(\bigcap_{i=1}^{n} A_n\right)^c \ = \ \bigcup_{i=1}^{n} A_n^c.$$

Definition 3.1 (Limit of a sequence of sets) *Given $\{A_n\}_{n\geq 1}$ a sequence of sets in $\mathcal{P}(X)$, we define*

$$\limsup_{n\to\infty} \ A_n \ := \ \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \qquad \text{and} \qquad \liminf_{n\to\infty} \ A_n \ := \ \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k.$$

If $\limsup_{n\to\infty} \ A_n = \liminf_{n\to\infty} \ A_n = A$ *then we set*

$$A \ := \ \lim_{n\to\infty} \ A_n.$$

Notice that the set

$$\limsup_{n\to\infty} \ A_n \ = \ \{x \in X : x \text{ belongs to infinitely many of the } \mathrm{A_n}\}.$$

Basic properties:

(a) $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

(b) If $(A_n)_{n\geq 1}$ is a increasing sequence, i.e. $A_n \subseteq A_{n+1}$, then

$$\lim_{n\to\infty} A_n \;=\; \bigcup_{n=1}^{\infty} A_n.$$

If $(A_n)_{n\geq 1}$ is a decreasing sequence, i.e. $A_n \supseteq A_{n+1}$, then

$$\lim_{n\to\infty} A_n \;=\; \bigcap_{n=1}^{\infty} A_n.$$

3.1.2 Algebras and σ-Algebra

Definition 3.2 *A nonempty subset $\mathcal{A}$ of $\mathcal{P}(X)$ is called an algebra in X if*

(i) $\varnothing, X \in \mathcal{P}(X)$

(ii) If $A, B \in \mathcal{A}$ then $A \bigcup B \in \mathcal{A}$

(iii) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.

For the definition, it is easy to see that

- If $A, B \in \mathcal{A}$ then
$$A\Delta B \;:=\; (A\backslash B)\bigcup(B\backslash A) \;\in\; \mathcal{A}.$$

- If $A_1, A_2, \ldots, A_n \in \mathcal{A}$ then
$$\text{both } \bigcap_{i=1}^{n} A_i \quad \text{and} \quad \bigcup_{i=1}^{n} A_i \;\text{ are in } \mathcal{A}.$$

Notice that the above statement does not hold for infinite sets in general.

Example 3.1 *Assume that $X = [0,1)$. The class $\mathcal{A}$ consisting $\varnothing$ and all sets of form*

$$\mathcal{A} \;=\; \bigcup_{i=1}^{n}[a_i, b_i) \qquad \text{with} \qquad 0 \leq a_1 \leq b_1 \leq \cdots \leq a_n \leq b_n \;\leq\; 1$$

is an algebra in $[0,1)$. However, if we consider

$$A_n \;=\; [1/n, 1) \;\in\; \mathcal{A}_n \qquad \textit{for all } n \geq 1,$$

then the union of A_n

$$\bigcup_{n=1}^{\infty} A_n \;=\; (0,1)$$

is not in $\mathcal{A}$.

Definition 3.3 *A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is an σ-algebra if*

(i) $\mathcal{A}$ is an algebra.

(ii) For any sequence of set $(A_n)_{n\geq 1} \subseteq \mathcal{A}$, it holds

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

In this case, we say that $(X, \mathcal{A})$ is a measurable space and $A \in \mathcal{A}$ is a measurable set.

By the De Morgan identity, one can see that if $\mathcal{A} \subseteq \mathcal{P}(X)$ is an σ-algebra then

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{A} \quad \text{for all } (A_n)_{n\geq 1} \subset \mathcal{P}(X).$$

Example 3.2 *Given a nonempty set X, then*

- *$\{\varnothing, X\}$ is the smallest σ-algebra on X.*
- *$\mathcal{P}(X)$ is the largest σ-algebra on X.*
- *The collection all subsets E of X such that E is countable or E^c is countable is a σ-algebra on X.*

Lemma 3.4 *Let $\mathcal{A}$ be an algebra on X. If*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$$

for all sequence $\{A_n\}_{n\geq 1}$ of mutually disjoint elements of $\mathcal{A}$ then $\mathcal{A}$ is a σ-algebra.

Proof. For any sequence $\{B_n\}_{n\geq 1}$ in $\mathcal{A}$, we need to show that

$$\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}.$$

Observing that

$$B_1 \bigcup B_2 = B_1 \bigcup [B_2 \backslash B_1] \quad \text{and} \quad B_1 \cap [B_2 \backslash B_1] = \varnothing,$$

we construct a sequence $(A_n)_{n\geq 1} \subset \mathcal{A}$ by induction

$$A_1 = B_1 \qquad \text{and} \qquad A_{n+1} = B_{n+1} \backslash \left(\bigcup_{i=1}^{n} A_i \right) \qquad \text{for all } n \geq 1.$$

One can see that $\{A_n\}_{n\geq 1}$ of mutually disjoint elements of $\mathcal{A}$ and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n,$$

and this yields

$$\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}.$$

$\square$

By the Definition 3.3, we can easily show that

Lemma 3.5 *Let $\mathcal{K}$ be a subset of $\mathcal{P}(X)$. Then*

$$\sigma(\mathcal{K}) := \bigcap \big\{\mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{K} \subset \mathcal{A} \text{ and } \mathcal{A} \text{ is a } \sigma - \text{algebra}\big\}$$

is the smallest σ-algebra on X which contains $\mathcal{K}$

The above lemma least to the following definition.

Definition 3.6 *The set $\sigma(\mathcal{K})$ is called a σ-algebra generated by $\mathcal{K}$.*

Borel σ-algebra. Assume that (X, d) is a metric space. Let $\mathcal{K}$ be a collection of open subsets in X, i.e.,

$$\mathcal{K} = \{\mathcal{O} \subset X : \mathcal{O} \text{ is open}\}.$$

In this case, $\sigma(\mathcal{K})$ is called *Borel σ-algebra* and denote by $\mathcal{B}(X)$. Moreover, a set $A \in \mathcal{B}(X)$ is called a *Borel set.*

From the property (iii) in the definition 3.2, one can show that

$$\mathcal{B}(X) = \sigma(\mathcal{H})$$

where $\mathcal{H} = \{F \subset X : F \text{ is closed}\}$ is a collection of closed subset of X.

Example 3.3 *Assume that $(X, d) = (\mathbb{R}, |\cdot|)$. Denote by*

$$\mathcal{F} = \{[a, b) : a < b\}, \quad \mathcal{F}_1 = \{[a, +\infty) : a \in \mathbb{R}\} \quad \text{and} \quad \mathcal{F}_2 = \{(-\infty, a] : a \in \mathbb{R}\},$$

we have

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{F}) = \sigma(\mathcal{F}_1) = \sigma(\mathcal{F}_2).$$

Proof. Let's show that

$$\mathcal{B}(\mathbb{R}) \;=\; \sigma(\mathcal{F}).$$

1. Given $a < b$, it holds

$$[a, b) \;=\; \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b \right) \;\in\; \mathcal{B}(\mathbb{R}).$$

This implies that

$$\sigma(\mathcal{F}) \;\subseteq\; \mathcal{B}(\mathbb{R}).$$

2. To complete the proof, we meed to show that

$$\mathcal{B}(\mathbb{R}) \;\subseteq\; \sigma(\mathcal{F}).$$

It is sufficient to show that for every open set V in $\mathbb{R}$, it holds

$$V \;\in\; \sigma(\mathcal{F}).$$

Since V is open, there exists open disjoint intervals (a_n, b_n) for $n = \overline{1, K}$ (K can be ∞) such that

$$\bigcup_{n=1}^{K} (a_n, b_n) \;=\; V.$$

Finally, observing that

$$(a_n, b_n) \;=\; \bigcup_{m=1}^{\infty} \left[a_n + \frac{1}{m}, b_n \right) \;\in\; \sigma(\mathcal{F}) \qquad \text{for all } n \geq 1,$$

we have $V \in \sigma(\mathcal{F})$. The proof in complete. □

Problem 5: Show that $\sigma(\mathcal{F}_2) = \mathcal{B}(\mathbb{R})$.

Problem 6: Given a function $f : \mathbb{R} \to \mathbb{R}$, is $\{f^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\}$ a σ-algebra? For $f(x) = x^2$ describe $\sigma(f) = \sigma(\{f^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\})$.

3.2 Measures

Given a measurable space $(X, \mathcal{A})$, let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(\varnothing) = 0$. We say that

- μ is additive if

  $$\mu\left(\bigcup_{i=1}^{n} A_i \right) \;=\; \sum_{i=1}^{n} \mu(A_i)$$

  for all $\{A_1, A_2, \ldots, A_n\}$ of mutually disjoint elements of $\mathcal{A}$.

- μ is σ-additive if
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$
for all $(A_n)_{n\geq 1}$ of mutually disjoint elements of $\mathcal{A}$.

- μ is σ-subadditive if
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n) \qquad \text{for all } A_n \in \mathcal{A}$$

Definition 3.7 (Measure) *The map $\mu : \mathcal{A} \to [0, +\infty]$ is called a measure if $\mu(\varnothing) = 0$ and μ is σ-additive. Moreover,*

- *if $\mu(X) < +\infty$ then μ is a finite measure;*
- *if $\mu(X) = 1$ then μ is a probability measure;*
- *if $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < +\infty$ for every $n \geq 1$ then μ is a σ-measure (In some sources, this is sometimes described as σ-finiteness).*

Remark 3.8 *The collection $(X, \mathcal{A}, \mu)$ is a triple that defines a* **measure space**. *This allows one to precisely define:*

(i) what is the set that the objects live in (i.e. X);

(ii) what are the objects that we know how to measure (i.e. $\mathcal{A}$);

(iii) what measure do we assign to each object.

This is a flexible abstract concept that allows us to address many different types of problems: from Lebesgue integration, to geometric measure theory, to probability and stochastic processes.

Definition 3.9 *Let $(X, \mathcal{A}, \mu)$ be a measure space. For any $E \in \mathcal{A}$, denote by*
$$\mathcal{A}_{|E} := \{A \cap E : A \in \mathcal{A}\} \qquad \text{and} \qquad \mu_E(A) := \mu(A \cap E)$$
In this case, $(E, \mathcal{A}_{|E}, \mu_E)$ is a restricted measure space on E of $(X, \mathcal{A}, \mu)$.

Example 3.4 (a). *Dirichlet measure*

$$\delta_x(A) \;=\; \begin{cases} 1 & \text{if} \quad x \in A \\ \\ 0 & \text{if} \quad x \notin A \end{cases} \qquad \textit{for all } A \in \mathcal{P}(X).$$

for a given $x \in X$. In this case, δ_x is concentrated at x.

(b). *Counting measure*

$$\mu^\sharp(A) \;=\; \begin{cases} \text{Card}(A) & \text{if} \quad A \text{ has a finite elements} \\ \\ +\infty & \text{if} \quad A \text{ has a infinite elements} \end{cases}$$

for every $A \in \mathcal{P}(X)$.

(c). *Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be a finite set. Given a numbers $0 \le p_1 \le \cdots \le p_n$ such that $\sum_{i=1}^{n} p_i = 1$, we define*

$$\mathbb{P}(A) \;=\; \sum_{k=1}^{m} p_{i_k} \qquad \textit{for all } A = \{p_{i_1,\ldots,p_{i_m}}\} \subseteq \Omega.$$

Then, $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ is a probability measure space.

Basics properties. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

- *(Finite additivity)* For finite disjoint set $(A_i)_{i=1}^n \subset \mathcal{A}$, it holds

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) \;=\; \sum_{i=1}^{n} \mu(A_i).$$

- *(Monotonicity)* For every $A \subseteq B$ in $\mathcal{A}$, it holds

$$\mu(A) \;\le\; \mu(B).$$

- *(Excision)* If $A \subseteq B$ in $\mathcal{A}$ and $\mu(A) < +\infty$, then

$$\mu(B \backslash A) \;=\; \mu(B) - \mu(A).$$

- *(Countable monotonicity)* Let $(A_n)_{n\ge}$ be a sequence of measurable sets. Then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \;\le\; \sum_{n=1}^{\infty} \mu(A_n).$$

Proof. (Countable monotonicity) Construct a new sequence of disjoint sets

$$B_1 := A_1, \quad B_2 = A_2 \backslash B_1, \quad \ldots \quad , B_n := A_n \backslash \bigcup_{i=1}^{n-1} B_i \,.$$

It is clear that $(B_n)_{n\geq 1}$ is mutually disjoint and

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad B_n \subseteq A_n.$$

Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)\,.$$

□

Proposition 3.9.1 (The continuity of μ) *For every sequence $(A_n)_{n\geq 1}$ of $\mathcal{A}$*

- *If $(A_n)_{n\geq 1}$ is increasing then*

$$\mu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

- *If $(A_n)_{n\geq 1}$ is decreasing and $\mu(A_1) < +\infty$ then*

$$\mu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. Assume that $(A_n)_{n\geq 1}$ is increasing. We have

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

Denote by

$$B_1 := A_1, \quad B_2 = A_2 \backslash B_1, \quad \ldots \quad , B_n := A_n \backslash \bigcup_{i=1}^{n-1} B_i \,.$$

It is clear that $(B_n)_{n\geq 1}$ is mutually disjoint and

$$A_n = \bigcup_{i=1}^{n} B_i \qquad \text{for all } n \geq 1.$$

Thus,

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mu(A_n)\end{aligned}$$

and yields the first statement. □

Problem 7: Prove the second statement of the above proposition. Is this still true if one remove the assumption $\mu(A_1) < +\infty$?

Corollary 3.10 *Let μ be a finite measure. Then for any sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$, it holds*

$$\mu\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mu(A_n) \leq \limsup_{n\to\infty} \mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n\right)$$

Proof. By the definition, we have

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} B_n \qquad \text{with} \qquad B_n = \bigcap_{k=n}^{\infty} A_k.$$

Since B_n is increasing and $B_n \subseteq A_n$, it holds

$$\mu\left(\liminf_{n\to\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n) \leq \liminf_{n\to\infty} \mu(A_n).$$

Similarly, one can show that

$$\limsup_{n\to\infty} \mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n\right)$$

and this complete the proof. □

Lemma 3.11 (Borel-Cantelli) *For any sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ such that*

$$\sum_{n=1}^{\infty} \mu(A_n) < +\infty.$$

Then it holds

$$\mu\left(\limsup_{n\to\infty} A_n\right) = 0.$$

Proof. By the definition, we have

$$A =: \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} B_n \qquad \text{with} \qquad B_n = \bigcup_{k=n}^{\infty} A_k.$$

Since B_n is decreasing and $\mu(B_1) \le \sum_{n=1}^{\infty} \mu(A_n) < +\infty$, it holds

$$\mu(A) = \lim_{n\to\infty} \mu(B_n) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(A_k) = 0.$$

The proof is complete. □

To complete this subsection, let us illustrate a typical use of the Borel-Cantelli Lemma.

Example 3.5 *Give a metric space (X,d), let $(X,\mathcal{B}(X),\mu)$ be a Borel measure on X. Assume that the sequence of continuous function $f_n : X \to \mathbb{R}$ converges in the sense of measure μ to a continuous function f, i.e.,*

$$\lim_{n\to\infty} \mu\left(x \in X : |f_n(x) - f(x)| > \varepsilon\right) = 0.$$

Then, there exists a subsequence f_{n_k} and $E \in \mathcal{B}(X)$ with $\mu(E^c) = 0$ such that

$$\lim_{n_k\to\infty} f_{n_k}(x) = f(x) \qquad \text{for all } x \in E.$$

Proof. For each $k \ge 1$, we select $n_k \ge 1$ sufficiently large such that

$$\mu\left(x \in X : \left|f_{n_k}(x) - f(x)\right| > \frac{1}{k}\right) \le \frac{1}{k^2}$$

with $n_{k-1} < n_k$. Set $A_k := \left\{x \in X : \left|f_{n_k}(x) - f(x)\right| > \frac{1}{k}\right\}$, we have

$$\sum_{k=1}^{\infty} \mu(A_k) \le \sum_{k=1}^{\infty} \frac{1}{k^2} < 1.$$

The Borel-Cantelli Lemma yields

$$\mu\left(\limsup_{k\to\infty} A_k\right) = 0.$$

Thus, set $E = X \backslash \limsup_{k\to\infty} A_k$ such that $\mu(E^c) = 0$, we obtain that

$$\lim_{n_k\to\infty} f_{n_k}(x) = f(x) \qquad \text{for all } x \in E$$

and this complete the proof. $\square$

Problem 8: Let $(X, \mathcal{A}, \mu)$ be a measure space. The symmetric difference of $A, B \subset X$ is defined by

$$A \Delta B \;=\; [A \backslash B] \cup [B \backslash A] \,.$$

Show that If A, B are measurable (i.e., $A, B \in \mathcal{A}$) and $\mu(A \Delta B) = 0$ then $\mu(A) = \mu(B)$.

3.3 Sets of measure zero and complete measure space

Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that

- $E \in \mathcal{A}$ is a set of measure zero (null set) if $\mu(E) = 0$.
- The statement (P) holds for almost everywhere in X if there exists a null set $E \subset X$ such that (P) holds for all $x \in E^c$.

It is clear that countable union of null sets is a null set.

Question. Is this true that if E is measurable with $\mu(E) = 0$ then every subset of E is measurable and has a zero measure?

NO in general. Indeed, let $X = \{-1, 0, 1\}$ and let $\mathcal{A} = \{\emptyset, \{0\}, \{-1, 1\}, X\}$ be a σ-algebra on X. Consider the measure $\mu : \mathcal{A} \to [0, 1]$ such that

$$\mu(X) \;=\; \mu(\{0\}) \;=\; 1 \qquad \text{and} \qquad \mu(\{-1, 1\}) \;=\; \mu(\emptyset) \;=\; 0.$$

In this case, $\{-1, 1\}$ is a null set but $\{-1\}$ and $\{1\}$ are not measurable. $\square$

Definition 3.12 *A measure space $(X, \mathcal{A}, \mu)$ is complete if every subset of null set is measurable.*

How to complete a measure space $(X, \mathcal{A}, \mu)$?

Theorem 3.13 *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then there exists a unique smallest complete measure space $(X, \overline{\mathcal{A}}, \bar{\mu})$ such that*

$$\mathcal{A} \subset \overline{\mathcal{A}} \qquad \text{and} \qquad \bar{\mu}(A) \;=\; \mu(A) \qquad \textit{for all } A \in \mathcal{A}.$$

Proof. Let's define

$$\overline{\mathcal{A}} \;=\; \{A \cup M : A \in \mathcal{A}, M \subseteq N \in \mathcal{A} \quad \text{with} \quad \mu(N) = 0\}$$

and

$$\overline{\mu}(M \cup A) \;=\; \mu(A).$$

We claim that $\overline{\mathcal{A}}$ is a σ-algebra. Indeed, it is clear that

$$\varnothing, X \in \overline{\mathcal{A}}.$$

For any $A \in \mathcal{A}$ and $M \subseteq N \in \mathcal{A}$ with $\mu(N) = 0$, we have

$$(A \cup M)^c \;=\; A^c \cap M^c \;=\; A^c \cap [N^c \cup (N \backslash M)] \;=\; (A^c \cap N^c) \bigcup (A^c \cap (N \backslash M)).$$

Since $A^c \cap N^c \in \mathcal{A}$ and $(A^c \cap (N \backslash M)) \subseteq N$, it holds

$$(A \cup M)^c \in \overline{\mathcal{A}}.$$

Countable union property. Assume that

$$A_i \in \mathcal{A}, \qquad M_i \subseteq N_i \in \mathcal{A} \qquad \text{with} \qquad \mu(N_i) = 0 \qquad \text{for all } i \geq 1.$$

Since $\mu\left(\bigcup_{i=1}^{\infty} N_i\right) = 0$, we have

$$\bigcup_{i=1}^{\infty} (A_i \cup M_i) \;\in\; \left(\bigcup_{i=1}^{\infty} A_i\right) \bigcup \left(\bigcup_{i=1}^{\infty} M_i\right) \;\in\; \overline{\mathcal{A}}.$$

Therefore, $\overline{\mathcal{A}}$ is a σ-algebra on X.

It is easy to check that $\overline{\mu}$ is a measure on $\overline{\mathcal{A}}$. To complete the proof, we need to show that if $(X, \mathcal{A}_1, \mu_1)$ is a complete measure space with

$$\mathcal{A} \subseteq \mathcal{A}_1 \qquad \text{and} \qquad \mu_1(E) \;=\; \mu(E) \qquad \text{for all } E \in \mathcal{A},$$

then

$$\overline{\mathcal{A}} \subseteq \mathcal{A}_1 \qquad \text{and} \qquad \mu_1(E) \;=\; \bar{\mu}(E) \qquad \text{for all } E \in \overline{\mathcal{A}}.$$

For any $A \cup M \in \overline{\mathcal{A}}$ with $A \in \mathcal{A}$ and $M \subseteq N \in \mathcal{A}$ with $\mu(N) = 0$, we have that $M \in \mathcal{A}_1$ since $\mathcal{A}_1$ is complete. This implies that

$$A \cup M \;\in\; \mathcal{A}_1 \qquad \text{and} \qquad \mu_1(A \cup M) = \mu_1(A) = \mu(A) = \bar{\mu}(A \cup M).$$

The proof is complete. $\square$

4 Lebesgue measures

Our goal is to construct a σ-algebra $\mathcal{L}(\mathbb{R})$ on $\mathbb{R}$ and a measure $m : \mathcal{L}(\mathbb{R}) \to [0, +\infty]$ such that

- $\mathcal{L}(\mathbb{R})$ contains all open and closed sets in $\mathbb{R}$.

- For any $a < b$, it holds

$$m([a,b]) \; = \; b - a.$$

- m is translation invariant

$$m(y+E) \; = \; m(E) \qquad \text{for all } y \in \mathbb{R}, E \in \mathcal{L}(\mathbb{R}).$$

We then extend our construction to $\mathcal{L}(\mathbb{R}^n)$.

How to construct $(\mathcal{L}(\mathbb{R}), m)$**?**

Step 1: Construct an outer measure $m^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty],$, i.e.,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \; \leq \; \sum_{k=1}^{\infty} m^*(E_k)$$

such that

$$m^*([a,b]) \; = \; b - a \qquad \text{for all } b \geq a.$$

Step 2. Using Caratheódory's approach to define $\mathcal{L}(\mathbb{R})$ relying on m^*.

Step 3. Restrict m^* on $\mathcal{L}(\mathbb{R})$ to obtain m.

4.1 Outer measures

Definition 4.1 *Given a nonempty set X, a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is called an outer measure if*

(i) $\mu^*(\varnothing) = 0$;

(ii) (Monotonicity) For any $E_1 \subseteq E_2$, it holds

$$\mu^*(E_1) \; \leq \; \mu^*(E_2).$$

(iii) (σ-subadditive) For any sequence $(E_n)_{n\geq 1} \subset \mathcal{P}(X)$, it holds

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \; \leq \; \sum_{n=1}^{\infty} \mu^*(E_n).$$

Notice that (ii)-(iii) can be rewritten by

$$E \; \subseteq \; \bigcup_{n=1}^{\infty} E_n \qquad \Longrightarrow \qquad \mu^*(E) \; \leq \; \sum_{n=1}^{\infty} \mu^*(E_n).$$

Remark 4.2 *An outer measure is not a measure on $\mathcal{P}(X)$ in general.*

Indeed, let $X = \{0, 1\}$ and let $\mu^* : \mathcal{P}(X) \to [0, 1]$ such that

$$\mu^*(X) \;=\; \mu^*(\{0\}) \;=\; 1 \qquad \text{and} \qquad \mu^*(\{1\}) \;=\; \frac{1}{2}.$$

In this case, μ^* is an outer measure but does not satisfy the finite additive property. Thus, it is not a measure on $(X, \mathcal{P}(X))$.

Question: *Can one find an outer measure on $\mathbb{R}$ such that the finite additive property does not hold?*

How to construct an outer measure? The most common way to obtain an outer measure is to start with "elementary sets".

Proposition 4.2.1 *Given an algebra $\mathcal{A}$ on X, let $\rho : \mathcal{A} \to [0, +\infty]$ be a σ-subadditive function with $\rho(\varnothing) = 0$. For any $E \in \mathcal{P}(X)$, define*

$$\mu^*(E) \;=\; \inf \left\{ \sum_{n=1}^{\infty} \rho(A_n) : E \subseteq \bigcup_{n=1}^{\infty} A_n \;\text{ with }\; (A_n)_{n\geq 1} \subseteq \mathcal{A} \right\},$$

then the following hold:

(i). μ^* is finite if ρ is finite.

(ii). μ^* is an extension of ρ, i.e.,

$$\mu^*(A) \;=\; \rho(A) \qquad \textit{for all } A \in \mathcal{A}.$$

(iii). μ^* is an outer measure on X.

Proof. (i) is trivial. Let's prove (ii). For any $A \in \mathcal{A}$, since $\mu^*(A) \leq \rho(A)$, it is sufficient to show that

$$\rho(A) \;\leq\; \mu^*(A).$$

Equivalently, for all $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ with $A \subseteq \bigcup_{n=1}^{\infty} A_n$, it holds

$$\rho(A) \;\leq\; \sum_{n=1}^{\infty} \rho(A_n).$$

Since $(A \cap A_n)_{n\geq 1} \subseteq \mathcal{A}$ and $\bigcup_{n=1}^{\infty} (A \cap A_n) = A$, the σ-subadditive of ρ implies that

$$\rho(A) \;=\; \rho\left(\bigcup_{n=1}^{\infty} A_n \cap A \right) \;\leq\; \sum_{n=1}^{\infty} \rho(A_n \cap A) \;\leq\; \sum_{n=1}^{\infty} \rho(A_n).$$

(iii). Let's show that μ^* is an outer measure. It is clear that $\mu^*(\varnothing) = 0$ and μ^* is monotone. It is enough to check that μ^* is σ-subadditive, i.e.,

$$\mu^*\left(E := \bigcup_{n=1}^{\infty} E_n\right) \;\leq\; \sum_{n=1}^{\infty} \mu^*(E_n) \qquad \text{for all } E_n \in \mathcal{P}(X). \tag{4.1}$$

By the definition of μ^*, for any $n \geq 1$ and $\varepsilon > 0$, there exists $(A_{n_k})_{k\geq 1} \subseteq \mathcal{A}$ such that

$$\mu^*(E_n) \;\geq\; \sum_{k=1}^{\infty} \rho(A_{n_k}) - \frac{\varepsilon}{2^n}, \quad E_n \subset \bigcup_{k=1}^{\infty} A_{n_k}$$

This implies that

$$\begin{aligned}\sum_{n=1}^{\infty} \mu^*(E_n) \;&\geq\; \sum_{n=1}^{\infty}\left[\sum_{k=1}^{\infty} \rho(A_{n_k}) - \frac{\varepsilon}{2^n}\right] \\ &= \sum_{(n,k)\in\mathbb{N}^2} \rho(A_{n_k}) - \varepsilon \;\geq\; \mu^*(E) - \varepsilon.\end{aligned}$$

Taking ε to 0, we obtain (4.1). □

Remark 4.3 *The previous proof actually can be made quite a bit sharper. More precise,*

- *the set $\mathcal{A}$ need not be an algebra (it mostly just needs to be able to cover any set);*
- *the only place where one needs the σ-subadditivity is in proving the extension property (i.e. this definition always will give an outer measure, even without subadditivity).*

Let us now provide some basic examples of outer measures.

Example 4.1 (Lebesgue-Stieltjes outer measure) *Given a non-decreasing function $f : [A, B] \to \mathbb{R}$, we define*

$$\rho((a,b)) \;=\; f(b) - f(a) \qquad \text{for all } (a,b) \subset [A,B].$$

The Lebesgue-Stieltjes outer measure is given by

$$\mu_f^*(E) := \inf\left\{\sum_{n=1}^{\infty} \rho((a_n,b_n)) : a_n, b_n \in [A,B], a_n \leq b_n, E \subset \bigcup_{n=1}^{\infty}(a_n,b_n)\right\}.$$

By the previous proposition μ_f^* is an outer measure.

Problem 9: Show that $\mu_f^*([a,b]) = f^+(b) - f^-(a)$, where f^+ is the limit from the right and f^- is the limit from the left of f.

Example 4.2 (Hausdorff outer measure) *For any $0 \le s < +\infty$, we define*

$$\alpha_s = \frac{\pi^{s/2}}{\Gamma(s/2+1)} \qquad \text{with} \qquad \Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx \ \ (\textit{Euler Gamma function}).$$

Notice that α_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$ for every $d \in \mathbb{N}$. For any given $0 \le s \le d$ and $\delta > 0$, we define for every $E \subset \mathbb{R}^d$ that

$$\mathcal{H}^s_\delta(E) := \inf\left\{ \sum_{n=1}^\infty \alpha_s \cdot \left(\frac{\operatorname{diam}(E_n)}{2}\right)^s : E \subset \bigcup_{n=1}^\infty E_n, diam(E_n) < \delta \right\}.$$

If $s = 0$ we only sum over non-empty E_n. By the previous proposition, $\mathcal{H}^s_\delta$ is an outer measure on $\mathbb{R}^d$.

Since the map $\sigma \to H^s_\delta(E)$ is non-increasing, we then define

$$\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E).$$

This is known as the Hausdorff *outer measure. Moreover,*

- *When $s = d$ this gives the $\mathbb{R}^d$ Lebesgue measure;*
- *When $s = 0$ this gives a counting measure;*
- *When $0 < s < n$ this gives a way of measuring the size of intermediate objects, such as curves and surfaces in three dimensions. It also allows one to measure the size of fractals and other more exotic objects.*

Problem 10: Show that both $\mathcal{H}^s_\delta$ are $\mathcal{H}^s$ is outer measures.

Problem 11: [Unexample: Jordan content] For every subset $E \subseteq \mathbb{R}$, we define

$$\mu_J(E) = \inf\left\{ \sum_{i=1}^k diam(E_k) : E \subset \bigcup_{i=1}^k E_k \right\}.$$

Show that μ_J is not an outer measure on $\mathbb{R}$.

Definition 4.4 (Carathéodory measurable set) *Given an outer measure μ^* on X, a set $A \in \mathcal{P}(X)$ is called a Carathéodory measurable set with respect to μ^* if*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $E \in \mathcal{P}(X)$.

Consider a collection of Carathéodory measurable subsets of X with respect to μ^*

$$\mathcal{F} \;=\; \{A \subseteq X : A \text{ is a Carathéodory measurable set w.r.t } \mu^*\}\,.$$

By the definition, it is clear that

$$X, \varnothing \;\in\; \mathcal{F} \qquad \text{and} \qquad A \in \mathcal{F} \quad \Longleftrightarrow \quad A^c \in \mathcal{F}.$$

Notice that thanks to the σ-sub-additive property of the outer measure μ^*, the set $A \in \mathcal{F}$ if and only if

$$\mu^*(E) \;\geq\; \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $E \in \mathcal{P}(X)$.

Example 4.3 *Let* $X = [0,1]$ *and* $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ *such that*

$$\begin{cases} \mu^*(A) \;=\; 0 \qquad \text{if} \qquad A \cap \mathbb{Q} = \varnothing \\ \\ \mu^*(A) \;=\; 1 \qquad \text{if} \qquad A \cap \mathbb{Q} \neq \varnothing. \end{cases}$$

In this case, one can check that μ^* *is an outer measure on* $[0,1]$. *Moreover,*

- $[0,1)$ *is not a Carathéodory measurable set w.r.t* μ^*.
- $\{1/\sqrt{2}\}$ *is a Carathéodory measurable set w.r.t* μ^*

Problem 12: Find the collection $\mathcal{F}$ of all Carathéodory measurable sets w.r.t μ^* in the above example.

Theorem 4.5 (Carathéodory) *Let* μ^* *be an outer measure on* X. *Then* $(X, \mathcal{F}, \mu^*)$ *is a complete measure space.*

Proof. 1. We first claim that $\mathcal{F}$ is an algebra. It is clear that

$$X, \varnothing \;\in\; \mathcal{F} \qquad \text{and} \qquad A \in \mathcal{F} \quad \Longleftrightarrow \quad A^c \in \mathcal{F}.$$

For any $A, B \in \mathcal{F}$, we need to show that $A \cup B \in \mathcal{F}$, i.e.,

$$\mu^*(E) \;\geq\; \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \qquad \text{for all } E \in \mathcal{P}(X). \tag{4.2}$$

Indeed, since $A, B \in \mathcal{F}$, it holds

$$\begin{cases} \mu^*(F) \;=\; \mu^*(F \cap A) + \mu^*(E \cap A^c) \\ \\ \mu^*(F) \;=\; \mu^*(F \cap B) + \mu^*(E \cap B^c) \end{cases} \qquad \text{for all } F \in \mathcal{P}(X).$$

Observe that

$$E \cap (A \cup B) \ = \ (E \cap A) \cup (E \cap A^c \cap B).$$

We have

$$\mu^*(E \cap (A \cup B)) \ \leq \ \mu^*(E \cap A) + \mu^*((E \cap A^c \cap B)).$$

Thus,

$$\begin{aligned}
\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \ &\leq \ \mu^*(E \cap A) + \mu^*((E \cap A^c \cap B)) \\
&\qquad\qquad + \mu^*(E \cap (A \cup B)^c) \\
&= \ \mu^*(E \cap A) + \mu^*((E \cap A^c \cap B)) \\
&\qquad\qquad + \mu^*(E \cap A^c \cap B^c) \\
&= \ \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \ \mu^*(E).
\end{aligned}$$

2. μ^* is is additive on $\mathcal{F}$. For any $A, B \in \mathcal{F}$ with $A \cap B = \varnothing$, it holds that

$$\begin{aligned}
\mu^*(E \cap (A \cup B)) \ &= \ \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) \\
&= \ \mu^*(E \cap A) + \mu^*(E \cap B).
\end{aligned}$$

Thus, by the induction method, one can show that for any $\{A_1, \ldots, A_n\}$ finite collection of disjoint sets in $\mathcal{F}$, it holds

$$\mu^*\left(E \cap \left(\bigcup_{k=1}^{n} A_k\right)\right) \ = \ \sum_{k=1}^{n} \mu^*(E \cap A_k) \qquad \text{for all } E \in \mathcal{P}(X).$$

In particular, choosing $E = X$, we obtain that

$$\mu^*\left(\bigcup_{k=1}^{n} A_k\right) \ = \ \sum_{k=1}^{n} \mu^*(A_k).$$

3. Let us now check that $\mathcal{F}$ is a σ-algebra. For a given be a sequence of mutually disjoint sets A_n in $\mathcal{F}$, we show that

$$S := \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

Equivalently,

$$\mu^*(E) \ \geq \ \mu^*(E \cap S) + \mu^*(E \cap S^c) \qquad \text{for all } E \in \mathcal{P}(X).$$

Using the σ-subadditive property of μ^*, we have

$$\mu^*(E \cap S) \ = \ \mu^*\left(\bigcup_{n=1}^{\infty} (E \cap A_n)\right) \ \leq \ \sum_{n=1}^{\infty} \mu^*(E \cap A_n) \ = \ \lim_{n \to \infty} \sum_{k=1}^{n} \mu^*(E \cap A_k)$$

$$= \lim_{n\to\infty} \mu^*\left(E \bigcap \left(S_n =: \bigcup_{k=1}^{n} A_k\right)\right).$$

On the other hand,

$$\mu^*(E \cap S^c) \;=\; \mu^*\left(E \cap \left(\bigcap_{k=1}^{\infty} A_k^c\right)\right) \;\leq\; \mu^*\left(E \cap \left(\bigcap_{k=1}^{n} A_k^c\right)\right) \;=\; \mu^*(E \cap S_n^c),$$

and it yields

$$\begin{aligned}\mu^*(E \cap S) + \mu^*(E \cap S^c) &\leq \limsup_{n\to\infty}\left[\mu^*(E \cap S_n) + \mu^*(E \cap S_n^c)\right] \\ &= \mu^*(E).\end{aligned}$$

Thus, $S \in \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra.

4. *One can show that if μ^* is additive and σ-sub-additive then μ^* is σ-additive.* Therefore, $(X, \mathcal{F}, \mu^*)$ is a measure space.

5. To conclude the proof, we will show that $(X, \mathcal{F}, \mu^*)$ is complete. Given $N \in \mathcal{F}$ with $\mu^*(N) = 0$, we claim that

$$M \;\in\; \mathcal{F} \qquad \text{for all } M \subseteq N.$$

Equivalently,

$$\mu^*(E) \;\geq\; \mu^*(E \cap M) + \mu^*(E \cap M^c) \qquad \text{for all } E \in \mathcal{P}(X).$$

By the monotonicity of μ^*, it holds

$$0 \;\leq\; \mu^*(M \cap E) \;\leq\; \mu^*(N \cap E) \;\leq\; \mu^*(N) \;=\; 0$$

and it yields $\mu^*(M \cap E) = \mu^*(N \cap E) = 0$. Thus,

$$\mu^*(M \cap E) + \mu^*(E \cap M^c) \;\leq\; 0 + \mu^*(E \cap N^c) \;\leq\; \mu^*(E)$$

and the proof is complete. □

By using Carathéodory theorem, we have created a large class of complete measures space from some small family of sets and their size. However, the constructed σ-algebra via this approach is abstract. A natural question is whether one can directly show that a set is Carathéodory measurable. To do that we shall consider a special class of outer measures.

Definition 4.6 *Let (X, d) be a metric space. An outer measure μ^* on X is called a metric outer measure if*

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F),$$

for all sets $E, F \subset X$ satisfying

$$d(E, F) := \inf\{d(x, y) : x \in E, y \in F\} > 0.$$

Proposition 4.6.1 *Let X be a metric space and μ^* be a metric outer measure on X. Then every Borel set is μ^* measurable.*

Proof: Since closed sets generate the Borel σ-algebra, we just need to prove that any closed set is Carathéodory measurable. Now let $C \subset X$ be closed, and let $F \subset X$ satisfy $\mu^*(F) < \infty$ (the other case is trivial), we show that

$$\mu^*(F) \ \geq \ \mu^*(F \cap C) + \mu^*(F \backslash C).$$

Consider a sequence of sets $E_n \subset F \backslash C$ such that

$$E_0 \ := \ \{x \in F \backslash C : d(x, C) \geq 1\}, \qquad E_n \ := \ \left\{x \in F \backslash C : \frac{1}{n+1} \leq d(x, C) < \frac{1}{n}\right\}.$$

Clearly these sets are disjoint and $\bigcup_{n=0}^{\infty} E_n = F \backslash C$. In particular,

$$\begin{aligned}
\mu^*(F \cap C) + \mu^*(F \backslash C) \ &= \ \mu^*(F \cap C) + \mu^*\left(\bigcup_{n=0}^{\infty} E_n\right) \\
&\leq \ \mu^*(F \cap C) + \mu^*\left(\bigcup_{i=0}^{n} E_i\right) + \sum_{i=n+1}^{\infty} \mu^*(E_i) \\
&= \ \mu^*\left((F \cap C) \cup \left(\bigcup_{i=0}^{n} E_i\right)\right) + \sum_{i=n+1}^{\infty} \mu^*(E_i) \\
&\leq \ \mu^*(F) + \sum_{i=n+1}^{\infty} \mu^*(E_i).
\end{aligned}$$

To complete the proof, we need to show that $\sum_{n=0}^{\infty} \mu^*(E_n) < +\infty$. By the definition of E_n, it holds

$$d(E_{2k}, E_{2h}), d(E_{2k+1}, E_{2h+1}) \ \geq \ 0 \qquad \text{for all } h \neq k.$$

Since μ^* be a metric outer measure on X, one has that

$$\sum_{k=0}^{N} \mu^*(E_{2k}) \ = \ \mu^*\left(\bigcup_{k=0}^{N} E_{2k}\right) \ \leq \ \mu^*(F)$$

and

$$\sum_{k=0}^{N} \mu^*(E_{2k+1}) \;=\; \mu^*\left(\bigcup_{k=0}^{N} E_{2k+1}\right) \;\leq\; \mu^*(F).$$

Thus,

$$\sum_{k=0}^{\infty} \mu^*(E_k) \;\leq\; 2\cdot\mu^*(F) \;<\; +\infty$$

and this complete the proof. □

Problem 13: Let $\mathcal{A}$ be a σ-algebra on X and let $\mu : \mathcal{A} \to [0, +\infty]$ be additive. Show that μ is a measure on $(X, \mathcal{A})$ if and only if one of the followings holds

(i) μ is σ-subadditive;

(ii) For every sequence of increasing sets $\{A_n\}_{n\geq 1}$ in $\mathcal{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \;=\; \lim_{n\to\infty} \mu(A_n)\,.$$

Problem 14: In $X = \mathbb{N}$, consider the algebra

$$\mathcal{A} \;=\; \{A \in \mathcal{P}(X) \mid A \text{ is finite or } A^c \text{ is finite}\}\,.$$

Show that the function $\nu : \mathcal{A} \to [0, +\infty]$ defined as

$$\nu(A) \;=\; \begin{cases} \displaystyle\sum_{n\in A} \frac{1}{2^n} & \text{if} \quad A \text{ is finite} \\ \\ +\infty & \text{if} \quad A \text{ is infinite} \end{cases}$$

is additive but not σ-additive.

4.2 Lebesgue measure on $\mathbb{R}$

Let $\mathcal{I}$ be a set of open intervals on $\mathbb{R}$. For any $I = (a, b)$ with $a \leq b$, denote by

$$\ell(I) \;=\; \begin{cases} b - a & -\infty < a \leq b < +\infty \\ \\ +\infty & \text{if} \quad a = -\infty \ \text{ or } \ b = +\infty. \end{cases}$$

Observe that for every $A \in \mathcal{P}(\mathbb{R})$, there exists $(I_n)_{n\geq 1} \subset \mathcal{I}$ such that

$$A \;=\; \bigcup_{n=1}^{\infty} I_n\,.$$

Introduce the function $m^* : \mathcal{P}(\mathbb{R}) \to [0 + \infty]$ be such that for every $A \in \mathbb{R}$

$$m^*(A) \ := \ \inf \ \left\{ \sum_{n=1}^{\infty} \ell(I_n) : I_n \in \mathcal{I} \quad \text{and} \quad A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

Proposition 4.6.2 *The function $m^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ is an outer measure. In addition,*

(i) for every $-\infty < a < b < +\infty$, it holds

$$m^*((a,b)) \ = \ \ell((a,b)) \ = \ b - a;$$

(ii) m^* is translation invariant, in the sense that $m^*(x + E) = m^*(E)$.

Proof. The fact that m^* is an outer measure follows immediately from Proposition 4.2.1 and m^* is translation invariant follows directly from the fact that ℓ is translation invariant. To verify that m^* extends ℓ on the open intervals (a, b), we need to prove that

$$m^*((a,b)) \ \geq \ \ell((a,b)) \ = \ b - a.$$

Equivalently, for any $(a,b) \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i)$, it holds

$$b - a \ \leq \ \sum_{i=1}^{\infty} \ell((a_i, b_i)).$$

For every $\varepsilon > 0$ sufficiently small, since $[a + \varepsilon/2, b - \varepsilon/2] \subset (a,b)$ is compact, there exists a finite sub-covering (a_{i_k}, b_{i_k}) such that

$$[a + \varepsilon/2, b - \varepsilon/2] \ \subset \ \bigcup_{k=1}^{N} (a_{i_k}, b_{i_k})$$

and this yields

$$b - a - \varepsilon \ \leq \ \sum_{k=1}^{N} (b_{i_k} - a_{i_k}) \ \leq \ \sum_{i=1}^{\infty} \ell((a_i, b_i)).$$

Taking $\varepsilon \to 0+$, we the complete the proof. $\square$

Using the Carathéodory's approach, we then define the Lebesgue measure space in $\mathbb{R}$.

Definition 4.7 *A set $E \in \mathbb{R}$ is Lebesgue measurable if*

$$m^*(A) \ = \ m^*(E \cap A) + m^*(E^c \cap A) \qquad \textit{for all } A \in \mathcal{P}(\mathbb{R}).$$

Denote by

$$\mathcal{M} \;=\; \{E \in \mathcal{P}(\mathbb{R}) \mid E \text{ is Lebesgue measurable}\}.$$

and $m : \mathcal{M} \to [0, \infty]$ such that

$$m(E) \;=\; m^*(E) \qquad \text{for all } E \in \mathcal{M}.$$

From Carathéodory's theorem, it holds

- $\mathcal{M}$ is a σ-algebra.
- m is a complete measure on $\mathcal{M}$.

We say that m is the *Lebesgue measure* on $\mathbb{R}$.

Basic properties of m.

(i). $m(\varnothing) = 0, \qquad m([a, b] = b - a \qquad \text{and} \qquad m(\{x\}) = 0.$

(ii). For all $a \in \mathbb{R}$ and $E \in \mathcal{M}$, it holds

$$m(a + E) \;=\; m(E)\,.$$

(iii) If $m^*(A) = 0$ then A is Lebesgue measurable and $m(A) = 0$.

Proposition 4.7.1 *It holds that*

$$\mathcal{B}(\mathbb{R}) \;\subseteq\; \mathcal{M}.$$

Proof. It is sufficiently to show that the set $(a, +\infty)$ is measurable for any given $a \in \mathbb{R}$. Equivalently, for any $A \in \mathcal{P}(\mathbb{R})$, it holds

$$m^*(A) \;\geq\; m^*(A \cap (a, +\infty)) + m^*(A \cap (-\infty, a]).$$

It is trivial if $m^*(A) = +\infty$. Assume that $m^*(A) < \infty$. For every $\varepsilon > 0$, there exists $\{I_n\}_{n \geq 1} \subseteq \mathcal{I}$ such that

$$A \;\subseteq\; \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad m^*(A) \;\geq\; \sum_{n=1}^{\infty} \ell(I_n) - \varepsilon\,.$$

For any $n \in \mathbb{N}$, denote by

$$J_n \;=\; I_n \cap (a, +\infty) \qquad \text{and} \qquad J'_n \;=\; I_n \cap \left(-\infty, a + \frac{\varepsilon}{2^n}\right).$$

It is clear that

$$\ell(I_n) \;\geq\; \ell(J_n) + \ell(J'_n) - \frac{\varepsilon}{2^n}.$$

Thus,

$$\begin{aligned} m^*(A)+\varepsilon \;&\geq\; \sum_{n=1}^{\infty} \ell(I_n) \;\geq\; \sum_{n=1}^{\infty}\Big(\ell(J_n)+\ell(J_n')-\frac{\varepsilon}{2^n}\Big) \\ &\geq\; m^*(A\cap(a,+\infty))+m^*(A\cap(-\infty,a])-\varepsilon. \end{aligned}$$

Taking $\varepsilon>0$ to $0+$, we obtain

$$m^*(A) \;\geq\; m^*(A\cap(a,+\infty))+m^*(A\cap(-\infty,a])$$

and this implies that $(-\infty,a)$ is Lebesgue measurable. □

Problem 15: Show that m^* is an outer metric measure on $\mathbb{R}$.

Problem 16: Construct a Lebesgue measurable set but not Borel set.

4.2.1 Outer and inner approximation

Let $(\mathbb{R},\mathcal{M},m)$ be the Lebesgue measure space on $\mathbb{R}$. It is known that

$$\mathcal{B}(\mathbb{R}) \;\subsetneq\; \mathcal{M}.$$

Question: *Is the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ "dense" in $\mathcal{M}$ w.r.t m, i.e., for every $\varepsilon>0$ there exists K_ε closed and O_ε open in $\mathbb{R}$ such that*

$$K_\varepsilon \;\subseteq\; E \;\subseteq\; O_\varepsilon \qquad \text{and} \qquad m\,(O_\varepsilon\backslash K_\varepsilon) \;<\; \varepsilon.$$

The following theorem is to answer the above question.

Theorem 4.8 *Let E be a subset of $\mathbb{R}$. Then E is Lebesgue measurable if and only if one of the following assertion holds*

(a) For any $\varepsilon>0$, there exists O_ε open subset of $\mathbb{R}$ such that

$$E\subseteq O_\varepsilon \qquad \text{and} \qquad m^*(O_\varepsilon\backslash E) \;\leq\; \varepsilon.$$

(b) For any $\varepsilon>0$, there exists K_ε closed subset of $\mathbb{R}$ such that

$$E\supseteq K_\varepsilon \qquad \text{and} \qquad m^*(E\backslash K_\varepsilon) \;\leq\; \varepsilon.$$

Proof. Let's show that E is measurable if any only if (a) holds.

1. Assume that E is measurable. Two cases are considered

Case 1: If $m(E)<+\infty$ then there exists $(I_n)_{n\geq 1}\subset\mathcal{I}$ such that

$$E \;\subseteq\; \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad m(E)+\varepsilon \;\geq\; \sum_{n=1}^{\infty}\ell(I_n).$$

The set $O_\varepsilon := \bigcup_{n=1}^{\infty} I_n \ \supseteq\ E$ is open. Moreover, the σ-additive property of m yields

$$m(O_\varepsilon \backslash E) \ = \ m(O_\varepsilon) - m(E) \ \leq \ \left(\sum_{n=1}^{\infty} \ell(I_n)\right) - m(E) \ \leq \ m(E) + \varepsilon - m(E) \ = \ \varepsilon.$$

Case 2: If $m(E) = +\infty$ then we can write

$$E \ = \ \bigcup_{n\in\mathbb{N}}^{\infty} E_n \qquad \text{with} \qquad E_n \ = \ E \cap \big([n, n+1) \cup [-n-1, -n)\big).$$

Since E_n is Lebesgue measure with $m(E_n) \leq 1$, from case 1 there exists a open set $E_n \subseteq O_{n,\varepsilon}$ such that

$$m(O_{n,\varepsilon} \backslash E_n) \ \leq \ \frac{\varepsilon}{2^{n+1}} \qquad \text{for all } n \in \mathbb{N}.$$

The set $O_\varepsilon = \bigcup_{n=0}^{\infty} O_{n,\varepsilon}$ is open, containing E, and satisfies

$$m(O_\varepsilon \backslash E) \ \leq \ \sum_{n=0}^{\infty} m(O_{n,\varepsilon} \backslash E) \ \leq \ \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{n+1}} \ = \ \varepsilon.$$

2. Assume that (a) holds, we show that E is Lebesgue measurable. For every $n \geq 1$, there exists an open set $O_n \supseteq E$ such that $m^*(O_n \backslash E) \leq \frac{1}{n}$. Set $G := \bigcap_{n=1}^{\infty} O_n \in \mathcal{M}$, we have

$$E \ \subseteq \ G \ \in \ \mathcal{M} \qquad \text{and} \qquad m^*(G \backslash E) \ = \ 0.$$

Since $m^*(G \backslash E) = 0$, it holds that $G \backslash E$ is in $\mathcal{M}$. Thus, $E = G \backslash (G \backslash E)$ is Lebesgue measurable.

3. To conclude the proof, let's show that E is Lebesgue measurable if and only if (b) holds. Indeed, E is measurable if any only if E^c is measurable. Equivalently, for every $\varepsilon > 0$, there exists O_ε open set such that

$$E^c \ \subseteq \ O_\varepsilon \qquad \text{and} \qquad m^*(O_\varepsilon \backslash E) \ \leq \ E.$$

The set $K_\varepsilon = O_\varepsilon^c$ is closed and satisfies

$$K_\varepsilon \ \subseteq \ E \qquad \text{and} \qquad m^*(E \backslash K_\varepsilon) \ = \ m^*(O_\varepsilon \backslash E) \ \leq \ \varepsilon.$$

The proof is complete. □

Corollary 4.9 *Let E be a measurable set with $m(E) < \infty$. Then there exists $(I_n)_{n=1}^N \subset \mathcal{I}$ finite disjoint open intervals such that*

$$m\left(\bigcup_{n=1}^N I_n \ \Delta E\right) \ = \ m\left(E \backslash \bigcup_{n=1}^N I_n\right) + m\left(\bigcup_{n=1}^N I_n \backslash E\right) \ \le \ \varepsilon.$$

Proof. From the above theorem, there exists any open set O_ε such that

$$E \ \subseteq \ O_\varepsilon \qquad \text{and} \qquad m(O_\varepsilon \backslash E) \ \le \ \frac{\varepsilon}{2}.$$

Since O_ε is open, one has

$$O_\varepsilon \ = \ \bigcup_{n=1}^\infty \ I_n$$

for $I_n \in \mathcal{I}$ and $I_m \cap I_n = \varnothing$ for every $m \neq n$. By the σ-additive property of m, one has

$$m(O_\varepsilon) \ = \ \sum_{n=1}^\infty \ m(I_n) \ = \ \lim_{n\to\infty} \ \sum_{k=1}^n \ m(I_k).$$

In particular, there exits $N_\varepsilon \in \mathbb{N}$ such that

$$m\left(\bigcup_{k=1}^{N_\varepsilon} I_k\right) \ = \ \sum_{k=1}^{N_\varepsilon} \ m(I_n) \ \ge \ m(O_\varepsilon) - \frac{\varepsilon}{2}.$$

This implies that

$$m\left(E \backslash \bigcup_{n=1}^{N_\varepsilon} I_n\right) + m\left(\bigcup_{n=1}^{N_\varepsilon} I_n \backslash E\right) \ \le \ m\left(O_\varepsilon \backslash \bigcup_{n=1}^{N_\varepsilon} I_n\right) + m\left(O_\varepsilon \backslash E\right) \ \le \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ = \ \varepsilon$$

and the proof is complete. □

Remark 4.10 *If E is a bounded set in $\mathbb{R}$ then for every $\varepsilon > 0$, there exists an open set O_ε such that $E \subseteq O_\varepsilon$ and*

$$m^*(O_\varepsilon) \ \le \ m^*(E) + \varepsilon.$$

However, it does not imply that

$$m^*(O_\varepsilon \backslash E) \ \le \ \varepsilon.$$

The first inequality is directly from the definition of outer measure. However, a bounded set E is not measurable in general. Thus, the second inequality fails if E is not Lebesgue measurable.

4.2.2 Uncountable set with zero Lebesgue measure and non-Lebesgue measurable set

Up to now, we has obtained the followings:

- Every at most infinite countable set in $\mathbb{R}$ has a zero Lebesgue measure.
- $\mathcal{B}(\mathbb{R})$ is a dense subset of $\mathcal{M}$.
- $(\mathbb{R}, \mathcal{M}, m)$ is a complete measure space.

One can ask several questions:

- Is there a uncountable set with zero Lebesgue measure?
- Construct a non-Lebesgue measurable set.
- Show that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ is not complete.
- Construct a Lebesgue measurable set but not Borel.

1. Cantor set. Define a sequence of closed set $(C_n)_{n\geq 1}$ by induction

$$C_0 = [0,1], \quad C_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right] \quad \text{and} \quad C_n = \frac{C_{n-1}}{3} \bigcup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right).$$

The cantor set $C = \bigcap_{n=1}^{\infty} C_n$ is nowhere dense and closed. In particular, C is measurable. Moreover, since $(C_n)_{n\geq 1}$ is decreasing and $m(C_0) = 1 < +\infty$, it holds

$$m(C) = m\left(\bigcap_{n=1}^{\infty} C_n\right) = m\left(\lim_{n\to\infty} C_n\right) = \lim_{n\to\infty} m(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

We now show that C is uncountable. Assume by a contradiction that the set C is at most countable, i.e.,

$$C = \{c_1, c_2, \ldots, c_n, \ldots\}.$$

Then, one can construct a decreasing sequence of compact subsets $(F_k)_{k\geq 1}$ such that

$$c_k \notin F_k \subset C_k.$$

In particular, one has

$$x \in \bigcap_{k=1}^{\infty} F_k \subseteq \bigcap_{k=1}^{\infty} C_k = C.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$x = c_{n_0} \in \bigcap_{k=1}^{\infty} F_k$$

and this yields a contradiction. $\square$

Problem 17: Let C be the cantor set which is constructed as above. Show that

$$\mathcal{H}^s(C) \;=\; \begin{cases} 0 & \text{if} \quad s \;>\; \dfrac{\ln 2}{\ln 3} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Question. *Notice that the cantor set C is a Borel set with zero Lebesgue measure. Can one construct a non Borel subset of C?*

2. Non-measurable sets. Thanks to the translation invariance of m, the following holds:

Lemma 4.11 *Let E be a bounded Lebesgue measurable. Suppose that there exists a bounded countable infinite set of real numbers Λ such that*

$$\lambda_1 + E \bigcap \lambda_2 + E \;=\; \varnothing \qquad \lambda_1 \neq \lambda_2 \in \Lambda.$$

Then, $m(E) = 0$.

Proof. By the σ-additivity of m, it holds

$$\sum_{\lambda\in\Lambda} m(\lambda + E) \;=\; m\left(\bigcup_{\lambda\in\Lambda} \lambda + E\right) \;<\; +\infty$$

Since $m(\lambda + E) = m(E)$ for all $\lambda \in \Lambda$ and Λ has infinite elements, one obtains that $m(E) = 0$. $\square$

As a consequence, any bounded set V satisfies the above property but having a positive outer measure is non-Lebesgue measurable. To construct V, one can think of dividing E into infinite countable disjoint subsets $(V_n)_{n\geq 1}$ such that every $n \geq 1$,

$$x - y \;\in\; \mathbb{Q} \qquad \text{for all } x, y \in V_n.$$

This leads to the following definition.

Definition 4.12 (Vitali sets) *Given $A \subset \mathbb{R}$, V is called a Vitali subset of A if*

$$\text{Card}\,(V \cap Q_a) \;=\; 1 \qquad \text{with} \qquad Q_a \;:=\; a + \mathbb{Q}$$

for all $a \in A$.

How to construct a Vitali set? For any $x \in \mathbb{R}$, set

$$Q_x := x + \mathbb{Q},$$

we have

$$Q_x \bigcap Q_y \neq \varnothing \iff Q_x = Q_y \iff x - y \in \mathbb{Q}.$$

Given any subset $A \subset [0,1]$, consider the collection of all subset Q_x

$$\mathcal{Q}_A = \{Q_x : x \in A\}.$$

Axiom of choice. There exists a choice function $f : \mathcal{Q}_A \to A$ such that

$$f(Q_x) \in Q_x \cap A \qquad \text{for all } x \in A.$$

It is easy to check that the set

$$V_A := \{f(Q_x) : x \in A\} \subseteq A$$

is a Vitali subset of A. Indeed, it is easy to see that

$$V_A \cap Q_a = \{f(Q_a)\} \qquad \text{for all } a \in A.$$

In the following theorem, we shall show that every set with positive Lebesgue measurable set contains a non Lebesgue measurable subset.

Theorem 4.13 *If $A \in \mathcal{M}$ has a positive Lebesgue measure then the set V_A is non Lebesgue measurable.*

Proof. Since V_A is a Vitali subset of A, one has that

$$q_1 + V_A \bigcap q_2 + V_A = \varnothing \qquad \text{for all } q_1 \neq q_2 \in \mathbb{Q}.$$

Moreover, for every $x \in A \subseteq [0,1]$, it holds

$$x \in f(Q_x) + [-1,1]$$

and this implies that

$$A \subseteq N_A := \bigcup_{q \in \mathbb{Q} \bigcap [-1,1]} q + V_A \subseteq [-1,2].$$

Assume that V_A is Lebesgue measurable. Then, the set N_A is also Lebesgue measurable and

$$0 < m(A) \leq m(N_A) \leq 3.$$

On the other hand, by the σ-additive property of m, one has

$$m(N_A) \ = \ m\left(\bigcup_{q\in[-1,1]\cap\mathbb{Q}} q+V_A\right) \ = \ \sum_{q\in[-1,1]\cap\mathbb{Q}} m(q+V_A) \ = \ \sum_{q\in[-1,1]\cap\mathbb{Q}} m(V_A).$$

Thus,

$$0 \ < \ m(A) \ \leq \ \sum_{q\in[-1,1]\cap\mathbb{Q}} m(V_A) \ \leq \ 3$$

and it yields a contradiction. □

3. A non-Borel Lebesgue measurable set. Our construction will be divided into two steps:

Step 1. The above cantor set C can be expressed by

$$C \ := \ \left\{x = \sum_{n=1}^{\infty} c_n\cdot 3^{-n} : c_n\in\{0,2\}\right\}.$$

For every $x = \sum_{n=1}^{\infty} c_n\cdot 3^{-n}$, we define

$$f(x) \ = \ \sum_{n=1}^{\infty} d_n\cdot 2^{-n} \qquad \text{with} \qquad d_n \ = \ \begin{cases} 1 & \text{if} \quad c_n=2\\ 0 & \text{if} \quad c_n=0. \end{cases}$$

The function $f : C\to[0,1]$ is increasing, and has the same value at the end of each of the intervals we've removed. It can be extended to a continuous function on $[0,1]$ such that $f(x)$ is constant on each removed interval

$$f'(x) \ = \ 0 \qquad a.e. \ \ x\in[0,1].$$

The following function

$$g(x) \ = \ x+f(x) \qquad \text{for all } x\in[0,1]$$

is strictly increasing, continuous, and $g([0,1]) = [0,2]$. Hence, it has a continuous inverse $g^{-1} : [0,2]\to[0,1]$. On the other hand, observe that the function g maps removed interval of $[0,1]$ to intervals of $[0,2]$ of the same length, we have

$$m\left(g\left([0,1]\backslash C\right)\right) \ = \ m\left([0,1]\backslash C\right) \ = \ 1$$

and this implies that

$$m\left(g(C)\right) \ = \ m([0,2]) - m\left(g\left([0,1]\backslash C\right)\right) \ = \ 1.$$

Step 2. Since $g(C)$ has positive Lebesgue measure, there exists $k \in \mathbb{Z}$ such that the set $g(C) \cap [k, k+1]$ has also positive Lebesgue measure. In particular, the set

$$A \ := \ g(C) \cap [k, k+1] - k \ \subseteq \ [0,1] \ \text{ has positive Lebesgue measure.}$$

From Theorem 4.13, the set $k + V_A \subseteq g(C)$ is not Lebesgue measurable. In particular, it is not Borel. Thanks to the increasing property of g, the set $g^{-1}(k + V_A)$ is also not Borel. On the other hand, by completeness of the Lebesgue space, since

$$g^{-1}(k + V_A) \ \subseteq \ C \qquad \text{with} \qquad m(C) \ = \ 0,$$

the set $g^{-1}(k + V_A)$ is Lebesgue measurable. □

Problem 18: Show that a strictly increasing function defined on an interval maps Borel sets to Borel sets.

Problem 19: Let $\gamma_1 : [c, d] \to \mathbb{R}^d$ be a smooth curve with $\gamma_1(c) = x, \gamma_1(d) = y$. Show that $\mathcal{H}^1(\gamma_1([c, d])) \geq |x - y|$.

Problem 20: Given $x, y \in \mathbb{R}^n$ and $S := \{tx + (1-t)y : t \in [0,1]\}$ show that $\mathcal{H}^1(S) = |x - y|$.

4.3 Lebesgue measure on $\mathbb{R}^d$

Given a closed and bounded rectangle

$$R \ = \ [a_1, b_1] \times [a_2, b_2] \times [a_d, b_d], \qquad -\infty < a_i \leq b_i < \infty.$$

The volume of R is

$$V(R) \ = \ (b_1 - a_1) \cdot (b_2 - a_2) \ldots (b_d - a_d).$$

Denote by $\mathcal{R}(\mathbb{R}^d)$ a collection of bounded and closed rectangle.

Lemma 4.14 (Outer measure) *The function $\mu^* : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty]$ defined by*

$$\mu^*(E) \ = \ \inf \ \left\{ \sum_{i=1}^{\infty} V(R_i) : E \subseteq \bigcup_{i=1}^{\infty}, R_i \in \mathcal{R}(\mathbb{R}^d) \right\}$$

is a metric outer measure.

Proof. The fact that this is an outer measure follows from an earlier proposition. The fact that it's a metric outer measure follows the same argument as for the Lebesgue measure on $\mathbb{R}$. □

Some basic properties: The followings hold

(i) $\mu^*(R) = V(R)$ for all $R \in \mathbb{R}^d$;

(ii) $\mu^*(a+E) = \mu^*(A)$ for all $a \in \mathbb{R}$ and $A \subseteq \mathbb{R}^d$;

(iii) $\mu^*(\{a\} \times \mathbb{R}^{d-1}) = 0$ for all $a \in \mathbb{R}$.

Definition 4.15 (Lebesgue measurable set) *A subset A of $\mathbb{R}^d$ is called Lebesgue measurable if*

$$\mu^*(E) \;=\; \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \textit{for all } E \in \mathbb{R}^d.$$

Denote by

$$\mathcal{L}(\mathbb{R}^d) \;=\; \big\{A \in \mathcal{P}(\mathbb{R}^d) : A \text{ is Lebesgue measurable}\big\}$$

and $\mu : \mathcal{L}(\mathbb{R}^d) \to [0, +\infty]$ such that

$$\mu(A) \;=\; \mu^*(A) \qquad \text{for all } A \in \mathcal{L}(\mathbb{R}^d).$$

With the same argument in one dimensional case, one can show that

Theorem 4.16 *The triple $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \mu)$ is a complete measure space in $\mathbb{R}^d$. Moreover,*

$$\mu(R) \;=\; V(R) \qquad \textit{for all } R \in \mathcal{R}(\mathbb{R}^d)\,,$$

and

$$\mu(a+A) \;=\; \mu(A) \qquad \textit{for all } a \in \mathbb{R}, A \in \mathcal{L}(\mathbb{R}^d).$$

4.3.1 Borel σ-algebra and regularity properties

Let $\mathcal{T}(\mathbb{R}^d)$ be a collection of all open subsets of $\mathbb{R}^d$. Denote by

$$\mathcal{B}(\mathbb{R}^d) \;=\; \sigma(\mathcal{T}(\mathbb{R}^d))$$

the σ-algebra generated by $\mathcal{T}(\mathbb{R}^n)$. Since μ^* is a metric outer measure on $\mathbb{R}^d$, from Proposition 4.6.1, it holds

$$\mathcal{B}(\mathbb{R}^d) \;\subset\; \mathcal{L}(\mathbb{R}^d).$$

Moreover, one also has that

Proposition 4.16.1 *The Borel σ-algebra is also generated by $\mathcal{R}(\mathbb{R}^d)$, i.e.,*

$$\mathcal{B}(\mathbb{R}^d) \;=\; \sigma\left(\mathcal{R}(\mathbb{R}^d)\right).$$

Proof. We only need to show that

$$\mathcal{B}(\mathbb{R}^d) \;\subseteq\; \sigma\left(\mathcal{R}(\mathbb{R}^d)\right).$$

Equivalently, for any O open and bounded subset of $\mathbb{R}^d$, it holds

$$O \;\in\; \sigma\left(\mathcal{R}(\mathbb{R}^d)\right).$$

In order to do so, let's construct a sequence $(G_n)_{n\geq 1} \subseteq \sigma\left(\mathcal{R}(\mathbb{R}^d)\right)$ such that

$$O \;=\; \bigcup_{n=1}^{\infty} G_n.$$

For $n = 1$, we divide $\mathbb{R}^d$ into cubes (almost disjoint) with side 2^{-1}. Let $\{C_{1,1}, C_{1,2}, \dots, C_{1,N_1}\}$ be all cubes contained in O and thus the set

$$O \;\supseteq\; G_1 \;:=\; \bigcup_{i=1}^{N_1} C_i \;\in\; \sigma\left(\mathcal{R}(\mathbb{R}^d)\right).$$

For $n = 2$, we set $O_1 := O \backslash G_1$ and divide $\mathbb{R}^d$ into cubes (almost disjoint) with side 2^{-2} and let $\{C_{2,1}, C_{2,2}, \cdot, C_{2,N_2}\}$ be all cubes contained in O_1. The set

$$O_1 \;\supseteq\; G_2 \;:=\; \bigcup_{i=1}^{N_1} C_{2,i} \;\in\; \sigma\left(\mathcal{R}(\mathbb{R}^d)\right).$$

Since O is an open set, by continuing this process, we obtain the sequence of mutually disjoint $(G_n)_{n\geq 1}$ such that

$$O \;=\; \bigcup_{n=1}^{\infty} G_n \;\in\; \sigma\left(\mathcal{R}(\mathbb{R}^d)\right)$$

and this complete the proof. □

Theorem 4.17 (Borel regularity) *Let A be a subset of $\mathbb{R}^d$. Then*

$$\mu^*(A) \;=\; \inf\left\{\mu(G) : A \subseteq G, G \text{ is open}\right\}. \tag{4.3}$$

In addition, if $A \in \mathcal{L}(\mathbb{R}^d)$ then

$$\mu(A) = \inf\left\{\mu(G) : A \subseteq G, G \text{ is open}\right\} = \sup\left\{\mu(K) : K \subseteq A, K \text{ is closed}\right\}. \tag{4.4}$$

Proof. 1. (4.3) is trivial if $\mu(A) = +\infty$. Assume that $\mu(A) < +\infty$. For every $\varepsilon > 0$, it holds

$$A \;\subseteq\; \bigcup_{i=1}^{\infty} R_i \qquad \text{and} \qquad \mu^*(A) \;\geq\; \sum_{i=1}^{\infty} V(R_i) + \frac{\varepsilon}{2}.$$

For every $i \geq 1$, there exists $S_i \in \mathcal{R}(\mathbb{R}^d)$ such that

$$R_i \;\subset\; \operatorname{int}(S_i) \qquad \text{and} \qquad V(S_i) \;\leq\; V(\mathbb{R}_i) + \frac{\varepsilon}{2^{i+1}}.$$

Thus, the set $G_\varepsilon := \bigcup_{i=1}^{\infty} \text{int}(S_i) \supset A$ is open and

$$\mu(G_\varepsilon) \;\leq\; \sum_{i=1}^{\infty} V(R_i) + \frac{\varepsilon}{2^{i+1}} \;\leq\; \mu^*(A) + \varepsilon.$$

and it yields (4.3).

2. Let's prove (4.4) for $A \in \mathcal{L}(\mathbb{R}^d)$. For every $\varepsilon > 0$, we need to find K_ε compact subset of $\mathbb{R}^d$ such that

$$K_\varepsilon \;\subseteq\; A \qquad \text{and} \qquad \mu(K_\varepsilon) \;\geq\; \mu(A) + \varepsilon.$$

If A is bounded then there exists $F \in \mathcal{R}(\mathbb{R}^d)$ such that $A \subseteq F$. From the previous result, there exists an open set G_ε such that

$$F \backslash A \;\subseteq\; G_\varepsilon \qquad \text{and} \qquad \mu(G_\varepsilon) \;\leq\; \mu(F \backslash A) + \varepsilon.$$

Thus, the set $K_\varepsilon := F \backslash G_\varepsilon \subseteq A$ is compact and satisfies

$$\mu(A) \;=\; \mu(F) - \mu(F \backslash A) \;\leq\; \mu(F) - \mu(G_\varepsilon) + \varepsilon \;=\; \mu(K_\varepsilon) + \varepsilon.$$

Otherwise, if A is unbounded then one can consider the set $A_R = A \cap B(0, R)$ and let R to $+\infty$. □

With the same argument in theorem 4.8, one can show that

Theorem 4.18 *The set $A \in \mathcal{P}(\mathbb{R}^d)$ is Lebesgue measurable if and only if for every $\varepsilon > 0$, there exists O_ε open set such that*

$$A \;\subseteq\; O_\varepsilon \qquad \text{and} \qquad \mu^*(O_\varepsilon \backslash A) \;\leq\; \varepsilon.$$

This theorem showed that $\mathcal{B}(\mathbb{R}^d)$ is "dense" is $\mathcal{L}(\mathbb{R}^d)$ with respect to Lebesgue measure μ. To conclude this subsection, we prove the following theorem.

Theorem 4.19 $\big(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), \mu\big)$ *is the completion of* $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu\big)$.

Proof. Let's consider the completion of $\mathcal{B}(\mathbb{R}^d)$

$$\overline{\mathcal{B}}(\mathbb{R}^d) \;:=\; \left\{A \cup M : A \in \mathcal{B}(\mathbb{R}^d), M \subseteq N \in \mathcal{B}(\mathbb{R}^d), \mu(N) = 0\right\}.$$

We show that

$$\overline{\mathcal{B}}(\mathbb{R}^d) \;=\; \mathcal{L}(\mathbb{R}^d).$$

Since $\mu^*(N) = \mu(N) = 0$, one has that M is in $\mathcal{L}(\mathbb{R}^d)$ for every $M \subseteq N$. Thus,

$$\overline{\mathcal{B}}(\mathbb{R}^d) \;\subseteq\; \mathcal{L}(\mathbb{R}^d).$$

Let A be a set in $\mathcal{L}(\mathbb{R}^d)$. For any $\ell \geq 0$, there exist K_ℓ closed and O_ℓ open sets such that

$$K_\ell \subseteq A \subseteq O_\ell \quad \text{and} \quad \mu(O_\ell \backslash K_\ell) < \frac{1}{\ell}.$$

The following Borel sets $K := \bigcup_{\ell \geq 1} K_\ell$ and $O := \bigcap_{\ell \geq 1} O_\ell$ satisfy

$$K \subseteq A \subseteq O \quad \text{and} \quad \mu(O \backslash K) = 0.$$

Thus,

$$A = K \bigcup (A \backslash K) \in \overline{\mathcal{B}}(\mathbb{R}^d).$$

and this complete the proof. □

Problem 21: Construct a set $E \subset \mathbb{R}^2$, where $E = E_1 \times E_2$ (Cartesian product) so that E is Lebesgue measurable in $\mathbb{R}^2$, but E_1 is not Lebesgue measurable in $\mathbb{R}$. Show that if E_1 and E_2 are both Lebesgue measurable in $\mathbb{R}$ then will $E_1 \times E_2$ be Lebesgue measurable in $\mathbb{R}^2$?

4.4 Hausdorff Measure

Here we state definitions and properties of the Hausdorff measure, without any detailed proofs. For any $0 \leq s < +\infty$, we define

$$\alpha_s = \frac{\pi^{s/2}}{\Gamma(s/2+1)} \quad \text{with} \quad \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \ \ (\textit{Euler Gamma function}).$$

Notice that α_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$ for every $d \in \mathbb{N}$. For any given $0 \leq s \leq d$ and $\delta > 0$, we define for every $E \subset \mathbb{R}^d$ that

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{n=1}^\infty \alpha_s \cdot \left(\frac{\mathrm{diam}(E_n)}{2} \right)^s : E \subset \bigcup_{n=1}^\infty E_n, diam(E_n) < \delta \right\}.$$

If $s = 0$ we only sum over non-empty E_n. Since the map $\sigma \to H^s_\delta(E)$ is non-increasing, we then define

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

This is known as the *Hausdorff* metric outer measure.

Proposition 4.19.1 *The Hausdorff outer measure H^s satisfies the following properties:*

(i) $\mathcal{H}^0$ is the counting measure (meaning it just returns the number of elements in a set).

(ii) $\mathcal{H}^s \equiv 0$ *for* $s > d$.

(iii) $\mathcal{H}^s(x+E) = \mathcal{H}^s(E), \qquad \mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$.

(iv) $\mathcal{H}^d = \mathcal{L}^d$.

We can use this outer measure to provide a notion of dimensionality of a set.

Proposition 4.19.2 *Let* $0 \le s < t < \infty$*, and* $E \subset \mathbb{R}^d$*. Then*

1. *If* $\mathcal{H}^s_\delta(E) = 0$ *for* $0 < \delta$*, then* $\mathcal{H}^s(E) = 0$.
2. *If* $\mathcal{H}^s(E) < \infty$ *then* $\mathcal{H}^t(E) = 0$.
3. *If* $\mathcal{H}^t(E) > 0$ *then* $\mathcal{H}^s(E) = \infty$.

With the previous proposition in mind, we can define

$$dim_{\mathcal{H}}(E) := \inf \left\{ 0 \le s \le \infty : \mathcal{H}^s(E) = 0 \right\}.$$

the Hausdorff dimension of E.

Example 4.4 *The Hausdorff dimension of the Cantor set* C *is* $ln(2)/ln(3)$.

Example 4.5 *Any hyperplane with dimension* k *will have Hausdorff dimension* k.

The Hausdorff measure is well-equipped to handle mappings between spaces of different dimension:

Proposition 4.19.3 *Suppose* $E \subset \mathbb{R}^k$ *and* $f : E \to \mathbb{R}^d$ *be a Hölder continuous function, meaning that for some* $L > 0$, $\theta \in]0,1]$ *we have*

$$\|f(x) - f(y)\| \;\le\; L\|x-y\|^\eta \qquad \text{for all } x, y \in E.$$

Then,

$$\mathcal{H}^{s/\eta}(f(E)) \le \frac{\alpha_{s/\eta} 2^s}{\alpha_s 2^{s/\eta}} L^{s/\eta} \mathcal{H}^s(E).$$

As many geometric objects (i.e. curves and surfaces) are defined in terms of smooth mappings from between $\mathbb{R}^d$ and $\mathbb{R}^m$, it turns out that $\mathcal{H}^s$ is well-suited to describe many geometric objects.

Proposition 4.19.4 *Given a continuous curve* γ *parametrized as* $f : [a,b] \to \mathbb{R}^d$*, we define*

$$length(\gamma) := \sup \left\{ \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\| : a = t_0 < t_1 < \cdots < t_n = b \right\},$$

where the supremum is taken over possible partitions $\{t_i\}$*. Then for any Lipschitz* f *we will have*

$$\mathcal{H}^1(f([a,b])) = length(\gamma).$$

In addition, if $f \in C^1$ *then* $length(\gamma) = \int_a^b |f'(t)|\, dt$.

The same idea holds for more general manifolds:

Proposition 4.19.5 *Let $M \subset \mathbb{R}^d$ be a k-dimensional manifold of class C^1. Let ϕ be a local chart of the manifold, meaning that for $A \subset \mathbb{R}^k$, $\phi : A \to M$ is a C^1 function, and $\nabla\phi$ has rank k in A. Then we have that*

$$\mathcal{H}^k(\phi(A)) = \textit{surface area}\Big(\phi(A)\Big) = \int_A \sqrt{det(\partial_{y_i}\phi \cdot \partial_{y_j}\phi)}\, dy.$$

In particular, if the manifold is the graph of a function $f : \mathbb{R}^{d-1} \to \mathbb{R}$ and $M = \{(x, f(x)) : x \in A\}$, then

$$\mathcal{H}^{d-1}(M) = \textit{surface area}\Big(M\Big) = \int_A \sqrt{1 + |\nabla f|^2}\, dx.$$

In turn, M has Hausdorff dimension k and $\mathcal{H}^k$ corresponds to the standard surface measure.

Problem 22: Let $E = \{(x, x^2) : x \in [0, 1]\}$, $E \subset \mathbb{R}^2$. Prove that $dim_H(E) = 1$. Compute $\mathcal{H}^1(E)$.

Problem 23: Prove Proposition 4.19.3

5 Measurable functions

Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces.

Definition 5.1 *A map $f : X \to Y$ is measurable if and only if*

$$f^{-1}(B) \in \mathcal{M} \qquad \textit{for all } B \in \mathcal{N}.$$

In the case, (Y, d) is a metric space and $\mathcal{N} = \mathcal{B}(Y)$ Borel σ-algebra, we will say that f is a Borel function.

Proposition 5.1.1 *Let $\mathcal{F}$ be such that $\sigma(\mathcal{F}) = \mathcal{N}$. Then $f : X \to Y$ is measurable if*

$$f^{-1}(F) \in \mathcal{M} \qquad \textit{for all } F \in \mathcal{F}.$$

Proof. Observe that the collection

$$\mathcal{N}_f := \{A \in \mathcal{N} : f^{-1}(A) \in \mathcal{M}\} \subseteq \mathcal{N}$$

is an σ-algebra on Y. Since $\mathcal{F} \subseteq \mathcal{N}_f$, it holds

$$\sigma(\mathcal{F}) \subseteq \mathcal{N}_f$$

and it yields

$$\mathcal{N}_f = \mathcal{N}.$$

The proof is complete. □

Corollary 5.2 *Let (X, d_X) and (Y, d_Y) be metric space and $\mathcal{M} = \mathcal{B}(X)$, $\mathcal{N} = \mathcal{B}(Y)$. Then any continuous function $f : X \to Y$ is measurable.*

Proof. Let $\mathcal{F}$ be a collection of all open sets in Y. By the continuity of f, it holds

$$f^{-1}(A) \text{ is open for all } A \in \mathcal{F},$$

and this particularly yields

$$f^{-1}(A) \in \mathcal{B}(X) \qquad \text{for all } A \in \mathcal{F}.$$

Since $\sigma(\mathcal{F}) = \mathcal{B}(Y)$, the above proposition implies that f is measurable. □

Problem 24: Assume that $f : X \to Y$ is measurable. Then

$$\mathcal{U}_f = \{f^{-1}(B) : B \in \mathcal{N}\}$$

is the smallest σ-algebra with respect to which f is measurable.

5.1 Real value functions

In the following discussion, we will be letting $\mathcal{N} = \mathcal{B}(\mathbb{R})$ (as is a common convention).

5.1.1 Basic properties

Given a measurable space $(X, \mathcal{M})$, Consider the real value function

$$f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}.$$

One can show that f is measurable if any only if one of the following holds

(a). $f^{-1}([-\infty, b)) \in \mathcal{M}$ for all $b \in \mathbb{R}$;

(b). $f^{-1}([-\infty, b]) \in \mathcal{M}$ for all $b \in \mathbb{R}$;

(c). $f^{-1}((a, +\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$;

(d). $f^{-1}([a, +\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

Lemma 5.3 *Assume that $f, g : X \to \overline{R}$ are measurable. Then the following functions*

$$cf, \quad f + g, \quad f^2, \quad f \cdot g, \quad |f|, \quad \max\{f, g\} \quad \text{and} \quad \min\{f, g\}$$

are measurable.

Proof. 1. Let's show that $f+g$ is measurable. For any $b \in \mathbb{R}$, it holds

$$\begin{aligned}(f+g)^{-1}([\infty,b)) &= \{x \in X : f(x)+g(x) < b\} \\ &= \bigcup_{q\in\mathbb{Q}} \Big[\{x \in X : f(x) < q\} \bigcap \{x \in X : g(x) < b-q\}\Big] \ \in \ \mathcal{M}.\end{aligned}$$

Thus, $f+g$ is measurable.

2. f^2 is measurable. For any $b \geq 0$, it holds

$$\begin{aligned}(f^2)^{-1}([-\infty,b)) &= \{x \in X : f^2(x) < b\} \ = \ \Big\{x \in X : |f(x)| < \sqrt{b}\Big\} \\ &= \{x \in X : -\sqrt{b} < f(x) < \sqrt{b}\} \ = \ f^{-1}(]-\sqrt{b},\sqrt{b}[) \ \in \ \mathcal{M}.\end{aligned}$$

As a consequence, the function

$$f \cdot g \ = \ \frac{1}{4}\left[(f+g)^2 - f^2 - g^2\right]$$

is measurable.

3. One can show that $|f|$ is measurable. Thus,

$$\max\{f,g\} \ = \ \frac{f+g+|f-g|}{2} \qquad \text{and} \qquad \min\{f,g\} \ = \ \frac{f+g-|f-g|}{2}$$

are measurable. □

Problem 25: Prove or give a counterexample: if $(X,\mathcal{M})$ is a measurable space and $f : X \to [-\infty,\infty]$ is a function such that

$$f^{-1}\left((a,\infty)\right) \ \in \ \mathcal{M} \qquad \text{for all } a \in \mathbb{R},$$

then f is is a measurable function.

Problem 26: Give an example of a measurable space $(X,\mathcal{M})$ and $f : X \to \mathbb{R}$ such that $|f|$ is measurable but f is not measurable.

Point-wise convergence. Measurability is preserved by limiting operations of sequences of functions. Here operations are understood in a point-wise sense.

Proposition 5.3.1 *If $f_n : X \to \overline{R}$ is measurable for all $n \geq 1$ then*

$$\sup_{n\geq 1} f_n, \qquad \inf_{n\geq 1} f_n, \qquad \limsup_{n\to\infty} f_n \qquad \text{and} \qquad \liminf_{n\to\infty} f_n$$

are measurable on X

Proof. For any $b \in \mathbb{R}$, it holds

$$\left(\sup_{n\geq 1} f_n\right)^{-1}([-\infty, b)) \;=\; \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) < b\} \;\in\; \mathcal{M}$$

and

$$\left(\inf_{n\geq 1} f_n\right)^{-1}([-\infty, b)) \;=\; \bigcup_{n=1}^{\infty} \{x \in X : f_n(x) < b\} \;\in\; \mathcal{M}$$

Thus, $\sup_{n\geq 1} f_n$ and $\inf_{n\geq 1} f_n$ are measurable. As a consequence, one has that

$$\limsup_{n\to\infty} f_n \;=\; \inf_{n\geq 1}\left(\sup_{k\geq n} f_k\right) \qquad \text{and} \qquad \liminf_{n\to\infty} f_n \;=\; \sup_{n\geq 1}\left(\inf_{k\geq n} f_k\right)$$

are measurable. □

Definition 5.4 *A sequence of function $(f_n)_{n\geq 1}$ converges pointwise to f if*

$$\lim_{n\to\infty} f_n(x) \;=\; f(x) \qquad \textit{for all } x \in X.$$

From the above proposition, pointwise convergence preserves the measurability.

Corollary 5.5 *If f_n are measurable and converges pointwise to f then f is also measurable.*

Problem 27: Give an example of a measurable space $(X, \mathcal{M})$ and a family $(f_t)_{t\in\mathbb{R}}$ such that $f_t : X \to [0,1]$ is measurable for every $t \in \mathbb{R}$ but the function $f : X \to [0,1]$ defined by

$$f(x) \;:=\; \sup_{t\in\mathbb{R}} f_t(x) \qquad \text{for all } x \in X$$

is not measurable.

5.1.2 Simple functions

Given $A \in \mathcal{P}(X)$, the function $X_A : X \to \mathbb{R}$ defined by

$$X_A(x) \;=\; \begin{cases} 1 & \text{for all } x \in A \\ \\ 0 & \text{for all } x \in A^c \end{cases}$$

is called the characteristic function of the set A. It is clear that X_A is a measurable function if A is a measurable set.

Definition 5.6 *The function $\varphi : X \to \mathbb{R}$ is simple if*

$$\varphi \;=\; \sum_{i=1}^{N} c_i \cdot \chi_{E_i}$$

for some $c_i \in \mathbb{R}$ and $E_i \in \mathcal{M}$.

It is clear that a simple function is measurable.

Remark 5.7 *If φ is a simple function then there exists $(A_i)_{i=1}^N \subset \mathcal{M}$ a collection of finite mutually disjoint measurable sets such that*

$$\varphi \;=\; \sum_{i=1}^{N} d_i \cdot \chi_{A_i}.$$

Theorem 5.8 (Approximation by simple functions) *If $f : X \to \overline{R}$ is a non-negative measurable function then there exists a sequence of non-negative simple functions $(\varphi_n)_{n\geq 1}$ such that*

$$\varphi_n \to f \quad \text{pointwise.}$$

In addition, if f is bounded then φ_n converges uniformly to f, i.e.,

$$\lim_{n\to\infty} \|\varphi_n - f\|_\infty \;=\; 0.$$

Proof. 1. For any $n \geq 1$, divide *Range*$(f) = f(X)$ into sub-intervals with length 2^{-n}. More precisely, let

$$B_n \;:=\; \{x \in X : f(x) \geq n\}$$

and

$$I_{k,n} \;:=\; \big[k \cdot 2^{-n}, (k+1)\cdot 2^{-n}\big) \qquad \text{for all } k \in \{0, 1, 2, \ldots, n2^n - 1\},$$

consider the function $\varphi_n : X \to [0, +\infty]$ such that

$$\varphi_n \;=\; \sum_{k=0}^{n\cdot 2^n - 1} k2^{-n} \cdot \chi_{A_{k,n}} + n \cdot \chi_{B_n} \qquad \text{with} \qquad A_{k,n} \;=\; f^{-1}(I_{k,n}).$$

One can see that

- φ_n are non-negative simple functions for every $n \geq 1$;
- $(\varphi_n)_{n\geq 1}$ is a increasing sequence of functions, i.e.,

$$\varphi_n(x) \;\leq\; \varphi_{n+1}(x) \qquad \text{for all } x \in X.$$

2. Let us know show that $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in X$. Two cases may occur

- If $f(x) = +\infty$ then $\varphi_n(x) = n$ for all $n \geq 1$ and it yields $\lim_{n\to\infty} \varphi_n(x) = +\infty$.
- If $f(x) \in [0, +\infty)$ then for every $n \geq f(x)$, we have

$$0 \ < \ f(x) - \varphi_n(x) \ < \ 2^{-n}.$$

  Thus, $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

Therefore, φ_n converges pointwise to f.

3. Assume that $|f(x)| < M$ for all $x \in X$. We then have that

$$|f(x) - \varphi_n(x)| \ < \ 2^{-n} \qquad \text{for all } x \in X, n \geq M.$$

Thus,

$$\lim_{n\to\infty} \sup_{x\in X} |f(x) - \varphi_n(x)| \ = \ 0$$

and it yields that φ_n converges uniformly to f. □

Corollary 5.9 *Let $f : X \to \mathbb{R}$ be measurable and uniformly bounded by M. Then for $\varepsilon > 0$ there exists simple functions φ_ε and ψ_ε such that*

$$\varphi_\varepsilon \ < \ f \ < \ \psi_\varepsilon \qquad \text{and} \qquad \|\varphi_\varepsilon - \psi_\varepsilon\|_\infty \ < \ \varepsilon.$$

Proof. Decompose f into two parts

$$f \ = \ f^+ - f^- \qquad \text{with} \qquad \begin{cases} f^+ \ = \ \max\{0, f\}, \\ \\ f^- \ = \ \max\{0, -f\} \end{cases}$$

and then apply the previous lemma. □

Problem 28: Given $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ measurable space and μ a measure on $\mathcal{M}$, let $f : X \to Y$ be measurable. Denote by $f_\sharp\mu : \mathcal{N} \to [0, +\infty]$ such that

$$f_\sharp\mu(A) \ = \ \mu(f^{-1}(A)) \qquad \text{for all } A \in \mathcal{N}.$$

Show that $f_\sharp\mu$ is a measure on $\mathcal{N}$ (called the push forward of μ under f).

Problem 29: Given $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ measurable space, let $f : X \to Y$ be such that $f(X)$ is countable. Show that f is measurable if

$$f^{-1}(y) \ \in \ \mathcal{M} \qquad \text{for all } y \in Y.$$

Problem 30: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function. Then there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for almost every $x \in \mathbb{R}$.

5.1.3 Convergence almost everywhere

Definition 5.10 *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\big(f_n : X \to \overline{\mathbb{R}}\big)_{n\geq 1}$ be a sequence of functions on X. We say that*

- *f_n converges to f almost everywhere ($f_n \xrightarrow{a.e.} f$) if there exists $E \in \mathcal{M}$ with $\mu(E) = 0$ such that*

$$\lim_{n\to\infty} f_n(x) \;=\; f(x) \qquad \text{for all } x \in X\backslash E.$$

- *f_n converges to f almost uniformly ($f_n \xrightarrow{a.u.} f$) if f is finite and for every $\varepsilon > 0$ there exists $E_\varepsilon \in \mathcal{M}$ with $\mu(E_\varepsilon) < \varepsilon$ such that*

$$\lim_{n\to\infty} \left(\sup_{x\in X\backslash E_\varepsilon} |f_n(x) - f(x)| \right) \;=\; 0.$$

Basic properties: Assume that $f_n : X \to \overline{\mathbb{R}}$ are Borel functions. The following statements hold

(i). If the measure space $(X, \mathcal{M}, \mu)$ is complete then

$$f_n \xrightarrow{\text{a.e.}} f \qquad \Longrightarrow \qquad f \text{ is measurable.}$$

(ii) If f_n converges to f almost uniformly then f_n converges to f almost everywhere.

(iii) If $f_n \xrightarrow{\text{a.e.}} f$ and $f_n \xrightarrow{\text{a.e.}} g$ then

$$f(x) \;=\; g(x) \quad \text{for a.e. } x \in X.$$

Theorem 5.11 (Severini-Egorov) *Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of Borel functions. Assume that $(X, \mathcal{M}, \mu)$ is a finite measure space, i.e., $\mu(X) < \infty$. If f is a finite Borel function then*

$$f_n \xrightarrow{a.e.} f \qquad \Longrightarrow \qquad f_n \xrightarrow{a.u.} f.$$

Proof. Assume that f_n converges to f almost everywhere. Then there exists $E \in \mathcal{M}$ with $\mu(E) = 0$ such that

$$\lim_{n\to\infty} f_n(x) \;=\; f(x) \qquad \text{for all } x \in X\backslash E.$$

For every $\varepsilon > 0$, we want to find $E_\varepsilon \in \mathcal{M}$ such that $\mu(E_\varepsilon) < \varepsilon$ and

$$\lim_{n\to\infty} \left(\sup_{x\in X\backslash E_\varepsilon} |f_n(x) - f(x)| \right) \;=\; 0.$$

1. For any $k, n \geq 1$, denote by

$$E_{k,n} \;=\; \bigcup_{i=n}^{\infty} \left\{ x \in X : |f - f_i| > \frac{1}{k} \right\}.$$

It is clear that $E_{k,n}$ is measurable. Moreover, the sequence of sets $(E_{k,n})_{n\geq 1}$ is decreasing for any $k \geq 1$. This implies that

$$\lim_{n\to\infty} E_{k,n} \;=\; \bigcap_{n=1}^{\infty} E_{k,n} \;=\; \left\{ x \in X : \limsup_{n\to\infty} \, |f_n(x) - f(x)| > \frac{1}{k} \right\} \;:=\; E_k.$$

Since $\mu(E_{k,1}) \leq \mu(X) < +\infty$, one has

$$\lim_{n\to\infty} \mu(E_{k,n}) \;=\; \mu(E_k).$$

By the definition of E_k, it holds

$$E_k \;\subseteq\; E \qquad \text{for all } k \geq 1$$

and it yields

$$\lim_{n\to\infty} \mu(E_{k,n}) \;=\; \mu(E_k) \;=\; 0 \qquad \text{for all } k \geq 1.$$

2. For every $\varepsilon > 0$, one can find a sequence $(n_k)_{k\geq 1}$ such that

$$\mu(E_{k,n_k}) \;\leq\; \frac{\varepsilon}{2^k} \qquad \text{for all } k \geq 1.$$

The set $E_\varepsilon := \displaystyle\bigcup_{k=1}^{\infty} E_{k,n_k}$ has measure

$$\mu(E_\varepsilon) \;\leq\; \sum_{k=1}^{\infty} \mu(E_{k,n_k}) \;\leq\; \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \;=\; \varepsilon.$$

Moreover, for any $k \geq 1$, it holds

$$\sup_{x\in X\setminus E_\varepsilon} |f(x) - f_i(x)| \;\leq\; \frac{1}{k} \qquad \text{for all } i \geq n_k.$$

Thus,

$$\lim_{n\to\infty} \left(\sup_{x\in X\setminus E_\varepsilon} |f_n(x) - f(x)| \right) \;=\; 0$$

and the proof is complete. □

Remark 5.12 *The above theorem will be false in general if $\mu(X) = +\infty$. Indeed, let's consider $X = \mathbb{R}$ and*

$$f_n = \chi_{[n,+\infty)} \qquad \text{for all } n \geq 1.$$

We have that f_n converges to 0 pointwise. On the other hand, for every $E \in \mathcal{M}$ with $m(E) = 0$, one has

$$\sup_{x\in\mathbb{R}\setminus E} f_n(x) = 1 \qquad \text{for all } n \geq 1.$$

and this implies that

$$\lim_{n\to\infty}\left(\sup_{x\in\mathbb{R}\setminus E} f_n(x)\right) = 1.$$

Thus, f_n does not converge to f almost uniformly.

Problem 31: Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n : X \to \mathbb{R}\}_{n\geq 1}$ be a sequence of measurable functions. For any $n \geq 1$, denote by

$$E_n \doteq \left\{x \in X : |f_{n+1}(x) - f_n(x)| > \frac{1}{2^n}\right\}.$$

Assume that

$$\mu(E_n) \leq \frac{1}{2^n} \qquad \text{for all } n \geq 1,$$

show that $\{f_n\}_{n\geq 1}$ is pointwise convergent a.e on X.

Problem 32: Let $(X, \mathcal{M}, \mu)$ be a measure space. A sequence of measurable functions $\{f_n : X \to \mathbb{R}\}_{n\geq 1}$ is said to *converge in measure* to a measurable function f if for every $\eta > 0$, it holds

$$\lim_{n\to\infty} \mu\{x \in X : |f_n(x) - f(x)| > \eta\} = 0.$$

(i) Show that if $\mu(X) < \infty$ and $\{f_n\}_{n\geq 1}$ converges point-wise a.e. on X to a measurable function f then $\{f_n\}_{n\geq 1}$ converges in measure to f.

(ii) Show that $\{f_n\}_{n\geq 1}$ converges in measure to f, then there exists a subsequences of $\{f_n\}_{n\geq 1}$ converges point-wise a.e. on X to f

Problem 33: Let g be a measurable function which is finite almost everywhere. Suppose that f_n converges in measure to f, and that μ is a finite measure. Prove that

(i) $f_n \cdot g$ converges in measure to $f \cdot g$.

(ii) if g_n converges to g in measure, with f, g are finite a.e., then $f_n \cdot g_n$ converges in measure to $f \cdot g$.

5.1.4 Approximation by continuous functions

Definition 5.13 *A measure $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, +\infty]$ is called a Radon measure if $\mu(K)$ is finite for every compact subset K.*

Theorem 5.14 (Lusin) *Let μ be a Radon measure on $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be Borel. Assume that*

$$\{x : f(x) \neq 0\} \ \subset \ A,$$

for some Borel set A with $\mu(A) < \infty$. Then for every $\varepsilon > 0$, there exits a continuous function $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ such that

$$supp(f_\varepsilon) \ = \ \{x \in \mathbb{R}^d : f_\varepsilon(x) \neq 0\}$$

is contained in a compact set, and

$$\mu\left\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq f(x)\right\} \ < \ \varepsilon.$$

Proof. The proof is divided into several steps.

1. Assume that A is compact and $0 \leq f < 1$. In this case, there exists a bounded open set V such that

$$A \ \subset \ V \qquad \text{and} \qquad \overline{V} \ \text{ is compact.}$$

Recalling the construction in theorem 5.8, the function f can be approximated uniformly by a sequence of simple functions $(\varphi_n)_{n\geq 1}$ defined as follows

$$\varphi_n \ = \ \sum_{k=0}^{2^n-1} k2^{-n} \cdot \chi_{A_{k,n}} \qquad \text{with} \qquad A_{k,n} \ = \ f^{-1}\left[k2^{-n}, (k+1)2^{-n}\right).$$

Observe that

$$\begin{cases} \varphi_1 & = \ \dfrac{1}{2} \cdot \chi_{B_1} \qquad \text{with} \qquad B_1 \ = \ \left\{x \in \mathbb{R}^d : f(x) \geq \dfrac{1}{2}\right\}, \\ \\ \varphi_n - \varphi_{n-1} & = \ \dfrac{1}{2^n} \cdot \chi_{B_n} \qquad \text{with} \qquad B_n \ = \ \left\{x \in \mathbb{R}^d \ : \ f(x) - \varphi_{n-1} \ \geq \ \dfrac{1}{2^n}\right\}, \end{cases}$$

we have

$$\varphi_n \ = \ \sum_{i=1}^{n} \frac{1}{2^n} \cdot \chi_{B_n} \qquad \text{and} \qquad \lim_{n\to\infty} \left(\sup_{x \in X} \ |\varphi_n(x) - f(x)| \right) \ = \ 0. \qquad (5.1)$$

Finally, since $\text{supp}(f) \subseteq A$, it holds

$$B_n \ \subseteq \ A \ \subseteq \ V \qquad \text{for all } n \geq 1.$$

2. For a fixed $\varepsilon > 0$, there exist compact sets K_n and open sets W_n such that

$$\begin{cases} K_n \subseteq B_n \subseteq W_n \\ \mu(W_n \backslash K_n) < \dfrac{\varepsilon}{2^n} \end{cases} \qquad \text{for all } n \geq 1.$$

Set $V_n = W_n \bigcap V$, we have

$$K_n \subseteq B_n \subseteq V_n \qquad \text{and} \qquad \mu(V_n \backslash K_n) < \frac{\varepsilon}{2^n} \qquad \text{for all } n \geq 1.$$

For any $n \geq 1$, let $g : \mathbb{R}^d \to [0,1]$ be such that

$$g_n(x) = \frac{d_{V_n^c}(x)}{d_{K_n}(x) + d_{V_n^c}(x)} \qquad \text{for all } x \in \mathbb{R}^d.$$

Since $d_{K_n} + d_{V_n^c} > d(K_n, V_n^c) > 0$, the function g_n is a continuous with value in $[0,1]$ and satisfies

$$g_n(x) = \begin{cases} 1 & \text{for all } x \in K_n \\ 0 & \text{for all } x \in V_n^c. \end{cases} \qquad \text{for all } n \geq 1.$$

Thus, g_n is an approximation of the characteristic function χ_{B_n} for every $n \geq 1$.

3. A continuous approximation f_ε of f is defined by

$$f_\varepsilon(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot g_n(x) \qquad \text{for all } x \in \mathbb{R}^d.$$

We first claim that f_ε is continuous. Indeed, given any $x \in \mathbb{R}^d$, we estimate

$$\begin{aligned} |f_\varepsilon(y) - f_\varepsilon(x)| &\leq \left(\sum_{n=1}^{N} \frac{1}{2^n} \cdot |g_n(y) - g_n(x)|\right) + \left(\sum_{n=N+1}^{\infty} \frac{1}{2^n} \cdot |g_n(y) - g_n(x)|\right) \\ &\leq \left(\sum_{n=1}^{N} \frac{1}{2^n} \cdot |g_n(y) - g_n(x)|\right) + \frac{1}{2^N} \end{aligned}$$

for all $y \in \mathbb{R}^d$ and $N \geq 1$. This implies that

$$\limsup_{y \to x} |f_\varepsilon(y) - f_\varepsilon(x)| \leq \limsup_{y \to x} \left(\sum_{n=1}^{N} \frac{1}{2^n} \cdot |g_n(y) - g_n(x)|\right) + \frac{1}{2^N} = \frac{1}{2^N}.$$

Taking $N \to \infty$, we obtain that

$$\limsup_{y \to x} |f_\varepsilon(y) - f_\varepsilon(x)| = 0.$$

Thus, f_ε is continuous at x. Moreover, since g_n is non-negative, it holds

$$\begin{aligned} \operatorname{supp}(f_\varepsilon(x)) &= \left\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq 0\right\} \subseteq \bigcup_{n=1}^{\infty} \left\{x \in \mathbb{R}^d : g_n(x) \neq 0\right\} \\ &\subseteq \bigcup_{n=1}^{\infty} V_n \subseteq V \subseteq \overline{V} \text{ compact.} \end{aligned}$$

Finally, we have

$$\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq f(x)\} \subseteq \bigcup_{n=1}^{\infty} \{x \in \mathbb{R}^d : \chi_{B_n}(x) \neq g_n(x)\} \subseteq \bigcup_{n=1}^{\infty} (V_n \backslash K_n),$$

and this yields

$$\mu(x \in \mathbb{R}^d : f_\varepsilon(x) \neq f(x)) \leq \sum_{n=1}^{\infty} \mu(V_n \backslash K_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

4. Let's extend our result to the case $|f(x)| < M$. In this case, we replace $f = \dfrac{f}{|M|}$ and has that $|f| < 1$. Then we decompose

$$f = f^+ - f^- \qquad \text{with} \qquad \begin{cases} f^+ = \max\{0, f\} \\ \\ f^- = \max\{0, -f\}. \end{cases}$$

Since f^+, f^+ take values in $[0, 1[$, for every $\varepsilon > 0$ there exist $f_\varepsilon^\pm$ continuous such that $\operatorname{supp}(f_\varepsilon^\pm)$ are contained in compact sets and

$$\mu\left(x \in \mathbb{R}^d : f_\varepsilon^\pm(x) \neq f^\pm(x)\right) \leq \frac{\varepsilon}{2}.$$

Then the approximation of f is $f_\varepsilon = f_\varepsilon^+ - f_\varepsilon^-$.

5. Let's now remove the compactness of A. From the density theorem (which we gave for the Lebesgue measure in Theorem 4.8, but which holds for Radon measures), for every $A \in \mathcal{B}(\mathbb{R}^d)$, there exists a compact set $\tilde{A}$ such that

$$\tilde{A} \subset A \qquad \text{and} \qquad \mu(A \backslash \tilde{A}) < \frac{\varepsilon}{2}.$$

Set $\tilde{f} := f \cdot \chi_{\tilde{A}}$. From the previous step, there exists $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ continuous with support contained in a compact set such that

$$\mu\left(\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq \tilde{f}(x)\}\right) \leq \frac{\varepsilon}{2}.$$

Thus,

$$\mu\left(\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq f(x)\}\right) \ \leq \ \frac{\varepsilon}{2} \ \leq \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ = \ \varepsilon.$$

6. Finally, let's us remove the boundedness of f. Consider the decreasing sequence of subsets of A

$$A_n \ = \ \{x \in \mathbb{R}^d : |f(x)| > n\} \qquad \text{for all } n \geq 1\,.$$

Since $\mu(A_1) \leq \mu(A) < \infty$ and $\bigcap_{n=1}^{\infty} A_n = \varnothing$, it holds

$$\lim_{n\to\infty} \mu(A_n) \ = \ \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \ = \ 0.$$

For any $\varepsilon > 0$, there exists $n_0 > 0$ sufficiently large such that $\mu(A_0) < \frac{\varepsilon}{2}$ and set

$$\tilde{f}_\varepsilon \ = \ (1 - \chi_{A_{n_0}}) \cdot f.$$

Since $\tilde{f}_\varepsilon$ is bounded by n_0, then exists f_ε continuous with compact support such that

$$\mu\left(\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq \tilde{f}_\varepsilon(x)\}\right) \ < \ \frac{\varepsilon}{2}.$$

This implies that

$$\mu\left(\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq f(x)\}\right) \ < \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ = \ \varepsilon$$

and the proof is complete. □

Corollary 5.15 *Let μ be a Radon measure on $\mathcal{B}(\mathbb{R}^d)$ and let A be a Borel set with $\mu(A) < +\infty$. Assume that $f : A \to \mathbb{R}$ is a Borel function. Then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq A$ with $\mu(A \backslash K_\varepsilon) < \varepsilon$ such that the restriction of f on K_ε, denote by $f_{|K_\varepsilon}$, is continuous.*

Problem 34: Give an example of a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that there does not exist a set $B \subset \mathbb{R}$ such that $m\left(B^c\right) = 0$ and $f_{|B}$ is a continuous function on B

Problem 35: Let $B \subset \mathbb{R}^d$ be a set of finite Lebesgue measure, with bounded support. Construct a function $\phi_\varepsilon \in C_c^\infty$ so that $m(\{\phi_\epsilon \neq \chi_B\}) < \varepsilon$.

6 Lebesgue Integration

6.1 Non-negative measurable function

Given a measure space $(X, \mathcal{M}, \mu)$, we denote a class of non-negative simple functions

$$S_+(X) = \Big\{\varphi : X \to [0, +\infty) : \varphi = \sum_{i=1}^{n} a_i \cdot \chi_{A_i} \quad \text{with} \quad a_i \in [0, +\infty) \quad \text{and}$$
$$A_1, A_2, \ldots, A_n \text{ are mutually disjoint sets in } \mathcal{M}\Big\}.$$

The Lebesgue integral of φ is defined as follows:

Definition 6.1 *The (Lebesgue) integral of f over X w.r.t μ is defined by*

$$\int_X \varphi d\mu(x) = \int_X \varphi d\mu = \sum_{i=1}^{n} a_i \cdot \mu(A_i).$$

Remark 6.2 *In the above definition, we use the convention that $0 \cdot \infty = 0$, i.e., if $a_i = 0$ and $\mu(A_i) = \infty$ or $a_i = \infty$ and $\mu(A_i) = 0$ then $a_i \cdot \mu(A_i) = 0$. Thus, one can verify that the integral is independent of how the simple function is represented as a linear combination of characteristic functions.*

Example 6.1 *The $\chi_{\mathbb{Q}}$ is not Riemann integral but*

$$\int_{\mathbb{R}} \chi_{\mathbb{Q}} \, dm = 1 \cdot m(\mathbb{Q}) = 0.$$

Moreover, $\displaystyle\int_{[0,1]} \chi_{[0,1]\setminus\mathbb{Q}} = 1$ but the lower Riemann integral of $\chi_{[0,1]\setminus\mathbb{Q}}$ on $[0,1]$ equals to 0.

Example 6.2 *Let μ be a counting measure on $\mathbb{Z}^+$. Given a sequence of nonnegative number $(a_k)_{k\geq 1}$, consider the function $f : \mathbb{Z}^+ \to [0, +\infty)$ such that*

$$f(k) = b_k \qquad \text{for all } k \geq 1.$$

Then

$$\int_{\mathbb{Z}^+} f d\mu = \sum_{k=1}^{\infty} b_k.$$

Given two simple functions $\varphi_1, \varphi_2 \in S_+(X)$ and $\alpha, \beta \in [0, +\infty]$, the followings hold:

(i) $\displaystyle\int_X \alpha \cdot \varphi_1 + \beta \cdot \varphi_2 \, d\mu = \alpha \cdot \int_X \varphi_1 d\mu + \beta \cdot \int_X \varphi_2 d\mu;$

(ii) $\int_X \varphi_1 d\mu \ \leq \ \int_X \varphi_2 d\mu \quad \text{if} \quad \varphi_1 \leq \varphi_2.$

Definition 6.3 *If the function $f : X \to [0, +\infty]$ is measurable then define*

$$\int_X f d\mu \ = \ \sup\left\{\int_X \varphi d\mu : \varphi \in S_+(X) \quad \text{and} \quad \varphi \leq f\right\}.$$

We say that f is integrable if

$$\int_X f \ d\mu \ < \ +\infty.$$

For any $A \in \mathcal{M}$, we define

$$\int_A f \ d\mu \ = \ \int_X f \cdot \chi_A \ d\mu.$$

Basic properties: Given two integrable function f, g, the followings hold

(i). $\int_X \alpha \cdot f \ = \ \alpha \cdot \int_X f \ d\mu$ for all $\alpha > 0$;

(ii). $\int_X f + g \ d\mu \ = \ \int_X f \ d\mu + \int_X g \ d\mu$;

(iii). $\int_X f \ d\mu \ \leq \ \int_X g \ d\mu \quad \text{if} \quad f \leq g$;

(iv). $\int_X f \ d\mu \ = \ \int_X g \ d\mu \quad \text{if} \quad f = g$ almost everywhere.

Notice that (ii) and (iv) are nontrivial. Indeed, $\varphi, \psi \in S_+(X)$ satisfy $\varphi + \psi \leq f + g$ does not imply that $\varphi \leq f$ and $\psi \leq g$. We will prove (ii) and (iv) later.

Theorem 6.4 (Monotone convergence theorem) *Let $(f_n)_{n\geq 1}$ be a increasing sequence of nonnegative measurable functions on X. Denote by*

$$f(x) \ = \ \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \in X.$$

Then,

$$\int_X f \ d\mu \ = \ \lim_{n\to\infty} \int_X f_n \ d\mu.$$

Proof. Since $(f_n(x))_{n\geq 1}$ is increasing, the function f is well-defined, measurable, and

$$f(x) \ \geq \ f_n(x) \qquad \text{for all } x \in X, n \geq 1.$$

By the monotonicity, it holds

$$\int_X f_n \, d\mu \;\leq\; \int_X f \, d\mu \qquad \text{for all } n \geq 1,$$

and it yields

$$\lim_{n\to\infty} \int_X f_n d\mu \;\leq\; \int_X f d\mu.$$

To conclude the proof, we will show that

$$\int_X f d\mu \;\leq\; \lim_{n\to\infty} \int_X f_n d\mu.$$

Equivalently,

$$(1-\varepsilon)\cdot \int_X f d\mu \;\leq\; \lim_{n\to\infty} \int_X f_n d\mu \qquad \text{for all } \varepsilon > 0.$$

For a fixed $\varepsilon > 0$, we claim that

$$\lim_{n\to\infty} \int_X f_n d\mu \;\geq\; (1-\varepsilon)\cdot \int_X \varphi d\mu \qquad \text{for all } S_+(X) \ni \varphi \leq f.$$

Indeed, for any $n \geq 1$, we consider the set

$$A_n \;=\; \{x \in X : f_n(x) \geq (1-\varepsilon)\cdot\varphi(x)\}.$$

Since $(f_n)_{n\geq 1}$ is increasing and converges to f pointwise, the sequence of sets $(A_n)_{n\geq 1}$ is increasing, i.e.,

$$A_1 \;\subseteq\; A_2 \;\subseteq\; \cdots \;\subseteq\; A_n \cdots$$

and

$$\lim_{n\to\infty} A_n \;=\; X.$$

On the other hand, we have

$$\int_X f_n \, d\mu \;\geq\; \int_{A_n} f_n d\mu \;\geq\; (1-\varepsilon)\cdot \int_{A_n} \varphi d\mu.$$

Assume that

$$\varphi \;=\; \sum_{i=1}^{N} a_i \cdot \chi_{E_i},$$

we compute

$$\lim_{n\to\infty} \int_{A_n} \varphi d\mu \;=\; \lim_{n\to\infty} \left(\sum_{i=1}^{N} a_i \cdot \mu(E_i \cap A_n) \right) \;=\; \sum_{i=1}^{N} a_i \cdot \mu(E_i) \;=\; \int_X \varphi d\mu.$$

Thus,

$$\lim_{n\to\infty} \int_X f_n d\mu \;\geq\; (1-\varepsilon)\cdot \int_X \varphi d\mu$$

and the proof is complete. □

Corollary 6.5 *Let $f : g : X \to [0, +\infty]$ be measurable. Then*

$$\int_X f + g d\mu \;=\; \int_X f d\mu + \int_x g d\mu$$

Proof. It is known that there exist increasing sequences of simple functions $(\varphi_n)_{n\geq 1}$ and $(\psi_n)_{n\geq 1}$ in $S_+(X)$ such that

$$\lim_{n\to\infty} \varphi_n(x) \;=\; f(x) \qquad \text{and} \qquad \lim_{n\to\infty} \psi_n(x) \;=\; g(x) \qquad \text{for all } x \in X.$$

This implies that $(\varphi_n + \psi_n)_{n\geq 1}$ is a increasing sequence of simple functions in $S_+(X)$ and

$$\lim_{n\to\infty} \varphi_n + \psi_n(x) \;=\; f(x) + g(x) \qquad \text{for all } x \in X.$$

Using the monotone convergence theorem, we obtain that

$$\begin{aligned}\int_X f + g d\mu \;&=\; \lim_{n\to\infty} \left(\int_X \varphi_n(x) + \psi_n(x)\ d\mu \right) \\ &=\; \lim_{n\to\infty} \int_X \varphi_n(x) d\mu + \lim_{n\to\infty} \int_X \psi_n(x) d\mu \;=\; \int_X f d\mu + \int_X g d\mu\end{aligned}$$

and the proof is complete. □

Problem 36: Show that the monotone convergence theorem fails for decreasing sequences of function. Does it hold for decreasing sequences on finite measure spaces?

Problem 37: Show that the monotone convergence theorem fails if the hypothesis that $f_1, f_2, \ldots$ are nonnegative functions is dropped.

Problem 38: Let μ be the counting measure on $\mathbb{N}$. Use the monotone convergence theorem to argue that for any $a_{n,k} \geq 0$ that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k}.$$

Problem 39: Let $(f_n)_{n\geq 1}$ be a sequence of nonnegative measurable functions. Show that

$$\int_X \sum_{n=1}^{\infty} f_n(x) d\mu \;=\; \sum_{n=1}^{\infty} \int_X f_n(x) d\mu.$$

Problem 40: Let $(f_n)_{n\geq 1}$ be a sequence of nonnegative measurable functions such that

$$\sum_{n=1}^{\infty} \int_X f_n(x) d\mu \;<\; +\infty.$$

Show that f_n converges to 0 almost everywhere.

Lemma 6.6 (Chebychev/Markov inequality) *Let $f : X \to [0, +\infty]$ be non-negative. Then for all $\lambda > 0$, it holds*

$$\mu(\{x \in X : f(x) \geq \lambda\}) \ \leq \ \frac{1}{\lambda} \cdot \int_X f d\mu.$$

Proof. Given $\lambda > 0$, we set

$$X_\lambda \ = \ \{x \in X : f(x) \geq \lambda\} \qquad \text{and} \qquad \varphi_\lambda \ = \ \lambda \cdot \chi_{X_\lambda}.$$

It is clear that

$$0 \ \leq \ \varphi_\lambda \ \leq \ f.$$

The monotonicity implies that

$$\int_X f \ d\mu \ \geq \ \int_X \varphi_\lambda d\mu \ = \ \lambda \cdot \mu(X_\lambda)$$

and it yields

$$\mu(\{x \in X \mid f(x) \geq \lambda\}) \ \leq \ \frac{1}{\lambda} \cdot \int_X f d\mu.$$

□

Corollary 6.7 *Let $f : X \to [0, +\infty]$ be integrable, i.e.,*

$$\int_X f \ d\mu \ < \ +\infty.$$

Then the function f is finite almost everywhere in X and $\{x \in X : f(x) > 0\}$ is σ-finite, i.e.,

$$\{x \in X : f(x) > 0\} \ \subseteq \ A_n \qquad \text{with} \qquad \mu(A_n) < +\infty.$$

Proof. Denote by

$$X_\infty \ = \ \{x \in X : f(x) = +\infty\}.$$

We have that

$$X_\infty \ \subseteq \ X_n \ = \ \{x \in X : f(x) \geq n\} \qquad \text{for all } n \geq 1.$$

The monotonicity and the Chybechev's inequality implies that

$$\mu(X_\infty) \ \leq \ \mu(X_n) \ \leq \ \frac{1}{n} \cdot \int_X f d\mu \qquad \text{for all } n \geq 1.$$

Taking n to $+\infty$, we obtain that

$$\mu(X_\infty) \ = \ 0.$$

To complete the proof, one observes that

$$\{x \in X : 0 \leq f(x) < +\infty\} \ \subseteq \ \bigcup_{n=1}^{\infty} X_{1/n}$$

with $\mu(X_{1/n}) \leq n \cdot \int_X f d\mu < +\infty$. □

Corollary 6.8 (Beppo Levi's lemma) *Let $(f_n)_{n\geq 1}$ be a increasing sequence of nonnegative measurable functions on X. Assume that*

$$\int_X f_n \, d\mu \ < \ M \qquad \textit{for all } n \geq 1$$

for some constant M. Then the function

$$f(x) \ = \ \lim_{n\to\infty} \ f_n(x) \qquad \textit{for all } x \in X$$

is almost finite and $\lim_{n\to\infty} \int_X f_n d\mu \ = \ \int_X f d\mu.$

Lemma 6.9 (Fatou's lemma) *Assume that $(X, \mathcal{M}, \mu)$ is a measure space. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative measurable functions such that $f_n \overset{a.e.}{\longrightarrow} f$. Then*

$$\int_X f \ d\mu \ \leq \ \liminf_{n\to\infty} \ \int_X f_n d\mu.$$

Proof. Consider the set of points where f_n converges to f

$$X_0 \ = \ \{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}.$$

By the above assumption, it holds that $\mu(X \backslash X_0) = 0$. Thus, we only need to prove that

$$\liminf_{n\to\infty} \ \int_{X_0} f_n d\mu \ \geq \ \int_{X_0} f d\mu.$$

For every $n \geq 1$, the following function

$$g_n(x) \ = \ \inf_{k\geq n} f_k(x) \qquad \text{for all } x \in X_0.$$

is increasing and satisfies

$$g_n(x) \ \leq \ f_n(x) \qquad \text{and} \qquad \lim_{n\to\infty} g_n(x) \ = \ f(x) \qquad \text{for all } x \in X_0.$$

Apply the monotone convergence theorem, we obtain that

$$\int_{X_0} f\ d\mu \ = \ \lim_{n\to\infty} \int_{X_0} g_n d\mu$$

and it yields

$$\int_{X_0} f\ d\mu \ = \ \liminf_{n\to\infty} \int_{X_0} g_n d\mu \ \leq \ \liminf_{n\to\infty} \int_{X_0} f_n d\mu$$

The proof is complete. □

Corollary 6.10 *For every sequence of non-negative measurable functions f_n, it holds*

$$\int_X \liminf_{n\to\infty} f_n d\mu \ \leq \ \liminf_{n\to\infty} \int_X f_n d\mu.$$

Problem 41: Show that Fatou's lemma does not hold for functions which take negative values.

6.2 Integration of general measurable functions

Given a measure space $(X, \mathcal{M}, \mu)$, let $f : X \to \overline{\mathbb{R}}$ be measurable. The functions

$$f^+ \ = \ \max\{f, 0\}, \qquad\qquad f^- \ = \ \max\{-f, 0\},$$

are measurable and satisfy

$$f \ = \ f^+ - f^- \qquad \text{and} \qquad |f| \ = \ f^+ + f^-.$$

Definition 6.11 *The Lebesgue integral of f is defined by*

$$\int_X f d\mu \ = \ \int_X f^+ d\mu - \int_X f^- d\mu$$

provided that $\int_X f^+ d\mu < +\infty$ *or* $\int_X f^- d\mu < +\infty$.

Notice that $\int_X f d\mu$ can take value $+\infty$ or $-\infty$. We say that f is *integrable* if and only if

$$\int_X |f| d\mu \ = \ \int_X f^+ + f^- d\mu \ < \ \infty.$$

Basic properties. Let $f, g : X \to \overline{\mathbb{R}}$ be integrable. Then the followings hold (i). $\displaystyle\int_X \alpha f d\mu = \alpha \cdot \int_X f \; d\mu$ for all $\alpha \in \mathbb{R}$;

(ii). $\displaystyle\int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu$;

(iii). $\displaystyle\int_X f d\mu \leq \int_X g d\mu \quad \text{if} \quad f \leq g$ a.e.

(iv). $\displaystyle\left| \int_X f d\mu \right| \; \leq \; \int_X |f| \; d\mu.$

Proof. (i), (iii) and (iv) are trivial. Let's prove (ii) by setting $h = f + g$. We have

$$h^+ - h^- \;=\; f^+ - f^- + g^+ - g^-$$

Equivalently,

$$h^+ + f^- + g^- \;=\; h^- + f^+ + g^+.$$

and it yields

$$\int_X h^+ + f^- + g^- d\mu \;=\; \int_X h^- + f^+ + g^+ \; d\mu.$$

The additive property of integral for non-negative function implies that

$$\int_X h^+ d\mu + \int_X f^- d\mu + \int_X g^- d\mu \;=\; \int_X h^- d\mu + \int_X f^+ d\mu + \int_X g^+ d\mu.$$

Thus,

$$\begin{aligned} \int_X h d\mu \;&=\; \int_X h^+ d\mu - \int_X h^- d\mu \;=\; \int_X f^+ d\mu - \int_X f^- d\mu + \int_X g^+ d\mu - \int_X g^- d\mu \\ &=\; \int_X f d\mu + \int_X g d\mu \end{aligned}$$

and the proof of (ii) is complete. □

Proposition 6.11.1 (Additivity and continuity of integration) *Let $f : X \to \overline{\mathbb{R}}$ be a integrable function and let $(X_n)_{n\geq 1}$ be a sequence of measurable sets. The following hold*

(i). If $(X_n)_{n\geq 1}$ is mutually disjoint and cover X then

$$\int_X f d\mu \;=\; \sum_{n=1}^{\infty} \int_{X_n} f \; d\mu.$$

(ii). If $(X_n)_{n\geq 1}$ is decreasing then

$$\int_{\bigcap_{n=1}^{\infty} X_n} f d\mu \;=\; \lim_{n\to\infty} \int_{X_n} f d\mu.$$

(iii). If $(X_n)_{n\geq 1}$ is increasing and $\bigcup_{n=1}^{\infty} X_n = X$ then

$$\lim_{n\to\infty} \int_{X_n} f d\mu \;=\; \int_X f d\mu.$$

Proof. We only need to prove the above proposition for non-negative measurable functions.

(i). For every $n \geq 1$, denote by

$$f_n \;=\; \sum_{i=1}^{n} f \cdot \chi_{X_i}.$$

One has that $(f_n)_{n\geq 1}$ is increasing of non-negative measurable functions and converges to f pointwise. By the monotone convergence theorem, it holds

$$\int_X f\ d\mu \;=\; \lim_{n\to\infty} \int_X f_n\ d\mu \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} \int_{X_i} f\ d\mu \;=\; \sum_{n=1}^{\infty} \int_{X_n} f\ d\mu.$$

(ii). Set $E = \bigcup_{n=1}^{\infty} X_n$, denote by

$$g \;=\; f \cdot \chi_E \qquad \text{and} \qquad g_n \;=\; f \cdot \chi_{X_n}.$$

We have that $(g_n)_{n\geq 1}$ is decreasing and converges to g pointwise. This implies that the sequence $(h_n := f - g_n)_{n\geq 1}$ is increasing of non-negative measurable functions and converges pointwise to $f - g$. Thus, the monotone convergence theorem implies that

$$\int_X f - g\ d\mu \;=\; \lim_{n\to\infty} \int_X (f - g_n)$$

Since f is integrable, one has that

$$\int_X g\ d\mu \;=\; \lim_{n\to\infty} \int_x g_n\ d\mu.$$

Thus,

$$\int_{\bigcap_{n=1}^{\infty} X_n} f d\mu \;=\; \lim_{n\to\infty} \int_{X_n} f d\mu.$$

(iii) is a consequence of (ii). □

Problem 42: Consider the set $X = [\pi, \infty)$. Is $f(x) = \dfrac{\sin(x)}{x}$ Lebesgue integrable on X? Does the improper Riemann integral of f exist?

Theorem 6.12 (Lebesgue dominated convergence theorem) *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n)_{n\geq 1}$ be a sequence of measurable functions which converges to a measurable function f almost everywhere. Assume that*

$$|f_n| \;\leq\; g \qquad a.e\ x \in X$$

for some integrable function g. Then f is integrable and

$$\lim_{n\to\infty}\int_X f_n d\mu \;=\; \int_X f\ d\mu.$$

Proof. Since $|f_n| \leq g$, it holds

$$g - f_n, g + f_n \;\geq\; 0 \quad a.e\ x \in X.$$

On the other hand, since f_n converges to f almost everywhere, one has

$$g - f_n \stackrel{\text{a.e.}}{\longrightarrow} g - f \qquad \text{and} \qquad g + f_n \stackrel{\text{a.e.}}{\longrightarrow} g + f.$$

The Fatou's lemma yields

$$\int_X g - f\ d\mu \;\leq\; \liminf_{n\to\infty}\int_X g - f_n\ d\mu$$

and

$$\int_X g + f\ d\mu \;\leq\; \liminf_{n\to\infty}\int_X g + f_n\ d\mu.$$

Since g is integrable, one has

$$\limsup_{n\to\infty}\int_X f_n\ d\mu. \;\leq\; \int_X f d\mu \;\leq\; \liminf_{n\to\infty}\int_X f_n\ d\mu.$$

Thus,

$$\lim_{n\to\infty}\int_X f_n d\mu \;=\; \int_X f\ d\mu.$$

and the proof is complete. □

Example 6.3 *Let $f_n : \mathbb{R} \to \mathbb{R}$ be such that*

$$f_n(x) \;=\; \begin{cases} n & \text{if} \quad 0 < x < \dfrac{1}{n} \\ \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $(f_n)_{n\geq 1}$ converges to $f = 0$ pointwise. However,

$$\int_{\mathbb{R}} f_n dm \;=\; 1 \;\text{ does not converge to }\; \int_{\mathbb{R}} f dm = 0.$$

In this case, the Lebesgue dominated convergence theorem can not be applied here. Indeed,

$$g(x) \ \geq \ \sup_{n\geq 1} f_n(x) \ \geq \ m \qquad \textit{for all } x \in \left[\frac{1}{m+1}, \frac{1}{m}\right[.$$

Thus,

$$\int_X gdm \ \geq \ \sum_{m=1}^{\infty} m \cdot \left(\frac{1}{m} - \frac{1}{m+1}\right) \ = \ \sum_{m=1}^{\infty} \frac{1}{m+1} \ = \ +\infty$$

and g is not integrable. □

Absolute continuity of the Lebesgue integral. Let's recall some basis facts:

(i) If $f : X \to \overline{\mathbb{R}}$ is integrable then f is finite almost everywhere;

(ii) If $\int_X |f|dm = 0$ then $f = 0$ almost everywhere in X.

Proposition 6.12.1 (Absolute continuity) *Suppose that $f : X \to \overline{\mathbb{R}}$ is integrable. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{M}$ with $\mu(A) < \delta$, it holds*

$$\int_A |f| \, d\mu \ \leq \ \varepsilon.$$

Moreover, there exists $X_0 \in \mathcal{M}$ such that $\mu(X_0) < \infty$ and

$$\int_{X\backslash X_0} |f|d\mu \ \leq \ \varepsilon.$$

Proof. It is sufficient to prove the above proposition for non-negative f. For any $n \geq 1$, we denote by

$$f_n(x) \ = \ \begin{cases} n & \text{if} \quad f(x) \geq n \\ \\ f(x) & \text{if} \quad f(x) < n. \end{cases}$$

It is clear that $(f_n)_n \geq 1$ is a increasing sequence of non-negative functions and converges pointwise to f. Applying the monotone convergence theorem, we get

$$\lim_{n\to\infty} \int_X f_n d\mu \ = \ \int_X f d\mu \ < \ +\infty.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\int_X (f - f_{n_0})d\mu \ \leq \ \frac{\varepsilon}{2}.$$

Thus, for any $A \in \mathcal{M}$, we have

$$\begin{aligned}\int_A f d\mu &= \int_A (f - f_{n_0}) d\mu + \int_A f_{n_0} d\mu \leq \int_X (f - f_{n_0}) d\mu + n_0 \cdot \mu(A) \\ &\leq \frac{\varepsilon}{2} + n_0 \cdot \mu(A).\end{aligned}$$

Choosing $\delta = \dfrac{\varepsilon}{2n_0}$, we obtain that

$$\int_A f \, d\mu \leq \varepsilon \qquad \text{for all } \mu(A) < \delta.$$

To prove the second part of the proposition, denote by

$$E_n = \left\{ x \in X : f(x) \in \left[\frac{1}{n+1}, \frac{1}{n} \right[\right\},$$

we have

$$\sum_{n=1}^{\infty} \int_{E_n} f d\mu \leq \int_X f d\mu < +\infty.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{n=n_1}^{\infty} \int_{E_n} f d\mu < \varepsilon.$$

Set

$$X_0 = \left\{ x : f(x) > \frac{1}{n_1} \right\}$$

we have that

$$\mu(X_0) \leq n \cdot \int_X f \, d\mu < +\infty$$

and

$$\int_{X \setminus X_0} f d\mu < \varepsilon.$$

The proof is complete □

Remark 6.13 *The above proposition fails if f is not integrable.*

Indeed, let's consider $X = (0,1)$ and μ Lebesgue measure on X. The function $f = \dfrac{1}{x}$ is not integrable and

$$\int_{(0,\delta)} f d\mu = \int_{(0,\delta)} \frac{1}{x} d\mu = +\infty.$$

Problem 43: Use the dominated convergence theorem to argue that

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n x^{a-1}\,dx \;=\; \int_0^\infty e^{-x}x^{a-1}\,dx.$$

Problem 44: For any C^1 function f, prove the identity

$$\frac{d}{dy}\int_0^1 f(x,y)dx \;=\; \int_0^1 \frac{d}{dy}f(x,y)dx,$$

Definition 6.14 *Consider the sequence of integrable function $(f_n)_{n\geq 1}$. We say that*

- $(f_n)_{n\geq 1}$ *is uniformly integrable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_E |f_n|d\mu \;<\; \varepsilon \qquad \textit{for all } n \geq 1, E \in \mathcal{M} \;\text{ with }\; \mu(E) < \delta;$$

- $(f_n)_{n\geq 1}$ *is tight if for every $\varepsilon > 0$, there exists $X_0 \in \mathcal{M}$ with $\mu(X_0) < +\infty$ such that*

$$\int_{X\setminus X_0} |f_n|\;d\mu \;<\; \varepsilon \qquad \textit{for all } n \geq 1.$$

Theorem 6.15 (Vitali convergence theorem) *Let $(f_n)_{n\geq 1}$ be a sequence of uniformly integrable and tight functions over X. Assume that*

$$f_n \xrightarrow{a.e.} f \qquad \text{and} \qquad f \;\text{ is integrable.}$$

Then,

$$\lim_{n\to\infty}\int_X |f_n - f|\;d\mu \;=\; 0.$$

Proof. 1. Let's first prove the theorem in the case

$$\mu(X) \;<\; +\infty.$$

Since f is integrable, f is finite almost everywhere. Thus, one can apply the Egorov's theorem to obtain that $(f_n)_{n\geq 1}$ converges to f almost uniformly on X, i.e., for every $\delta > 0$ there exists $E_\delta > 0$ such that

$$\mu(E_\delta) < \delta \qquad \text{and} \qquad \lim_{n\to\infty}\left(\sup_{x\in X\setminus E_\delta}|f_n(x) - f(x)|\right) \;=\; 0.$$

This implies that

$$I \;=\; \limsup_{n\to\infty} \int_X |f-f_n|\,d\mu \;\le\; \limsup_{n\to\infty} \int_{E_\delta} |f|+|f_n|\,d\mu + \limsup_{n\to\infty} \int_{X\setminus E_\delta} |f-f_n|\,d\mu$$

$$\le\; \limsup_{n\to\infty} \int_{E_\delta} |f|+|f_n|\,d\mu + \mu(X)\cdot \lim_{n\to\infty} \sup_{x\in X\setminus E_\delta} |f_n(x)-f(x)|$$

$$=\; \limsup_{n\to\infty} \int_{E_\delta} |f|+|f_n|\,d\mu \qquad \text{for all } \delta>0.$$

The uniform integrability of $(f_n)_{n\ge 1}$ and the absolute continuity of f implies that

$$I \;=\; \limsup_{n\to\infty} \int_X |f-f_n|\,d\mu \;<\; \varepsilon \qquad \text{for all } \varepsilon>0.$$

Thus, taking ε to $0+$, we get

$$\lim_{n\to\infty} \int_X |f_n-f|\,d\mu \;=\; 0.$$

2. Let's now remove the boundedness of $\mu(X)$. Observe that

- The tightness property of $(f_n)_{n\ge 1}$ implies that for every $\varepsilon>0$, there exists X^1_ε such that
$$\mu(X^1_\varepsilon) \;<\; +\infty \qquad \text{and} \qquad \int_{X\setminus X^1_\varepsilon} |f_n|d\mu \;<\; \frac{\varepsilon}{2}.$$

- The absolute continuity of f implies that for every $\varepsilon>0$, there exists X^2_ε such that
$$\mu(X^2_\varepsilon) \;<\; +\infty \qquad \text{and} \qquad \int_{X\setminus X^2_\varepsilon} |f|d\mu \;<\; \frac{\varepsilon}{2}.$$

Set $X_\varepsilon = X^1_\varepsilon \bigcup X^2_\varepsilon$, we have

$$\mu(X_\varepsilon) \;<\; \infty \qquad \text{and} \qquad \int_{X\setminus X_\varepsilon} |f_n|+|f|d\mu \;<\; \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \;=\; \varepsilon.$$

Therefore,

$$\limsup_{n\to\infty} \int_X |f-f_n|\,d\mu \;\le\; \varepsilon + \limsup_{n\to\infty} \int_{X_\varepsilon} |f-f_n|\,d\mu \;=\; \varepsilon$$

for all $\varepsilon>0$, and the proof is complete. □

Corollary 6.16 *Let $(f_n)_{n\ge 1}$ be a sequence of non-negative integrable functions which converges to 0 almost everywhere. Then*

$$\lim_{n\to\infty} \int_X f_n\,d\mu \;=\; 0$$

if any only if $(f_n)_{n\ge 1}$ is uniformly integrable and tight.

The Vitali convergence theorem can be stated in a stronger way.

Theorem 6.17 *A sequence of integrable f_n converges in $\mathbf{L}^1$ to an integrable function f if and only if*

(a). f_n converges to f in measure

(b). f_n is uniformly integrable.

(c). For every $\epsilon > 0$ there exists a set $\mu(E) < \infty$ so that for all n

$$\int_{X\setminus E} |f_n|\, d\mu \leq \epsilon.$$

There are also other ways to characterize uniform integrability.

Theorem 6.18 *Let $\mathcal{F}$ be a family of integrable functions. Consider the following conditions.*

1. *$\mathcal{F}$ is uniformly integrable.*
2. $\lim_{t\to\infty} \sup_{\mathcal{F}} \int_{x:|f(x)|>t} |f|\, d\mu = 0.$
3. *(De la Vallée Poussin) There exists an increasing function $\gamma : [0,\infty) \to [0,\infty]$, with*
$$\lim_{t\to\infty} \frac{\gamma(t)}{t} = \infty$$
such that
$$\sup_{f\in\mathcal{F}} \int_X \gamma(|f|)\, d\mu < \infty.$$

Then (2) and (3) are equivalent and imply (1). If, in addition, $\sup_{f\in\mathcal{F}} \int_X |f|\, d\mu < \infty$, then all three are equivalent.

Problem 45: For $X = [0,1]$, show that $f_n(x) = \sin(nx)$ is uniformly integrable and tight, but that it does not converge in $\mathbf{L}^1$.

Problem 46: For $X = [0,1]$ give an example of a sequence which converges in measure to 0 but does not converge almost everywhere.

Problem 47: Let $f : X \to \mathbb{R}$ be integrable. Show that

$$\lim_{n\to+\infty} \int_X |f|^{1/n}\, d\mu \;=\; \mu\{x \mid f(x) \neq 0\}\,.$$

Problem 48: Let $f : X \to \mathbb{R}$ be integrable and $|f| \leq 1$. Show that

$$\liminf_{n\to+\infty} \int_X |f|^n\, d\mu \;=\; \mu\{x \mid |f(x)| = 1\}\,.$$

6.2.1 The Radon-Nikodym theorem

For any $f : X \to [0,\infty]$ measurable function, let $\nu : \mathcal{M} \to [0,+\infty]$ be such that

$$\nu(E) \;=\; \int_E f d\mu \qquad \text{for all } E \in \mathcal{M}.$$

It is easy to check that ν is a measure on $(X,\mathcal{M})$. Thanks to the absolute continuity of the integral, we have that

$$\mu(E) \;=\; 0 \qquad \Longrightarrow \qquad \int_E f d\mu \;=\; 0.$$

This leads to the following definition.

Definition 6.19 *Given μ and ν measures on $(X,\mathcal{M})$, we say that ν is absolutely continuous with respect to μ, denote by $\nu \ll \mu$, if*

$$\mu(E) \;=\; 0 \qquad \Longrightarrow \qquad \nu(E) \;=\; 0.$$

In the other words, ν is absolutely continuous with respect to μ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\nu(E) \;<\; \varepsilon \qquad \text{for all } E \in \mathcal{M} \;\text{ with }\; \mu(E) \;<\; \delta.$$

The following holds:

Theorem 6.20 (Radon-Nykodym theorem) *Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space, i.e.,*

$$X \;=\; \bigcup_{n=1}^{\infty} A_n \qquad \text{with} \qquad \mu(A_n) \;<\; +\infty \quad \textit{for all } n \geq 1.$$

For every σ-finite measure ν on $(X,\mathcal{M})$ such that $\nu \ll \mu$, there exists a unique measurable function $f : X \to [0,\infty)$ such that

$$\nu(E) \;=\; \int_E f \; d\mu \qquad \textit{for all } E \in \mathcal{M}.$$

The function f is called **the Radon-Nikodym derivative** *and is denoted by* $\dfrac{d\nu}{d\mu}$.

Proof. 1. Suppose that both μ and ν are finite measure. The following collection of extended-value measurable functions

$$\mathcal{F} \;=\; \left\{ f : X \to [0,+\infty] : \int_A f d\mu \leq \nu(A) \quad \text{for all } A \in \mathcal{M} \right\}$$

is non-empty and satisfies

$$\max\{f,g\} \ \in \ \mathcal{F} \qquad \text{for all } f,g\in\mathcal{F}.$$

Since ν is finite, it holds that

$$\sup_{f\in\mathcal{F}}\left(\int_X f d\mu\right) \ \le \ \nu(X) \ < \ \infty.$$

Let $(f_n)_{n\ge1}\subseteq\mathcal{F}$ be such that

$$\lim_{n\to\infty}\int_X f_n d\mu \ = \ \sup_{f\in\mathcal{F}}\left(\int_X f d\mu\right).$$

Thus, the sequence of functions

$$g_n \ = \ \max\{f_1,\cdots,f_n\} \ \in \ \mathcal{F}$$

is increasing and satisfies

$$\lim_{n\to\infty} g_n(x) \ = \ \bar{g}(x) \qquad \text{and} \qquad \lim_{n\to\infty}\int_X g_n d\mu \ = \ \sup_{f\in\mathcal{F}}\left(\int_X f d\mu\right).$$

By Lebesgue's monotone convergence theorem, one has

$$\int_E \bar{g}\ d\mu \ = \ \lim_{n\to\infty}\int_E g_n d\mu \ \le \ \nu(E) \quad \text{for all } E\in\mathcal{M} \tag{6.1}$$

and this implies that

$$\bar{g}\in\mathcal{F}, \qquad \int_X \bar{g}d\mu \ = \ \sup_{f\in\mathcal{F}}\left(\int_X f d\mu\right).$$

2. To obtain the equality in (6.1), we shall prove that the non-negative measure ν_0 defined by

$$\nu_0(E) \ = \ \nu(E) - \int_E \bar{g}\ d\mu \qquad \text{for all } E\in\mathcal{M}$$

is 0. Suppose $\nu_0\neq0$, then there exists a constant $\delta>0$ sufficiently small such that

$$\nu_0(X)-\delta\cdot\mu(X) \ > \ 0.$$

By the *Hahn decomposition theorem*, there exists a set $P,N\in\mathcal{M}$ with

$$P\bigcap N \ = \ \varnothing, \qquad P\bigcup N \ = \ X,$$

satisfy the following properties

$$\begin{cases} \nu_0(P') - \delta \cdot \mu(P') \ \geq \ 0 & \text{for all } \mathcal{M} \ni P' \subseteq P, \\ \\ \nu_0(N') - \delta \cdot \mu(N') \ \leq \ 0 & \text{for all } \mathcal{M} \ni N' \subseteq N. \end{cases}$$

In particular, we have

$$\nu_0(P) - \delta \cdot \mu(P) \ = \ \big[\nu_0(X) - \delta\mu(X)\big] - \big[\nu_0(N) - \delta\mu(N)\big] \ > \ 0$$

and this implies that

$$\nu(E) \ \geq \ \delta \cdot \mu(P) + \int_P \bar{g} d\mu \ > \ \delta \cdot \mu(P).$$

Since $\nu \ll \mu$, one then gets that $\mu(P) > 0$.

To complete this step, we shall show that the function $\bar{g} + \delta \cdot \chi_P$ is in $\mathcal{F}$. Indeed, for every $E \in \mathcal{M}$, it holds

$$\begin{aligned} \nu(E) \ &= \ \int_E \bar{g} d\mu + \nu_0(E) \ \geq \ \int_E \bar{g} d\mu + \nu_0(E \cap P) \\ &\geq \ \int_E \bar{g} d\mu + \delta \cdot \mu(E \cap P) \ = \ \int_E (\bar{g} + \delta \cdot \chi_P)\, d\mu. \end{aligned}$$

Thus,

$$\int_X g d\mu + \delta \cdot \mu(P) \ \leq \ \int_X (\bar{g} + \delta \cdot \chi_P)\, d\mu \ \leq \ \sup_{f \in \mathcal{F}} \left(\int_X f d\mu\right) \ = \ \int_X g d\mu$$

and this yields a contradiction.

3. Students should be able to prove the uniqueness and also remove the finite measure assumption on μ, ν. □

We remark that σ-finiteness in the previous theorem is in fact necessary, as demonstrated by the following example:

Example 6.4 *Let μ be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let ν be the Lebesgue measure. It is easy to see that $\nu \ll \mu$. However, if we had a Radon Nikodym derivative, we would have*

$$\nu(A) = \int_A f(x)\, d\mu(x) \qquad \textit{for all } A \in \mathcal{B}(\mathbb{R}).$$

If we let $A = \{a\}$, i.e. the set given by a singleton, we obtain that $f(a) = 0$. Since this holds for all a, we have that $f \equiv 0$, and that hence $\nu \equiv 0$. This is false, so such a Radon Nikodym derivative does not exist.

Example 6.5 *Suppose that f is a non-decreasing, continuously differentiable function on $[0,1]$. Consider the Lebesgue-Stieltjes measure Df and the Lebesge measure m. We note that*

$$Df(E) = \int_E f'\, dm,$$

where f' is continuous on $[0,1]$, and hence is integrable. This immediately implies that Df is absolutely continuous with respect to m and that the Radon-Nikodym derivative satisfies $\dfrac{dDf}{dm} = f'$.

Theorem 6.21 (The Lebesgue decomposition theorem) *Let μ and ν be σ-finite measure on $(X, \mathcal{M})$. Then,*

$$\nu \;=\; \nu_a + \nu_s$$

such that

- *ν_a is absolutely continuous w.r.t μ.*
- *ν_s and μ are singular (denote by $\mu \perp \nu_s$), i.e., there exists $A, B \in \mathcal{M}$ such that $A \cap B = \varnothing$, $A \cup B = X$, and*

$$\nu_s(A) \;=\; \mu(B) \;=\; 0.$$

Proof. Observe that the following measure is σ-finite

$$\lambda \;=\; \mu + \nu \qquad \text{and} \qquad \mu \;\ll\; \lambda.$$

By the Radon-Nikodym theorem, there exists $f : X \to [0, \infty)$ measurable such that

$$\mu(E) \;=\; \int_E f d\lambda \;=\; \int_E f d\mu + \int_E f d\nu \qquad \text{for all } E \in \mathcal{M}.$$

Denote by

$$X_+ \;=\; \{x \in X : f(x) > 0\} \qquad \text{and} \qquad X_0 \;=\; \{x \in X : f(x) = 0\},$$

we have that

$$X_+ \bigcap X_0 \;=\; \varnothing, \qquad X_+ \bigcup X_0 \;=\; X.$$

Define

$$\nu_a(E) \;=\; \nu(E \cap X_+) \qquad \text{and} \qquad \nu_s(E) \;=\; \nu(E \cap X_0).$$

It is easy to check that

$$\nu_s(X_+) \;=\; \nu(X_+ \cap X_0) \;=\; \nu(\varnothing) \;=\; 0$$

and
$$\mu(X_0) \;=\; \int_{X_0} f\; d\lambda \;=\; \int_{X_0} 0 d\lambda \;=\; 0.$$
Thus, μ and ν_s are singular.

To complete the proof, we will show that $\nu_a \ll \mu$. Indeed, for any $E \in \mathcal{M}$ with $\mu(E) = 0$, we have
$$\mu(E) \;=\; \int_E f d\mu + \int_E f d\nu \;=\; 0.$$
In particular,
$$\int_E f d\nu \;=\; \int_{E\cap X_0} f\; d\nu + \int_{E\cap X_+} f\; d\nu \;=\; 0.$$
and this implies that
$$\int_{E\cap X_+} f\; d\nu \;=\; 0.$$
Thus, $\nu_a(E) = \nu(E \cap X_+) = 0$ and this complete the proof. □

Corollary 6.22 *As a consequence, it holds*
$$\nu(E) \;=\; \int_E g\; d\mu + \nu_s \qquad \text{with} \qquad \nu_s \perp \mu.$$

Example 6.6 *Let μ be the Lebesgue measure on $[0, 1]$ and let*
$$f(x) \;=\; \begin{cases} x & \text{if } x \in [0, 1/2] \\ \\ 1 + 2x & \text{if } x \in (1/2, 1] \end{cases}.$$
Consider the Lebesgue-Stieltjes measure Df associated with f. By the Lebesgue decomposition theorem, we can decompose
$$Df = \nu_a + \nu_s \qquad \text{with} \qquad \begin{cases} \nu_a \;<<\; \mu \\ \nu_s \;\perp\; \mu. \end{cases}$$
In this case, we can also easily give explicit formulas for these measures. For every $E \in \mathcal{M} \cap [0, 1]$, it holds
$$\nu_a(E) \;=\; \int_{E\cap([0,1]\setminus\{1/2\})} f'(x)\, dx = \int_{E\cap[0,1/2)} 1\, dx + \int_{E\cap(1/2,1)} 2\, dx,$$
and
$$\nu_s(E) = Df(E \cap \{1/2\}) = \frac{3}{2} \cdot \delta_{1/2}(E).$$

Example 6.7 *Consider* $f : [0,1] \to \mathbb{R}$ *given by*

$$f = x + f_C(x)$$

where C *is be the Cantor set and* f_C *is the Cantor function. Here* f *is continuously differentiable on intervals which don't intersect the Cantor set. Consider the Lebesgue-Stieltjes measure* Df*, we have*

$$Df(E \backslash C) = \int_{E \backslash C} f'(x)\, dx = \int_{E \backslash C} 1\, dx.$$

On the other hand, since the Lebesgue measure of C *is zero, we can write*

$$Df(E) \;=\; Df(E \cap C) + Df(E \backslash C) \;=:\; \nu_s(E) + \nu_a(E).$$

Here the Radon-Nikodym derivative of ν_a *is exactly* 1 *(given the formula above using the fundamental theorem of calculus). On the other hand, the support of* ν_s *is exactly the Cantor set.*

6.2.2 Signed measure

Let $(X, \mathcal{M})$ be a measurable space. A signed measure $\nu : \mathcal{M} \to \overline{R}$ satisfies the following properties

(i). $\nu(\varnothing) = 0$;

(ii). ν attains at most one of the values ∞ or $-\infty$;

(iii.) For all sequences of mutually disjoint sets $(E_n)_{n \geq 1}$ in $\mathcal{M}$, it holds

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) \;=\; \sum_{n=1}^{\infty} \nu(E_n),$$

where the series $\sum_{n=1}^{\infty} \nu(E_n)$ is absolutely convergent if $\nu\left(\bigcup_{n=1}^{\infty} E_n\right)$ is finite.

Example 6.8 *Let* ν^+, ν^- *be non-negative measure on* $(X, \mathcal{M})$ *such that one of them is finite. Then*

$$\nu \;=\; \nu^+ - \nu^- \qquad (*)$$

is a signed measure.

Question: Given a signed measure ν, can we decompose ν at in (*)?

Definition 6.23 *Let* ν *be a signed measure on* $(X, \mathcal{M})$*. We say that*

- *$A \in \mathcal{M}$ is positive if*

$$\nu(B) \ \geq \ 0 \qquad \textit{for all } B \in \mathcal{M}, B \subseteq A;$$

- *$A \in \mathcal{M}$ is negative if*

$$\nu(B) \ \leq \ 0 \qquad \textit{for all } B \in \mathcal{M}, B \subseteq A;$$

- *$A \in \mathcal{M}$ is null if*

$$\nu(B) \ = \ 0 \qquad \textit{for all } B \in \mathcal{M}, B \subseteq A;$$

Lemma 6.24 (Hahn Lemma) *For any $A \in \mathcal{M}$ with $0 < \nu(A) < \infty$, there exists a positive measurable set $P \subseteq A$ with*

$$0 \ < \ \nu(P) \ < \ \infty.$$

Proof. Assume that A contains sets of negative measure. Let $n_1 \in \mathbb{Z}^+$ be the smallest positive integer such that there exists $\mathcal{M} \ni E_1 \subset A$ such that

$$\nu(E_1) \ < \ -\frac{1}{n_1}.$$

The set $A_1 = A \backslash E_1 \subset A$ satisfies

$$\nu(A_1) \ = \ \nu(A) - \nu(E_1) \ > \ \nu(A) + \frac{1}{n_1} \ > \ 0.$$

If A_1 is positive then the proof is complete. Inductively, if $A_k := A \backslash \left(\bigcup_{i=1}^{k} E_i \right)$ is not positive, then let $n_{k+1} \in \mathbb{Z}^+$ be the smallest such that there exists $\mathcal{M} \ni E_{k+1} \subset A_k$ with

$$\nu(E_{k+1}) \ < \ -\frac{1}{n_{k+1}}.$$

By the definition of n_{k+1}, it holds

$$\nu(B) \ \geq \ -\frac{1}{n_{k+1} - 1} \qquad \text{for all } \mathcal{M} \ni B \subseteq A_k$$

If we never stop then the set

$$P \ = \ A \backslash \left(\bigcup_{i=1}^{\infty} E_i \right) \ \subset \ A.$$

satisfies

$$0 \ < \ \nu(P) + \sum_{i=1}^{\infty} \nu(E_i) \ = \ \nu(A) \ < \ +\infty \qquad \text{and} \qquad \nu(P) \ < \ +\infty.$$

In particular, we have

$$\sum_{i=1}^{\infty} \frac{1}{n_i} \leq \sum_{i=1}^{\infty} |\nu(E_i)| = \nu(P) - \nu(A) < +\infty$$

and this implies that $\lim_{i\to\infty} n_i = +\infty$. Thus, for every $B \subseteq P$, it holds that

$$B \subseteq A_k = A \backslash \left(\bigcup_{i=1}^{k} E_i\right) \qquad \text{for all } k \geq 1,$$

and this yields

$$\nu(B) \geq \lim_{k\to\infty} \left[-\frac{1}{n_{k+1} - 1}\right] = 0.$$

The proof is complete. □

Theorem 6.25 (Hahn decomposition theorem) *Let ν be a signed measure on $(X, \mathcal{M})$. Then there exists $P \in \mathcal{M}$ positive and $N \in \mathcal{M}$ negative such that*

$$P \bigcap N = \varnothing \qquad \text{and} \qquad P \bigcup N = X.$$

Proof. Assume that

$$\nu(A) < +\infty \qquad \text{for all } A \in \mathcal{M}.$$

By Hahn lemma, we can define

$$\gamma = \sup\{\nu(A) : A \in \mathcal{M} \text{ is positive}\}.$$

There exists a sequence of positive sets $(A_n)_{n\geq 1} \subseteq \mathcal{M}$ such that $\lim_{n\to\infty} \nu(A_n) = \gamma$. The following set

$$P := \bigcup_{i=1}^{\infty} A_i \in \mathcal{M},$$

is positive and

$$\nu(P) = \nu(A_k) + \nu(P \backslash A_k) \geq \nu(A_k).$$

and this implies that

$$\gamma = \nu(P) < \infty.$$

Set $N = X \backslash P \in \mathcal{M}$, we have that

$$P \bigcap N = \varnothing \qquad \text{and} \qquad P \bigcup N = X.$$

To complete the proof, we show that N is negative. Indeed, assume by a contradiction that N is not negative. Then there exists a set $E \subseteq N$ such that $\nu(E) > 0$. In this case, by Hahn lemma, there exists $\overline{P} \in \mathcal{M}$ such that

$$\overline{P} \subseteq E, \qquad \nu(P) > 0 \qquad \text{and} \qquad \overline{P} \text{ is positive.}$$

Set $\tilde{P} = P \bigcup \overline{P}$, we have that $\tilde{P}$ is positive and

$$\nu(\tilde{P}) = \nu(P) + \nu(\overline{P}) > \gamma$$

and it yields a contradiction. □

Theorem 6.26 (Jordan decomposition theorem) *Let ν be a signed measure. Then there exist two nonnegative measure ν^+, ν^- on $(X, \mathcal{M})$ such that one of them is finite and*

$$\nu = \nu^+ - \nu^- \qquad \text{and} \qquad \nu^+ \perp \nu^-. \tag{6.2}$$

Proof. From Hahn's theorem, there exist a positive measurable set P and a negative measurable set N such that

$$X = P \bigcup N \qquad \text{and} \qquad P \bigcap N = \varnothing.$$

The following measure $\nu^{\pm} : \mathcal{M} \to [0, +\infty]$

$$\nu^+(E) = \nu(E \cap P) \qquad \text{and} \qquad \nu^-(E) = -\nu(E \cap N) \qquad \text{for all } E \in \mathcal{M}.$$

satisfy (6.2) □

Notice that $\nu^{\pm}$ is unique. We say that

- ν^+ is the positive part of ν;
- ν^- is the negative part of ν;
- $|\nu| = \nu^+ + \nu^-$ is the total variation of ν.

Corollary 6.27 *Let ν and μ be σ-finite signed measures on $(X, \mathcal{M})$. There exist σ-finite measures ν_a and ν_s such that*

$$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \begin{cases} \nu_a \ll \mu \\ \nu_s \perp \mu. \end{cases}$$

Proof. Using Jordan decomposition theorem, we have

$$\nu \;=\; \nu^+ - \nu^- \qquad \text{with} \qquad \nu^+ \perp \nu^-$$

and $\nu^\pm$ positive. By the Lebesgue decomposition theorem, it holds

$$\nu^\pm \;=\; \nu_a^\pm + \nu_s^\pm \qquad \text{with} \qquad \begin{cases} \nu_a^\pm \;\ll\; \mu \\ \\ \nu_s^\pm \;\perp\; \mu. \end{cases}$$

Set $\nu_a = \nu_a^+ - \nu_a^-$ and $\nu_s = \nu_s^+ - \nu_s^-$, we have

$$\nu_a \;\ll\; \mu \qquad \text{and} \qquad \nu_s \;\perp\; \mu$$

and the proof is complete. □

Problem 49: Let $f : [a, b] \to \mathbb{R}$ satisfy

$$\sup_{P \in \mathcal{P}_{[a,b]}} |f(x_{i+1}) - f(x_i)| < \infty.$$

Define a function (mapping the set of closed subintervals of $[a, b]$ to $\mathbb{R}$) by $Df([a, b]) = f(b) - f(a)$. Argue that one can extend this function as a signed measure on the Borel sets. How would we interpret Df? What would the Lebesgue decomposition mean?

Example 6.9 *Consider the function $f(x) = \cos(2\pi x)$ on the set $x \in [0, 1]$. Consider the signed measure Df which satisfies $Df([a, b]) = f(b) - f(a)$. By the Hahn decomposition, there exist two positive measure such that*

$$Df \;=\; Df^+ - Df^-, \qquad Df^+ \;\perp\; Df^-$$

Let us now seek for formulas of these measures in this explicit case. Observe that

- *On the interval $[0, 1/2]$, f is a decreasing function, and hence $-Df$ is also a Lebesgue-Stieltjes measure;*
- *On the interval $(1/2, 1]$, f is an increasing function, and hence Df is exactly the Lebesgue-Stieltjes measure.*

Thus we may write

$$Df^+(E) = Df(E \cap (1/2, 1]), \qquad -Df^-(E) = Df(E \cap [0, 1/2]).$$

If we wanted we could make these measures even more explicit, by writing out their Radon-Nikodym derivative:

$$Df^+(E) = -\int_{E \cap (1/2,1]} 2\pi \sin(2\pi x)\, dx.$$

6.2.3 Relation between Riemann and Lebesgue integrals

Let f be a real valued function on $[a,b]$. We first show that

Lemma 6.28 *If f is Riemann integrable, i.e., f is bounded and*

$$\sup_{P\in\mathcal{P}_{[a,b]}} L(f,P) \;=\; \inf_{P\in\mathcal{P}_{[a,b]}} U(f,P)\,,$$

then f is Lebesgue integrable and

$$(R)\int_{[a,b]} f(x)\;dx \;=\; \int_{[a,b]} fdm.$$

Proof. Assume that f is Riemann integrable. Then there exists $(P_n)_{n\geq 1}$ an increasing sequence of partitions of $[a,b]$ such that

$$\lim_{n\to\infty} L(f,P_n) \;=\; \lim_{n\to\infty} U(f,P_n) \;=\; (R)\int_{[a,b]} f(x)\;dx$$

For any $n\geq 1$, assume that

$$P_n \;=\; \{a=x_0,x_1,\dots,x_{m-1},x_m=b\},$$

we define

$$\varphi_n \;=\; \sum_{i=0}^{m-1} m_i\cdot\chi_{[x_i,x_{i+1}]}, \qquad\qquad m_i \;=\; \inf_{x\in[x_i,x_{i+1}[} f(x),$$

and

$$\psi_n \;=\; \sum_{i=0}^{m-1} M_i\cdot\chi_{[x_i,x_{i+1}]}, \qquad\qquad M_i \;=\; \sup_{x\in[x_i,x_{i+1}[} f(x).$$

It holds that $(\varphi_n)_{n\geq1}$ is increasing sequence of functions, $(\psi_n)_{n\geq1}$ is decreasing sequence of functions, and

$$\varphi_n \;\leq\; f \;\leq\; \psi_n \qquad \text{for all } n\geq 1$$

and

$$\lim_{n\to\infty}\int_{[a,b]}\varphi_n dm \;=\; \lim_{n\to\infty}\int_{[a,b]}\psi_n dm \;=\; (R)\int_{[a,b]} f(x)\;dx.$$

The following functions

$$g \;=\; \lim_{n\to\infty}\varphi_n \;\leq\; f \;\leq\; G \;=\; \lim_{n\to\infty}\psi_n$$

are Lebesgue integrable. Using the dominated convergence theorem, we have

$$\int_{[a,b]} gdm \;=\; \lim_{n\to\infty}\int\varphi_n dm \;=\; (R)\int_{[a,b]} f(x)\;dx \;=\; \lim_{n\to\infty}\psi_n dm \;=\; \int_{[a,b]} Gdm.$$

and this yields

$$\int_{[a,b]} [G-g]dm \ = \ 0 \qquad \Longrightarrow \qquad G(x) = f(x) = g(x) \ \ a.e \ x \in [a,b].$$

This implies that f is Lebesgue measurable and

$$\int_{[a,b]} f dm \ = \ \int_{[a,b]} g dm \ = \ (R)\int_{[a,b]} f(x) \ dx.$$

The proof is complete. □

Theorem 6.29 *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if*

$$\mathcal{D}_f \ = \ \{x \in [a,b] : f \text{ is discontinuous}\}$$

has zero Lebesgue measure.

Sketch of proof. For any $\delta > 0$, denote by

$$h^\delta(x) \ = \ \inf_{y\in[x-\delta,x+\delta]} f(y), \qquad H^\delta(x) \ = \ \sup_{y\in[x-\delta,x+\delta]} f(y)$$

and

$$h(x) \ = \ \lim_{\delta\to 0} h^\delta(x), \qquad H(x) \ = \ \lim_{\delta\to 0} H^\delta(x).$$

We have that

(i) $h(x) \leq f(x) \leq H(x)$ for all $x \in [a,b]$;

(ii) f is continuous at x if and only if $h(x) = H(x)$;

(iii) h is lower semi-continuous and H is upper semi-continuous. Thus, h, H are Lebesgue measurable.

Key point. *For any simple functions such that*

$$\varphi(x) \ \leq \ f(x) \ \leq \ \psi(x)$$

it holds

$$\varphi(x) \ \leq \ h(x) \ \leq \ H(x) \ \leq \ \psi(x) \qquad a.e \ x \in [a,b].$$

• If f is Riemman integrable then there exist $(\varphi_n)_{n\geq 1}$ and $(\psi_n)_{n\geq 1}$ sequences of simple functions

$$\varphi_n \ \leq \ f \ \leq \ \psi_n \quad \text{and} \quad \lim_{n\to\infty}\int_{[a,b]} \varphi_n dm \ = \ (\mathcal{R})\int_a^b f(x)dx \ = \ \lim_{n\to\infty}\int_{[a,b]} \psi_n dm.$$

This implies that

$$\int_{[a,b]} h dm \;=\; \int_{[a,b]} H dm$$

and it yields $h = H$ a.e. $x \in [a, b]$.

• If $\mathcal{D}_f$ has zero Lebesgue measure then f is continuous almost everywhere. Consider the partition

$$P_n \;=\; \{a \;=\; x_0 \;<\; x_1 \;<\; \dots \;<\; x_{2^n-2} \;<\; x_{2^n-1} \;=\; b\}$$

with

$$x_i \;=\; a + i \cdot \frac{b-a}{2^n} \qquad \text{for all } i \in \{0, 1, \dots, 2^n - 1\}$$

and

$$\varphi_n \;=\; \sum_{i=0}^{2^n-1} m_i \cdot \chi_{[x_i, x_{i+1}[}, \qquad\qquad \psi_n \;=\; \sum_{i=0}^{2^n-1} M_i \cdot \chi_{[x_i, x_{i+1}[}.$$

By the definition of h and H, one has

$$\varphi_n \xrightarrow{\text{a.e.}} h \qquad \text{and} \qquad \psi_n \xrightarrow{\text{a.e.}} H.$$

Thus,

$$\varphi_n, \psi_n \xrightarrow{\text{a.e.}} f.$$

The dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_{[a,b]} \varphi_n dm \;=\; \lim_{n\to\infty} \int_{[a,b]} \psi_n dm \;=\; \int_{[a,b]} f dm$$

and it yields

$$(R) \int_a^b f dx \;=\; \int_{[a,b]} f dm.$$

The proof is complete. □

7 $\mathbf{L}^p$-spaces

7.1 $\mathbf{L}^p$-spaces with $1 \leq p < +\infty$

Let $(X, \mathcal{M}, \mu)$ be a measure space. For every given $p \geq 1$, we define

$$\mathcal{L}^p(X, \mu) \;=\; \left\{ f : X \to \overline{\mathbb{R}} \text{ is measurable} : \int_X |f|^p d\mu < +\infty \right\}$$

and

$$\|f\|_p \;=\; \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} \qquad \text{for all } f \in \mathcal{L}^p(X, \mu).$$

Every function $f \in \mathcal{L}^p(X,\mu)$ is finite almost everywhere. Moreover, $\mathcal{L}^P(X,\mu)$ is a vector space since

$$\alpha \cdot f, f+g \in \mathcal{L}^P(X,\mu) \qquad \text{for all } f,g \in \mathcal{L}^P(X,\mu), \alpha \in \mathbb{R}.$$

Indeed,

$$\|\alpha \cdot f\|_p \;=\; |\alpha| \cdot \|f\|_p \;<\; +\infty$$

and

$$\|f+g\|_p^p \;=\; \int_X (|f+g|^p)\, d\mu \;\leq\; 2^{p-1} \cdot \left(\int_X |f|^p d\mu + \int_X |g|^p d\mu \right) \;<\; +\infty.$$

Notice that $\big(\mathcal{L}^P(X,\mu), \|\cdot\|_p\big)$ is NOT a normed space. Indeed, the function

$$f \;=\; \begin{cases} 0 & x \in X \backslash \{a\} \\ \\ 1 & x = a \end{cases}$$

is not 0 but $\|f\|_p = 0$.

Equivalent relation in $\mathcal{L}^p(X,\mu)$: For any $f,g \in \mathcal{L}^p(X,\mu)$, we say that

$$f \;\sim\; g \qquad \Longleftrightarrow \qquad f \;=\; g \qquad a.e \;\; x \in X.$$

Denote by

$$\mathbf{L}^p(X,\mu) \;=\; \mathcal{L}^P(X,\mu)/\sim$$

and for any $f \in \mathcal{L}^P(X,\mu)$

$$\tilde{f} \;=\; \{g \in \mathcal{L}^P(X,\mu) : g = f \;\; a.e\}.$$

It is clear that

$$\|f\|_p \;=\; \|g\|_p \qquad \text{for all } g \in \tilde{f}.$$

The function $\|\cdot\|_p : \mathbf{L}^p(X,\mu) \to [0,+\infty)$ such that

$$\|\tilde{f}\|_p \;:=\; \|f\|_p$$

is well-defined.

We claim that $(\mathbf{L}^p(X,\mu), \|\cdot\|_p)$ is a norm space. Indeed, one can see that

(i). $\|f\|_p = 0$ if any only if $f = 0 \in \mathbf{L}^p(X,\mu)$;

(ii). For any $\alpha \in \mathbb{R}$ and $f \in \mathbf{L}^p(X,\mu)$, it holds

$$\|\alpha \cdot f\|_p \;=\; |\alpha| \cdot \|f\|_p.$$

Therefore, we only need to show that

$$\|f+g\|_p \ \leq\ \|f\|_p + \|g\|_p \qquad f,g\in L^p(X,\mu).$$

In order to do so, let us recall some basic inequalities.

Lemma 7.1 (Holder's inequality) *Let $p,q\in[1,+\infty)$ be conjugate. If $f\in\mathbf{L}^p(X,\mu)$ and $g\in\mathbf{L}^q(X,\mu)$ then $fg\in\mathbf{L}^1(X,\mu)$ and*

$$\|fg\|_1 \ \leq\ \|f\|_p\cdot\|g\|_q. \tag{$*$}$$

Proof. If f or g is 0 then it is trivial. Otherwise, we define

$$F(x)\ =\ \frac{f(x)}{\|f\|_p} \qquad \text{and} \qquad G(x)\ =\ \frac{g(x)}{\|g\|_q}.$$

One has that

$$\|F\|_p\ =\ \|G\|_q\ =\ 1$$

and $(*)$ is equivalent to

$$\|FG\|_1\ \leq\ 1.$$

Using Young's inequality, we have

$$|F(x)\cdot G(x)|\ \leq\ \frac{|F(x)|^p}{p}+\frac{|G(x)^q}{q|}.$$

Thus,

$$\int_X |FG|d\mu\ \leq\ \frac{1}{p}\cdot\int_X |F|^p d\mu+\frac{1}{q}\cdot\int_X |G|^q d\mu\ =\ \frac{1}{p}+\frac{1}{q}\ =\ 1$$

and the proof is complete □

Corollary 7.2 *If $\mu(X)<+\infty$ then it holds*

$$\mathbf{L}^{p_2}(X,\mu)\ \subset\ \mathbf{L}^{p_1}(X,\mu) \qquad \text{for all } 1\leq p_1<p_2<+\infty.$$

Proof. Consider the conjugate pair

$$(p,q)\ =\ \left(\frac{p_2}{p_1},\frac{p_2}{p_2-p_1}\right).$$

For any $f\in\mathbf{L}^{p_2}(X,\mu)$, it holds

$$|f|^{p_1}\ \in\ \mathbf{L}^p(X,\mu) \qquad \text{and} \qquad 1\ \in\ \mathbf{L}^q(X,\mu).$$

Applying the Holder's inequality, we get

$$\int_X |f|^{p_1}d\mu\ \leq\ \||f|^{p_1}\|_p\cdot\|1\|_q\ =\ \mu(X)^{1-\frac{1}{p_2}}\cdot\|f\|_{p_2}^{p_1}\ <\ +\infty$$

and it yields $f\in\mathbf{L}^{p_1}(X,\mu)$. □

Lemma 7.3 (Minskowki's inequality) *Let* $p \in [1, +\infty)$ *and* $f, g \in \mathbf{L}^p(X, \mu)$. *Then* $f + g \in \mathbf{L}^p(X, \mu)$ *and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. It is easy to prove the lemma for $p = 1$. Assume that $p > 1$ and q is its conjugate. Since $pq = pq(1/p + 1/q) = p + q$, it holds

$$|f + g|^{p-1} \in \mathbf{L}^q(X, \mu) \qquad \text{for all } f, g \in \mathbf{L}^p(X, \mu).$$

Applying the Hölder's inequality, we get

$$\int_X |f + g|^{p-1} \cdot |f| d\mu \leq \left\| |f + g|^{p-1} \right\|_q \cdot \|f\|_p = \|f + g\|_p^{p-1} \cdot \|f\|_p$$

and

$$\int_X |f + g|^{p-1} \cdot |g| d\mu \leq \left\| |f + g|^{p-1} \right\|_q \cdot \|g\|_p = \|f + g\|_p^{p-1} \cdot \|g\|_p.$$

Thus,

$$\int_X |f + g|^p d\mu \leq (\|f\|_p + \|g\|_p) \cdot \|f + g\|_p^{p-1}.$$

Dividing by $\|f + g\|_p^{p-1}$, and noting that the left hand side is $\|f + g\|^p$, we then obtain

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

The proof is complete. □

Problem 50: Assume that $p \geq 2$. Use the triangle inequality and the convexity of x^p to show that

$$\left\| \frac{f + g}{2} \right\|_p^p + \left\| \frac{f - g}{2} \right\|_p^p \leq \frac{1}{2}(\|f\|_p^p + \|g\|_p^p).$$

This inequality is related to the *uniform convexity* of the $\mathbf{L}^p$ norm. Note that the $p = 2$ case is related to the parallelogram identity.

Problem 51: Use Hölder's inequality (several times) to prove the following:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(x - y) h(y) \, dx \, dy \leq \|f\|_p \|g\|_q \|h\|_r,$$

where

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

Problem 52: Suppose that $f \in L^p \cap L^r$, with $1 \leq p < s < r \leq \infty$. Show that

$$\|f\|_s \leq \|f\|_p^\theta \|f\|_r^{1-\theta}.$$

Minkowski's inequality shows that $(\mathbf{L}^p(X, \mu), \| \cdot \|_p)$ is a normed space. We are now going to show that it is actually a Banach space.

Theorem 7.4 (Riesz-Fischer) *Let* $(f_n)_{n\geq 1}$ *be a Cauchy sequence in* $\mathbf{L}^p(X,\mu)$. *Then there exists* $f \in \mathbf{L}^p(X,\mu)$ *such that*

(i) There exists a subsequence $(f_{n_k})_{k\geq 1}$ *of* $(f_n)_{n\geq 1}$ *which converges to* f *almost everywhere.*

(ii) f_n *converges to* f *in* $\mathbf{L}^p(X,\mu)$, *i.e.,*

$$\lim_{n\to\infty} \|f_n - f\|_p \;=\; 0.$$

Proof. Since $(f_n)_{n\geq 1}$ is Cauchy, we can find $(f_{n_k})_{k\geq 1} \subseteq (f_n)_{n\geq 1}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p \;\leq\; 2^{-k} \qquad \text{for all } k \geq 1.$$

Denote by

$$g_k(x) \;:=\; \sum_{i=1}^{k} |f_{n_{i+1}}(x) - f_{n_i}(x)| \qquad \text{and} \qquad g(x) \;=\; \sum_{i=1}^{\infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|,$$

we have

$$\begin{cases} \|g_k\|_p \;\leq\; \displaystyle\sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p \;\leq\; \sum_{i=1}^{k} 2^{-i} \;\leq\; 1, \\ \\ (g_n)_{n\geq 1} \;\text{ is increasing and }\; g_n \xrightarrow{\text{a.e.}} g. \end{cases}$$

Thus, the monotone convergence theorem implies that

$$\int_X |g|^p d\mu \;=\; \lim_{n\to\infty} \int_X |g_n|^p d\mu \;\leq\; 1$$

and it yields $g \in \mathbf{L}^p(X,\mu)$. In particular,

$$g(x) \;=\; \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

is finite almost everywhere. Thus,

$$f_{n_1}(x) + \sum_{k=1}^{\infty} \left[f_{n_{k+1}}(x) - f_{n_k}(x)\right] \;\text{ is convergent a.e.}$$

and we define

$$f(x) \;:=\; f_{n_1}(x) + \sum_{k=1}^{\infty} \left[f_{n_{k+1}}(x) - f_{n_k}(x)\right].$$

It is clear that $f_{n_k} \xrightarrow{\text{a.e.}} f$ and

$$\|f\|_p \;\leq\; \|f_{n_1}\|_p + \|g\|_p \;<\; +\infty \qquad \Longrightarrow \qquad f \;\in\; \mathbf{L}^p(X,\mu).$$

To complete the proof, since $(f_n)_{n\geq 1}$ is Cauchy, we only to show that f_{n_k} converges to f in $\mathbf{L}^p(X,\mu)$. For any $k \geq 1$, consider the function

$$h_k(x) \;=\; |f_{n_k}(x) - f(x)|^p$$

we have that $h_k \xrightarrow{\text{a.e.}} 0$, and

$$|h_k| \;\leq\; (|f| + |f_{n_1}| + |g|)^p \;\in\; \mathbf{L}^1(X,\mu).$$

The dominated convergence theorem implies that

$$\lim_{k\to\infty} \int_X |f_{n_k} - f|^p d\mu \;=\; \lim_{k\to\infty} \int_X |h_k|^p d\mu \;=\; 0$$

and the proof is complete. □

As a consequence of Fatou's lemma, dominated convergence theorem, the Vitali convergence theorem, and the previous proposition we have the following:

Corollary 7.5 *Let $(f_n)_{n\geq 1}$ be a sequence in $\mathbf{L}^p(X,\mu)$. Assume that*

$$f_n \xrightarrow{a.e.} f.$$

(i) If $(f_n)_{n\geq 1}$ is bounded in $\mathbf{L}^p(X,\mu)$ then $f \in \mathbf{L}^p(X,\mu)$ and

$$\|f\|_p \;\leq\; \liminf_{n\to\infty} \|f_n\|_p.$$

(ii) If there exists $g \in \mathbf{L}^p(X,\mu)$ such that

$$|f_n(x)| \;\leq\; g(x) \qquad \text{for all a.e. } x \in X, \text{ for all } n \geq 1,$$

then f_n converges to f in $\mathbf{L}^p(X,\mu)$.

(iii) $(f_n)_{n\geq 1}$ is uniformly integrable and tight then f_n converges to f in $\mathbf{L}^p(X,\mu)$.

Notice that

$$f_n \xrightarrow{\mathbf{L}^p} f \qquad \Longrightarrow \qquad \|f_n\|_p \;\to\; \|f\|_p.$$

Proposition 7.5.1 *Let $(f_n)_{n\geq 1}$ be a sequence in $\mathbf{L}^p(X,\mu)$. If*

$$f_n \xrightarrow{a.e.} f \in \mathbf{L}^p(X,\mu) \qquad \text{and} \qquad \|f_n\|_p \;\to\; \|f\|_p$$

then $f_n \xrightarrow{\mathbf{L}^p} f$.

Proof. Consider the function

$$g_n(x) \;=\; \frac{|f_n(x)|^p + |f(x)|^p}{2} - \frac{|f_n(x) - f(x)|^p}{2^p}.$$

By convexity of t^p, we have that

$$g \;\geq\; 0 \qquad \text{and} \qquad g_n \;\xrightarrow{\text{a.e.}}\; |f|^p.$$

The Fatou's lemma implies

$$\liminf_{n\to\infty} \int_X |g_n| d\mu \;\geq\; \int_X |f|^p d\mu.$$

Recalling that $\|f_n\|_p \;\to\; \|f\|_p$, we then obtain that

$$\limsup_{n\to\infty} \int_X |f_n - f|^p d\mu \;\leq\; 0.$$

and it yields $f_n \;\xrightarrow{\mathbf{L}^p}\; f$. □

7.2 $\mathbf{L}^\infty$-space

Given a measure space $(X, \mathcal{M}, \mu)$, let $f : X \to \overline{\mathbb{R}}$ be Borel. We define the essential supremum of f as the following:

- If $\mu(\{|f| > M\}) > 0$ for all $M > 0$ then $\|f\|_\infty = \infty$.
- Otherwise,

$$\|f\|_\infty \;=\; \inf\{M > 0 : \mu\left(|f| > M\right) = 0\} \;<\; +\infty$$

We say that f is essentially bounded if $\|f\|_\infty$ is finite. In this case, we write

$$f \;\in\; \mathcal{L}^\infty\left(X, \mu\right).$$

Notice that for any $f \in \mathcal{L}^\infty\left(X, \mu\right)$, it holds

(a). $|f(x)| \leq \|f\|_\infty$ for a.e. $x \in X$;

(b). The essential supremum of f can be defined by

$$\|f\|_\infty \;:=\; \min\{M > 0 : |f(x)| \leq M \;\; a.e.\}.$$

Example 7.1 *Consider $f : [0, 1] \to \mathbb{R}$ to be the Dirichlet function (i.e. the function which is 1 on the rationals and 0 on the irrationals). Then $\|f\|_\infty = 0$, but f is not equal to the zero function.*

We denote by

$$\mathbf{L}^{\infty}(X,\mu) \;=\; \mathcal{L}^{\infty}(X,\mu)\big/\sim,$$

the following holds:

Proposition 7.5.2 $(\mathbf{L}^{\infty}(X,\mu), \|\cdot\|_{\infty})$ *is a Banach space.*

Proof. Given any Cauchy sequence $(f_n)_{n\geq 1}$ in $\mathbf{L}^{\infty}(X,\mu)$, we set

$$A_n \;=\; \{x \in X : |f_n(x)| > \|f_n\|_{\infty}\}$$

and

$$B_{n,m} \;=\; \{x \in X : |f_m(x) - f_n(x)| > \|f_m - f_n\|_{\infty}\}.$$

By the definition of $\|\cdot\|_{\infty}$, it holds

$$\mu(A_n) \;=\; \mu(B_{n,m}) \;=\; 0 \qquad \text{for all } n,m \geq 1,$$

and thus the set

$$X_0 \;:=\; \left(\bigcup_{n=1}^{\infty} A_n\right) \bigcup \left(\bigcup_{n,m=1}^{\infty} B_{n,m}\right)$$

has a zero measure. Set $\tilde{X} := X \backslash X_0$, we have

$$|f_m(x) - f_n(x)| \;\leq\; \|f_m - f_n\|_{\infty} \qquad \text{for all } x \in \tilde{X}.$$

In particular, $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in $\tilde{X}$. Thus, it is converges. Define

$$f(x) \;=\; \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{if} \qquad x \in \tilde{X}, \\ \\ 0 & \text{if} \qquad x \in X_0. \end{cases}$$

One can easily check that

$$\|f\|_{\infty} \;\leq\; \limsup_{n\to\infty} \|f_n\|_{\infty} \;<\; +\infty$$

and it yields $f \in \mathbf{L}^{\infty}(X,\mu)$. To complete the proof, we will show that

$$\lim_{n\to\infty} \|f_n - f\|_{\infty} \;=\; 0$$

Observe that

$$|f_n(x) - f(x)| \;\leq\; \|f_m - f_n\|_{\infty} + |f_m(x) - f(x)| \qquad \text{for all } x \in \tilde{X}.$$

Since $(f_n)_{n\geq 1}$ is Cauchy in $\mathbf{L}^{\infty}(X,\mu)$, there exists $N_{\varepsilon} > 0$ such that

$$\|f_m - f_n\|_{\infty} \;\leq\; \varepsilon \qquad \text{for all } m,n \geq N_{\varepsilon},$$

and this implies that

$$|f_n(x) - f(x)| \le \varepsilon + |f_m(x) - f(x)| \qquad \text{for all } m, m \ge N_\varepsilon.$$

For any $x \in \tilde{X}$, we take m to $+\infty$ and obtain that

$$|f_n(x) - f(x)| \le \varepsilon \qquad \text{for all } n \ge N_\varepsilon$$

and it yields

$$\|f_n - f\|_\infty \le \varepsilon \qquad \text{for all } n \ge N_\varepsilon.$$

The proof is complete. □

Proposition 7.5.3 (Relation between $\|\cdot\|_\infty$ and $\|\cdot\|_p$) *Assume that $f \in \mathbf{L}^p \cap \mathbf{L}^\infty$ for some $1 \le p < +\infty$. Then,*

$$f \in \bigcap_{q \ge p} \mathbf{L}^q \qquad \text{and} \qquad \lim_{q \to +\infty} \|f\|_q = \|f\|_\infty.$$

Proof. For any $p \le q$, we have

$$\int_X |f|^q d\mu = \int_X |f|^p \cdot |f|^{q-p} d\mu \le \|f\|_\infty^{q-p} \cdot \int_X |f|^p d\mu$$

and it yields

$$\|f\|_q \le \|f\|_\infty^{1-\frac{p}{q}} \cdot \|f\|_p^{\frac{p}{q}} < +\infty.$$

Thus, f is in $\mathbf{L}^q$ and

$$\limsup_{q \to \infty} \|f\|_q \le \limsup_{q \to \infty} \left(\|f\|_\infty^{1-\frac{p}{q}} \cdot \|f\|_p^{\frac{p}{q}} \right) = \|f\|_\infty.$$

To complete the proof, we need to show that

$$\|f\|_\infty \le \liminf_{q \to \infty} \|f\|_q.$$

Equivalently, for any $a < \|f\|_\infty$, it holds that

$$\liminf_{q \to \infty} \|f\|_q \ge a.$$

Applying Markov's inequality, we get

$$0 < \mu(\{|f| > a\}) = \mu(\{|f|^q > a^q\}) \le \frac{1}{a^q} \cdot \int_X |f|^q d\mu = a^{-q} \cdot \|f\|_q^q$$

and this implies that

$$\|f\|_q \ge a \cdot \mu(\{|f| > a\})^{\frac{1}{q}}.$$

Taking q to $+\infty$, we then obtain

$$\liminf_{q \to \infty} \|f\|_q \ge a \cdot \liminf_{q \to \infty} \mu(\{|f| > a\})^{\frac{1}{q}} = a.$$

The proof is complete □

Definition 7.6 (Convergence in measure) *Let $(f_n)_{n\geq 1}$ and f be measurable functions from X to $\mathbb{R}$. We say that $(f_n)_{n\geq 1}$ converges in measure to f if for every $\eta > 0$, it holds*

$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n - f| > \eta\}\right) \;=\; 0.$$

Proposition 7.6.1 *Let $(f_n)_{n\geq 1}$ and f be Borel functions from X to $\mathbb{R}$. The following statements hold:*

(i). If $\mu(X) < \infty$ and $\{f_n\}_{n\geq 1}$ converges point-wise a.e. on X to a measurable function f then $\{f_n\}_{n\geq 1}$ converges in measure to f.

(ii). If $\{f_n\}_{n\geq 1}$ converges in measure to f, then there exists a subsequences of $\{f_n\}_{n\geq 1}$ converges point-wise a.e. on X to f.

Proof. *(i).* By the Severini–Egorov theorem, for any $\varepsilon > 0$, there exists some set E_ε such that

$$\mu(E_\varepsilon) \;<\; \varepsilon \qquad \text{and} \qquad \lim_{n\to\infty} \sup_{x\in X\setminus E_\varepsilon} |f_n(x) - f(x)| \;=\; 0 \tag{1}$$

For any $\eta > 0$, set

$$A_\eta^n \;=\; \{x \in X : |f_n(x) - f(x)| > \eta\},$$

we have

$$\mu(A_\eta^n) \;=\; \mu(A_\eta^n \cap E_\varepsilon) + \mu(A_\eta^n \setminus E_\varepsilon) \;\leq\; \varepsilon + \mu(A_\eta^n \cap (X\setminus E_\varepsilon))$$

From (1), there exists $N_\eta > 0$ such that

$$|f_n(x) - f(x)| \;<\; \eta \qquad \text{for all } n > N_\eta, x \in X\setminus E_\varepsilon\,.$$

Thus,

$$A_\eta^n \cap (X\setminus E_\varepsilon) \;=\; \emptyset \qquad \text{for all } n > N_\eta$$

and it yields

$$\limsup_{n\to+\infty} \mu(A_\eta^n) \;\leq\; \varepsilon + \limsup_{n\to\infty} \mu(A_\eta^n \cap (X\setminus E_\varepsilon)) \;=\; \varepsilon\,.$$

Taking ε to $0+$, we obtain that

$$\limsup_{n\to+\infty} \mu(A_\eta^n) \;=\; 0$$

and the proof is complete. $\square$

7.3 Dense subsets in $\mathbf{L}^p$

In this subsection, we will show that the set of simple functions and the set of of continuous functions with compact support are dense in $\mathbf{L}^p$.

Proposition 7.6.2 *Assume that*

$$\mathcal{S} \ = \ \{\varphi : X \to \mathbb{R} : \varphi \ \text{ is simple}\}.$$

For any $1 \leq p < +\infty$, it holds

$$\overline{\mathcal{S}} \ = \ \mathbf{L}^p.$$

Proof. For any $f \in \mathbf{L}^p$, it holds that f is finite almost everywhere, and

$$f \ = \ f^+ - f^-$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are non-negative functions. Thus, we can assume that f is non-negative. Recalling the approximation theorem, there exists a increasing sequence of simple functions $(\varphi_n)_{n\geq 1} \subset \mathcal{S}$ such that

$$\varphi_n \xrightarrow{\text{a.e.}} f \qquad \text{and} \qquad \varphi_n(x) \ \leq \ f(x) \qquad a.e. \ \ x \in X.$$

Thus, the dominated convergence theorem implies that φ_n converges to f in $\mathbf{L}^p$. $\square$

Let Ω be an open subset in $\mathbb{R}^d$. The set

$$\mathcal{C}_c(\Omega) \ = \ \Big\{f : \Omega \to \mathbb{R} \ \ \textit{continuous functions with } \overline{\textit{supp}(f)} \textit{ compact}\Big\}$$

is a vector space.

Theorem 7.7 *Let $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ be a Radon measure, i.e.,*

$$\mu(K) \ < \ +\infty \qquad \textit{for all } K \subset \Omega \ \ \textit{compact}.$$

The set $\mathcal{C}_c(\Omega)$ is dense in $\mathbf{L}^p(\Omega, \mu)$.

Proof. It is sufficient to show that for any $f \in \mathbf{L}^p(\Omega, \mu)$ non-negative and $\varepsilon > 0$, there exists $f_\varepsilon \in \mathcal{C}_c(\Omega)$ such that

$$\|f_\varepsilon - f\|_p \ \leq \ \varepsilon.$$

This is divided into several steps:

1. Assume that $\Omega = \mathbb{R}^d$ and

$$0 \ \leq \ \|f\|_\infty \ \leq \ M \qquad \text{and} \qquad \text{supp}(f) \ \subset \ B(0, r).$$

Since μ is radon, we have that

$$\mu(B(0,r)) \ \leq \ \mu(\overline{B(0,r)}) \ < \ +\infty.$$

By Lusin's theorem, there exists $f_\varepsilon \in C_c(\mathbb{R}^d)$ such that

$$\|f_\varepsilon\|_\infty \ \leq \ M \qquad \text{and} \qquad \mu\left(E_\varepsilon := \left\{x \in \mathbb{R}^d : f_\varepsilon(x) \neq f(x)\right\}\right) \ \leq \ \frac{\varepsilon^p}{2^p M^p}.$$

Thus,

$$\int_{\mathbb{R}^d} |f_\varepsilon - f|^p d\mu \ = \ \int_{E_\varepsilon} |f_\varepsilon - f|^p d\mu \ \leq \ (2M)^p \cdot \mu(E_\varepsilon) \ = \ \varepsilon^p$$

and it yields

$$\|f_\varepsilon - f\|_p \ \leq \ \varepsilon.$$

2. Let's now remove the assumption

$$\mathrm{supp}(f) \ \subseteq \ B(0,r).$$

For every $n \geq 1$, we set

$$g_n \ = \ f \cdot \chi_{B(0,n)}.$$

Using the dominated convergence theorem, one can show that

$$\lim_{n\to\infty} \|g_n - f\|_p \ = \ 0.$$

In particular, there exists $n_\varepsilon > 0$ sufficiently large such that

$$\|g_{n_\varepsilon} - f\|_p \ \leq \ \frac{\varepsilon}{2}.$$

From the previous step, on can find $f_\varepsilon \in C_c(\mathbb{R}^d)$ such that

$$\|f_\varepsilon - g_{n_\varepsilon}\|_p \ \leq \ \frac{\varepsilon}{2}$$

and it yields

$$\|f_\varepsilon - f\|_p \ \leq \ \frac{\varepsilon}{2}.$$

3. To remove the assumption

$$0 \ \leq \ \|f\|_\infty \ \leq \ M,$$

we consider the sequence of functions

$$g_n \ = \ f \cdot \chi_{E_n} \qquad \text{with} \qquad E_n \ = \ \{x \in X : f(x) \leq n\}.$$

which converges to f in $\mathbf{L}^p$. Thus, there exists $n_\varepsilon > 0$ sufficiently large such that

$$\|g_{n_\varepsilon} - f\|_p \ \leq \ \frac{\varepsilon}{2}.$$

From the previous step, on can find $f_\varepsilon \in C_c(\mathbb{R}^d)$ such that

$$\|f_\varepsilon - g_{n_\varepsilon}\|_p \ \leq \ \frac{\varepsilon}{2} \qquad \Longrightarrow \qquad \|f_\varepsilon - f\|_p \ \leq \ \frac{\varepsilon}{2}.$$

4. Let's go back to the case $\Omega \subset \mathbb{R}^d$ open. We denote by

$$\bar{f} \ = \ \begin{cases} f(x), & x \in \Omega \\ \\ 0, & x \in \Omega^c \end{cases}$$

and $\bar{\mu} : \mathcal{B}(\mathbb{R}^d) \to [0, +\infty]$ such that

$$\bar{\mu}(A) \ = \ \mu(A \cap \Omega) \qquad \text{for all } A \in \mathcal{B}(\mathbb{R}^d).$$

From the previous steps, there exists $f_\varepsilon \in \mathcal{C}_c(\mathbb{R}^d)$ such that

$$\|\bar{f}_\varepsilon - \bar{f}\|_p \ \leq \ \frac{\varepsilon}{2} \qquad \Longrightarrow \qquad \left\|(\bar{f}_\varepsilon)_{|\Omega} - f\right\|_p \ \leq \ \frac{\varepsilon}{2}.$$

To complete the proof, we modify $\bar{f}_\varepsilon$ so that its support is contained in Ω. Let $(V_n)_{n\geq 1}$ be a sequence of open bounded subsets of Ω such that

$$\overline{V}_n \ \subseteq \ V_{n+1} \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} V_n \ = \ \Omega.$$

We set

$$g_n(x) \ = \ \bar{f}_\varepsilon(x) \cdot \frac{d_{V_{n+1}^c}(x)}{d_{V_{n+1}^c}(x) + d_{V_n}(x)}.$$

It is clear that

$$\overline{supp(g_n)} \ \subset \ \overline{V_{n+1}}, \qquad |g_n| \ \leq \ \bar{f}_\varepsilon \qquad \text{and} \qquad g_n \xrightarrow{\text{a.e.}} f_\varepsilon \ \ \in \ \Omega.$$

Hence, g_n converges to $\bar{f}_\varepsilon$ in $L^p(\Omega, \mu)$. In particular, there exists $n_\varepsilon > 0$ sufficiently large such that

$$\|g_{n\varepsilon} - \bar{f}\|_p \ \leq \ \frac{\varepsilon}{2}.$$

Therefore, set $f_\varepsilon = g_{n_\varepsilon}$ we have that $f_\varepsilon \in C_c(\Omega)$ and

$$\|f_\varepsilon - f\|_p \ \leq \ \varepsilon$$

and the proof is complete. □

Corollary 7.8 (Translation continuity in $\mathbf{L}^p$) *Let f be in $\mathbf{L}^p(\mathbb{R}^d)$, $1 \le p < \infty$. Then*

$$\lim_{|h|\to 0} \int_{\mathbb{R}^d} |f(x+h) - f(x)|^p dx \;=\; 0$$

Proof. From the above theorem, for every $\varepsilon > 0$ there exists $f_\varepsilon \in C_c(\mathbb{R}^d)$ such that

$$\|f_\varepsilon - f\|_p \;\le\; \frac{\varepsilon}{4}.$$

Let K_ε be the compact support of f_ε. Consider the compact set

$$K_\varepsilon \;\subseteq\; \tilde{K}_\varepsilon \;=\; \{x \in \mathbb{R}^d : d_{K_\varepsilon}(x) \le 1\},$$

we have

$$supp(f_\varepsilon(\cdot + h)) \;\subseteq\; \tilde{K}_\varepsilon \qquad \text{for all} \quad |h| \le 1.$$

For every $|h| \le 1$, one estimates

$$\begin{aligned}\int_{\mathbb{R}^d} |f_\varepsilon(x+h) - f_\varepsilon(x)|^p dx \;&=\; \int_{\tilde{K}_\varepsilon} |f_\varepsilon(x+h) - f_\varepsilon(x)|^p dx \\ &\le\; \left(\sup_{|x-y|\le|h|} |f_\varepsilon(x) - f_\varepsilon(y)|^p\right) \cdot \mathcal{L}^d(\tilde{K}_\varepsilon).\end{aligned}$$

Notice that f_ε is uniformly continuous in $\mathbb{R}^d$, we have

$$\lim_{h\to 0+} \int_{\mathbb{R}^d} |f_\varepsilon(x+h) - f_\varepsilon(x)|^p dx \;=\; 0.$$

In particular, there exists $\delta_\varepsilon > 0$ small such that $|h| < \delta_\varepsilon$

$$\|f_\varepsilon(\cdot + h) - f_\varepsilon(\cdot)\|_p \;\le\; \frac{\varepsilon}{4}$$

and this implies that

$$\|f(\cdot + h) - f(\cdot)\|_p \;\le\; \|f_\varepsilon(\cdot + h) - f_\varepsilon(\cdot)\|_p + 2 \cdot \|f_\varepsilon - f\|_p \;\le\; \frac{3\varepsilon}{4}.$$

The proof is complete. □

Problem 53: Assume that $\mu(X) < \infty$ and $1 < p, q < +\infty$ are a conjugate pair. Show that if $f : X \to \mathbb{R}$ is a Borel function such that

$$fg \;\in\; \mathbf{L}^1(X, \mu) \qquad \text{for all } g \in \mathbf{L}^p(X, \mu)$$

then f is in $\mathbf{L}^r(X, \mu)$ for $r \in [1, q]$.

Problem 54: Let $1 \leq p < \infty$. Show that if $f \in \mathbf{L}^p(\mathbb{R}^d, \mu)$ *(μ is the d-Lebesgue measure)* and f is uniformly continuous, then

$$\lim_{\|x\| \to \infty} f(x) \;=\; 0\,.$$

Is it still true if f is just continuous?

Problem 55: Let $\{f_n\}_{n\geq 1}$ be the sequence defined by

$$f_n(x) \;=\; \frac{n}{e^{n\sqrt{x}} - 1} \qquad \text{for all } x \in (0,1)\,.$$

Show that $f_n \in \mathbf{L}^p((0,1), m)$ and $f_n \to 0$ in $\mathbf{L}^p$ for every $1 \leq p < 2$.

Problem 56: Prove Hardy's inequality, namely that if $F(t) = t^{-1}\displaystyle\int_0^t f(t)\,dt$ then for $1 < p < \infty$

$$\|F\|_{\mathbf{L}^p([0,\infty))} \leq \frac{p}{p-1} \cdot \|f\|_{\mathbf{L}^p([0,\infty))}.$$

Problem 57: One can define the Fourier transform of an $\mathbf{L}^1$ function via

$$\mathcal{F}(f)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-i\lambda t}\,dt.$$

This definition makes sense for any λ by Holder's inequality. However, one often would like to make sense of the Fourier transform of an $\mathbf{L}^2$ function, and this point-wise definition does not make sense. However, one can directly compute that

$$\langle \mathcal{F}(\chi_{[b,c]}), \mathcal{F}(\chi_{[d,e]})\rangle = \langle \chi_{[b,c]}, \chi_{[d,e]}\rangle.$$

Argue that this formula allows us to define a Fourier transform on $\mathbf{L}^2$ and that the Fourier transform is an isometry on $\mathbf{L}^2$.

7.4 Dual space of $\mathbf{L}^p$

Given a real Banach space $(B, \|\cdot\|)$, the dual space of B is defined by

$$B^* \;=\; \{T : B \to \mathbb{R} \;:\; T \;\; \textit{is linear and bounded}\}$$

with norm

$$\|T\|_* \;=\; \sup_{B\setminus\{0\}} \frac{|T(x)|}{\|x\|} \;=\; \sup_{\|x\|=1} \frac{|T(x)|}{\|x\|}.$$

Notice that if $T \in B^*$ then T is continuous.

Goal: Given a measure space $(X, \mathcal{M}, \mu)$ and $1 \leq p < +\infty$, find $[\mathbf{L}^p(X,\mu)]^*$.

Let $f \in \mathbf{L}^q(X, \mu)$ where q is the conjugate of p, i.e.,

$$\frac{1}{p} + \frac{1}{q} \ = \ 1 \qquad \Longleftrightarrow \qquad p + q \ = \ pq.$$

Consider the linear functional $T_f : \mathbf{L}^p(X, \mu) \to \mathbb{R}$ such that

$$T_f(g) \ = \ \int_X f \cdot g \ d\mu \qquad \text{for all } g \in \mathbf{L}^p(X, \mu).$$

The followings hold:

Proposition 7.8.1 *Given any $f \in \mathbf{L}^q$ with $p > 1$, the map T_f is bounded and*

$$\|T_f\|_* \ = \ \|f\|_q.$$

If X is σ-finite then the same result holds for $p = 1$ and $q = \infty$.

Proof. Two cases are considered:

1. Assume that $p > 1$. From Holder's inequality, we have

$$|T_f(g)| \ \leq \ \int_X |fg| d\mu \ \leq \ \|f\|_q \cdot \|g\|_p \qquad \text{for all } g \in \mathbf{L}^p$$

Thus, the map T_f is bounded and

$$\|T_f\|_* \ \leq \ \|f\|_q. \tag{7.1}$$

We need to show that $\|f\|_q \leq \|T_f\|_*$. If $f = 0$ then it is trivial. Otherwise, we consider we consider

$$g \ = \ \text{sign}(f) \cdot \left(\frac{|f|}{\|f\|_q} \right)^{\frac{q}{p}} \ \in \ \mathbf{L}^p \qquad \text{with} \qquad \|g\|_p \ = \ 1.$$

We compute that

$$T_f(g) \ = \ \frac{1}{\|f\|_q^{\frac{q}{p}}} \cdot \int_X |f|^{1+\frac{q}{p}} d\mu \ = \ \frac{1}{\|f\|_q^{\frac{q}{p}}} \cdot \int_X |f|^q d\mu \ = \ \frac{\|f\|_q^q}{\|f\|_q^{\frac{q}{p}}} \ = \ \|f\|_q$$

and this yields (7.1).

2. Assume that $p = 1, q = +\infty$ and X is σ-finite. In this case, we also have that

$$|T_f(g)| \ \leq \ \int_X |fg| d\mu \ \leq \ \|f\|_\infty \cdot \int_X |g| d\mu \qquad \text{for all } g \in \mathbf{L}^1.$$

Thus, the map T_f is bounded with $\|T_f\|_* \leq \|f\|_\infty$. To compute the proof, we need to show that

$$\|T_f\|_* \ \geq \ \|f\|_\infty. \tag{7.2}$$

Notice that the function

$$g \;=\; \text{sign}(f) \;\notin\; \mathbf{L}^1.$$

For every $\varepsilon > 0$, one needs to find $g_\varepsilon \in \mathbf{L}^1$ such that

$$\|g\|_1 \;=\; 1 \qquad \text{and} \qquad |T_f(g_\varepsilon)| \;>\; \|f\|_\infty - \varepsilon.$$

By the definition of $\|\cdot\|_\infty$, the following set

$$A_\varepsilon \;=\; \{x \in X : |f(x)| > \|f\|_\infty - \varepsilon\}$$

have a positive measure. Since μ is σ-finite, there exists $B_\varepsilon \subseteq A_\varepsilon$ such that

$$0 \;<\; \mu(B_\varepsilon) \;<\; +\infty.$$

Consider the function $g_\varepsilon : X \to \mathbb{R}$ such that

$$g_\varepsilon \;=\; \text{sign}(f) \cdot \frac{\chi_{B_\varepsilon}}{\mu(B_\varepsilon)}.$$

We compute

$$\|g_\varepsilon\|_1 \;=\; \int_X \frac{\chi_{B_\varepsilon}}{\mu(B_\varepsilon)} d\mu \;=\; 1$$

and

$$|T_f(g_\varepsilon)| \;=\; \left|\int_X fg d\mu\right| \;=\; \frac{1}{\mu(B_\varepsilon)} \int_{B_\varepsilon} |f| d\mu \;\geq\; \|f\|_\infty - \varepsilon.$$

Thus,

$$\|T_f\|_* \;>\; \|f\|_\infty - \varepsilon \qquad \text{for all } \varepsilon > 0,$$

and this yields (7.2). □

As a consequence, the linear map $T : \mathbf{L}^q \to [\mathbf{L}^p]^*$ defined by

$$T[f] \;=\; T_f \qquad \text{for all } f \in \mathbf{L}^q$$

is isometry, .i.e.,

$$\|f\|_q \;=\; \|T[f]\|_* \;=\; \|T_f\|_* \qquad \text{for all } f \in \mathbf{L}^q.$$

In particular, it is linear, bounded and injective. The next theorem will show that T is actually onto for $p > 1$.

Theorem 7.9 (Riesz representation theorem for $(\mathbf{L}^p)^*$) *Given any $p > 1$, the map $T : \mathbf{L}^q \to [\mathbf{L}^p]^*$ such that*

$$T[f](g) \;=\; \int_X fg d\mu \qquad \textit{for all } f \in \mathbf{L}^q, g \in \mathbf{L}^p$$

is an isometric isomorphism of $\mathbf{L}^q$ onto $[\mathbf{L}^p]^*$, i.e., T is bijective and

$$\|T[f_1] - T[f_2]\|_* \;=\; \|f_1 - f_2\|_q.$$

Proof. From the previous proposition, we only need to show that T is onto, i.e., for any $F \in [\mathbf{L}^p]^*$, find $f \in \mathbf{L}^q$ such that

$$T[f] \; = \; F.$$

Equivalently,

$$T[f](g) \; = \; \int_X f \cdot g d\mu \; = \; F(g) \qquad \text{for all } g \in \mathbf{L}^p. \tag{$*$}$$

The proof of (*) is divided into several steps:

1. Assume that $\mu(X) < +\infty$. Recalling that the set

$$\mathcal{S} \; = \; \{\varphi : X \to \mathbb{R} : \varphi \;\; is \; a \; simple \; function\}$$

is dense in $\mathbf{L}^p$. Thus, by the continuity of $T[f]$ and F, it is sufficient to check $(*)$ for all $g = \varphi \in \mathcal{S}$. Indeed, if $(*)$ holds for all $g = \varphi \in S$ then for every $g \in \mathbf{L}^p$, there exists $(\varphi_n)_{n\geq 1} \subset \mathcal{S}_p$ such that

$$\varphi_n \;\; converges \; to \;\; g \;\; in \;\; \mathbf{L}^p.$$

and thus

$$T[f](g) \; = \; \lim_{n\to\infty} T[f](\varphi_n) \; = \; \lim_{n\to\infty} F(\varphi_n) \; = \; F(g).$$

On the other hand, since T_f and F are linear, $(*)$ holds for all simple functions if any only if it holds for all measurable characteristic functions. Therefore, we need to find $f \in \mathbf{L}^q$ such that

$$\int_X f \cdot \chi_A \, d\mu \; = \; F(\chi_A) \qquad \text{for all } A \in \mathcal{M}.$$

Let $\nu : \mathcal{M} \to \mathbb{R}$ be such that

$$\nu(A) \; = \; F(\chi_A) \qquad \text{for all } A \in \mathcal{M}.$$

One can check that ν is a sign measure on $(\mathcal{M}, \mu)$. Indeed,

$$\nu(\varnothing) \; = \; F(0) \; = \; 0 \qquad \text{and} \qquad \nu(A) \; < \; +\infty \qquad \text{for all } A \in \mathcal{M}.$$

Given $(A_n)_{n\geq 1}$ a disjoint sequence of sets, we have

$$\chi_A \; = \; \sum_{i=1}^{\infty} \chi_{A_i}, \qquad A \; = \; \bigcup_{n=1}^{\infty} A_n.$$

Using the dominated convergent theorem, we get

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} \chi_{A_i} - \chi_A \right\|_p \; = \; 0.$$

and this implies that

$$\begin{aligned}\nu(A) &= F(\chi_A) = \lim_{n\to\infty} F\left(\sum_{k=1}^{n}\chi_{A_k}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}F(\chi_{A_k})\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\nu(A_k) = \sum_{k=1}^{\infty}\nu(A_k).\end{aligned}$$

On the other hand, if $\mu(A) = 0$ then $\chi_A = 0$ in $\mathbf{L}^p$ and this implies that

$$\nu(A) = F(\chi_A) = 0.$$

The measure ν is absolutely continuous with respect to μ. Using Radon-Nikodym theorem, there exists $f : X \to \mathbb{R}$ measurable such that

$$F(\chi_A) = \nu(A) = \int_X f\cdot\chi_A d\mu \qquad \text{for all } A\in\mathcal{M}$$

and it yields

$$F(\varphi) = \int_X f\cdot\varphi\, d\mu \qquad \text{for all } \varphi\in\mathcal{S}.$$

Choosing $\varphi = \text{sign}(f)\in\mathcal{S}$, we have

$$\int_X |f|\, d\mu = F(\varphi) \le \|F\|_*\cdot\|\varphi\|_p = \|F\|_*\cdot\mu(X)^{\frac{1}{p}} < +\infty$$

and thus $f\in\mathbf{L}^1$. To show that $f\in\mathbf{L}^q$, we consider a sequence of simple functions $(\varphi_n)_{n\ge 1}\in\mathcal{S}$ such that

$$\varphi_n \xrightarrow{\text{a.e.}} f \qquad \text{and} \qquad |\varphi_n| \le |f|.$$

The following function

$$g_n = \text{sign}(f)\cdot\left(\frac{|\varphi_n|}{\|\varphi_n\|_q}\right)^{\frac{q}{p}}$$

is in $\mathbf{L}^\infty$ and satisfies

$$\|g_n\|_p = 1, \qquad f\cdot g_n = |f\cdot g_n| \qquad \text{and} \qquad \int_X |\varphi_n\cdot g_n|\, d\mu = \|\varphi_n\|_q\,.$$

Using the Fatou's lemma, one gets

$$\begin{aligned}\|f\|_q &\le \liminf_{n\to\infty}\|\varphi_n\|_q = \liminf_{n\to\infty}\int_X|\varphi_n\cdot g_n|d\mu \le \liminf_{n\to\infty}\int_X|f\cdot g_n|d\mu\\ &= \liminf_{n\to\infty}\left|\int_X f\cdot g_n d\mu\right| = \liminf_{n\to\infty}|F(g_n)| \le \|F\|_*.\end{aligned}$$

Thus,

$$f \ \in \ \mathbf{L}^q \qquad \text{and} \qquad \|f\|_q \ \leq \ \|F\|_*.$$

Using the previous proposition, we obtain that

$$\|F\|_* \ = \ \|f\|_q.$$

2. X is σ-finite. Then there exists $(A_n)_{n\geq X}$ increasing such that

$$X \ = \ \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \mu(A_n) \ < \ +\infty.$$

From step 1, let $f_n \in \mathbf{L}^q(A_n,\mu)$ be such that

$$F(g) \ = \ \int_{A_n} f_n \cdot g d\mu \qquad \text{for all } g \in \mathbf{L}^p(A_n,\mu).$$

Since $(A_n)_{n\geq 1}$ is increasing, one has that

$$f_m \ = \ f_n \qquad a.e.\ x \in A_n, m \geq n.$$

Denote by

$$f(x) \ = \ \lim_{n\to\infty} f_n(x) \qquad a.e.\ x \in X$$

The Fatou's lemma implies that

$$\|f\|_q \ \leq \ \liminf_{n\to\infty} \ \|f_n\|_q \ \leq \ \|F\|_*$$

and the dominated convergence theorem yields

$$F(g) \ = \ \int_X f \cdot g d\mu \qquad \text{for all } g \in \mathbf{L}^p.$$

3. In general, X is non σ-finite then for any σ-finite set A, let $f_A \in \mathbf{L}^q(A,\mu)$ be such that

$$F(g) \ = \ \int_A f_A \cdot g d\mu \qquad \text{for all } g \in \mathbf{L}^p(A,\mu).$$

Introduce the constant

$$M \ = \ \sup\{\|f_A\|_q : A \subset X \text{ is } \sigma-\text{finite}\} \ \leq \ \|F\|_\infty.$$

Choosing $(A_n)_{n\geq 1}$ increasing sequence of sets such that $\lim_\infty \ \|f_{A_n}\|_q = M$, and we define

$$B \ = \ \bigcup_{n=1}^{\infty} A_n, \qquad f_B \ = \ \lim_{n\to\infty} f_{A_n}.$$

By a contradiction argument, one can show that

$$F(g \cdot \chi_{X\setminus B}) \ = \ 0 \qquad \text{for all } g \in \mathbf{L}^p$$

and this implies that

$$\int_X f_B \cdot g \, d\mu \;=\; F(g) \qquad \text{for all } g \in \mathbf{L}^p.$$

The proof is complete. □

Corollary 7.10 *Given $(X, \mathcal{M}, \mu)$ a measure space, and $1 < p < \infty$, then $\mathbf{L}^p(X, \mu)$ is reflexive, i.e.,*

$$[\mathbf{L}^p(X,\mu)]^{**} \;\simeq\; \mathbf{L}^p(X,\mu).$$

It means that the identical map

$$i : \mathbf{L}^p(X,\mu) \;\to\; [\mathbf{L}^p(X,\mu)]^{**}$$

such that

$$i(f)(F) \;=\; F(f) \qquad \text{for all } F \in [\mathbf{L}^p(X,\mu)]^{*}$$

is isometric isomorphism.

7.5 Weak convergence

Given $p \geq 1$, let q is its conjugate, i.e., $1/p + 1/q = 1$.

Definition 7.11 (Weak convergence) *We say that a sequence $(f_n)_{n\geq 1} \subset \mathbf{L}^p(X,\mu)$ converges weakly to $f \in \mathbf{L}^p(X,\mu)$ (denote by $f_n \xrightarrow{\mathbf{L}^p} f$) if*

$$\lim_{n\to\infty} \int_X f_n \varphi \, d\mu \;=\; \int_X f \varphi \, d\mu \qquad \text{for all } \varphi \in \mathbf{L}^q(X,\mu).$$

Example 7.2 *Let f_n be an orthonormal sequence in $\mathbf{L}^2(X,\mu)$. Then, f_n converges weakly to zero.*

Lemma 7.12 (The Riemann-Lebesgue lemma) *For any $f \in L^1(\mathbb{R})$, it holds*

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} f(t)\sin(at)\, dt = 0.$$

Sketch of proof. 1. Suppose f is an indicator function of an interval $[b, c]$. Then, by explicit computation

$$\lim_{a\to\infty} \int_{\mathbb{R}} f(t)\cdot \sin(at)dt \;=\; \lim_{a\to\infty} \frac{1}{a} \cdot (-\cos(ab) + \cos(ac)) \;=\; 0.$$

Given any set E with finite Lebesgue measure, for any $\varepsilon > 0$, there exists a finite collection of find disjoint intervals I_i such that $m\left(E\Delta\left(\bigcup_{j=1}^K I_j\right)\right) \;\leq\; \varepsilon$. This then gives

$$\left|\int \chi_E(t)\sin(at)\, dt\right| \;\leq\; \left|\int (\chi_E - \chi_{\cup I_j})\sin(at)\, dt\right| + \left|\int \chi_{\cup I_j} \sin(at)\, dt\right|.$$

The first term is bounded by ε, while the second goes to zero by the explicit computation above. Hence we have Riemann Lebesgue for indicator functions. By linearity, we also have the Riemann-Lebesgue lemma for simple functions.

2. By linearity, we also have the Riemann-Lebesgue lemma for simple functions. Now let f_n be a sequence of simple functions converging to f in $\mathbf{L}^1$. We then have

$$\left|\int f(t)\sin(at)\,dt\right| \le \left|\int (f(t)-f_n(t))\sin(at)\,dt\right| + \left|\int f_n(t)\sin(at)\,dt\right|.$$

The first term goes to zero in n (uniformly in a), while by what we've already proven we know that the second term goes to zero as $a \to \infty$. This then proves the desired result. □

As a consequence, the following holds:

Example 7.3 *The sequence $f_n = \sin(nx)$ converges weakly to zero in $\mathbf{L}^\infty(\mathbb{R})$.*

Problem 58: Prove that the sequence $f_n(x) = e^{inx}$, $n \in \mathbb{Z}$, is a maximal orthogonal sequence in $\mathbf{L}^2_{\mathbb{C}}((-\pi,\pi))$ (complex-valued, $\mathbf{L}^2$ functions on the interval $(-\pi,\pi)$).

Some basis facts.

(a) *If $(f_n)_{n\ge1}$ converges to f in $\mathbf{L}^p(X,\mu)$ then $(f_n)_{n\ge1}$ converges weakly to f in $\mathbf{L}^p(X,\mu)$.*

Indeed, using Holder's inequality, we have

$$\limsup_{n\to\infty}\left|\int_X (f_n - f)\cdot\varphi\,d\mu\right| \;\le\; \limsup_{n\to\infty}\|f_n - f\|_p\cdot\|\varphi\|_q \;=\; 0$$

for all $\varphi \in \mathbf{L}^q$. Thus,

$$\lim_{n\to\infty}\int_X (f_n - f)\cdot\varphi\,d\mu \;=\; 0.$$

(b) *If $(f_n)_{n\ge1}$ converges weakly to f in $\mathbf{L}^p(X,\mu)$ then*

$$\liminf_{n\to\infty}\;\|f_n\|_p \;\ge\; \|f\|_p.$$

Indeed, using Riesz-representation theorem, we have

$$\begin{aligned}\|f\|_p \;&=\; \|T[f]\|_* \;=\; \sup_{\|\varphi\|_q=1}\left|\int_X f\varphi d\mu\right| \;=\; \sup_{\|\varphi\|_q=1}\left(\lim_{n\to\infty}\left|\int_X f_n\varphi d\mu\right|\right)\\ &\le\; \sup_{\|\varphi\|_q=1}\left(\liminf_{n\to\infty}\|f_n\|_p\|\varphi\|_q\right) \;=\; \liminf_{n\to\infty}\;\|f_n\|_p\,.\end{aligned}$$

(c) *If* $(f_n)_{n\geq 1}$ *converges weakly to* f *in* $\mathbf{L}^p(X,\mu)$ *then the sequence* $(f_n)_{n\geq 1}$ *is bounded.*

Recalling that $\mathbf{L}^p(X,\mu)$ is Banach and reflexive, the Banach Alaoglu's lemma yields the following compactness result.

Theorem 7.13 (Weak compactness in $\mathbf{L}^p$) *If* $1 < p < \infty$ *and* $(f_n)_{n\geq 1}$ *is a bounded sequence in* $\mathbf{L}^p$, *i.e.,*

$$\sup_{n\geq 1} \|f_n\|_p \ \leq\ M < +\infty.$$

Then there exists $f \in \mathbf{L}^p$ *and a subsequence* $(f_{n_k})_{k\geq 1}$ *of* $(f_n)_{n\geq 1}$ *such that* $f_{n_k} \xrightarrow{\mathbf{L}^p} f$.

Remark 7.14 *The weak compactness theorem does not hold when* $p = 1$. *For example, if we consider the sequence in* $\mathbf{L}^1(\mathbb{R})$ *given by* $f_n(x) = n\phi(x/n)$, *where* ϕ *is a non-negative function satisfying* $\int \phi = 1$, *then* f_n *is bounded in* $\mathbf{L}^1$ *but does not converge weakly to anything. In particular, we have, for* $g \in \mathbf{L}^\infty$, *which is also continuous, we get*

$$\int f_n g \to g(0).$$

This cannot be represented as the integral of g *times an* $\mathbf{L}^1$ *function. This indicates that* f_n *actually converges weakly to a measure: namely the Dirac mass centered at zero. Indeed, it is possible to recover a sort of compactness for* $\mathbf{L}^1$ *functions if one is willing to relax to consider measures.*

Let's provide a sufficient condition of convergence in $\mathbf{L}^p$.

Proposition 7.14.1 (Radon-Riesz Theorem) *For* $1 < p < \infty$, *assume that*

$$f_n \xrightarrow{\mathbf{L}^p} f \qquad \text{and} \qquad \|f_n\|_p \to \|f\|.$$

Then, f_n *converges to* f *in* $\mathbf{L}^p$.

Proof. 1. To get an idea, let us start with $p = 2$. In this case, we have

$$\|f_n - f\|_2^2 \ =\ \|f_n\|_2^2 + \|f\|_2^2 - 2\langle f_n, f\rangle,$$

where $\langle f_n, f\rangle = \int f_n f d\mu$ is the $\mathbf{L}^2$ inner product. Taking $n \to \infty$ and using both $f_n \rightharpoonup f$ and $\|f_n\| \to \|f\|$, we obtain that

$$\lim_{n\to\infty} \left(\|f_n\|_2^2 + \|f\|_2^2 - 2\langle f_n, f\rangle\right) \ =\ 2\cdot \|f\|_2^2 - 2\langle f, f\rangle \ =\ 0.$$

2. Two sources for the proof of the case $1 < p < \infty$ are page 78 in Riesz's functional analysis book, or at this link. Two cases are considered:

- If $p \geq 2$ then one can show that

$$|1+t|^p \;\geq\; 1+pt+C_1|t|^p$$

for some constant $C_1>0$. In particular, applying this inequality for $t=\frac{f_n-f}{f}$, we have

$$\left|1+\frac{f_n-f}{f}\right|^p \;\geq\; 1+p\cdot\frac{f_n-f}{f}+C_1\cdot\left|\frac{f_n-f}{f}\right|^p.$$

Thus,

$$|f_n|^p \;\geq\; |f|^p+p(f_n-f)|f|^{p-1}\text{sign}(f)+C_1\cdot|f_n-f|^p$$

and this yields

$$\int_X|f_n|^pd\mu \;\;\geq\;\; \int_X|f|^pd\mu+p\cdot\int_X(f_n-f)\cdot|f|^{p-1}\text{sign}(f)d\mu+C_1\int_X|f_n-f|^pd\mu.$$

Taking $n\to+\infty$, and using both $f_n\rightharpoonup f$ and $\|f_n\|\to\|f\|$, we obtain that

$$\int_X|f|^pd\mu \;\geq\; \int_X|f|^pd\mu+0+C_1\cdot\limsup_{n\to\infty}\int_X|f_n-f|^pd\mu.$$

This implies that

$$\lim_{n\to\infty}\int_X|f_n-f|^pd\mu \;=\; 0.$$

- To complete the proof, we will consider the case $1<p<2$. In this case, we have the inequality (for some $c_1>0$)

$$|1+t|^p-1-pt \;\geq\; \begin{cases} c_1|t|^p & \text{for } |t|>1,\\ \\ c_1t^2 & \text{for } |t|\leq 1.\end{cases}$$

Applying this inequality to $t=\dfrac{f_n-f}{f}$, and then multiplying by $|f|^p$, we obtain

  - If $|f_n-f|<f$ then

$$|f_n|^p \;\geq\; |f|^p+p(f_n-f)|f|^{p-2}f+c_1\cdot|f_n-f|^2|f|^{p-2}$$

  - If $|f_n-f|\geq f$ then

$$|f_n|^p \;\geq\; |f|^p+p(f_n-f)|f|^{p-2}f+c_1\cdot|f_n-f|^p$$

This implies that

$$\int_{|f_n-f|<f}|f_n-f|^pd\mu \;\;\leq\;\; \int_{|f_n-f|<f}|f|^{p-1}|f_n-f|d\mu$$

$$\leq \left(\int_{|f_n-f|<f} |f|^p\right)^{\frac{1}{2}} \cdot \left(\int_{|f_n-f|<f} |f|^{p-2} \cdot |f_n - f|^2\right)^{\frac{1}{2}}$$

$$\leq C_2 \cdot \|f\|_p^{p/2} \cdot \left(\int_{|f_n-f|<f} |f_n|^p - |f|^p - p(f_n - f)|f|^{p-2}f\right)^{\frac{1}{2}}$$

On the set where $|f_n - f| > f$, we obtain the same integral inequality as in the case where $p > 2$, namely

$$\int_{|f_n-f|>f} |f_n|^p d\mu \geq \int_{|f_n-f|>f} |f|^p d\mu + p \cdot \int_{|f_n-f|>f} (f_n - f) \cdot |f|^{p-1}\text{sign}(f) d\mu$$
$$+ c_1 \int_{|f_n-f|>f} |f_n - f|^p d\mu.$$

Hence,

$$\int_X |f_n - f|^p d\mu \leq C_3 \cdot \max_{s\in\{1,1/2\}} \left(\int_X |f_n|^p - |f|^p - p(f_n - f)|f|^{p-2} f d\mu\right)^s$$

and it yields

$$\lim_{n\to\infty} \int_X |f_n - f|^p d\mu = 0.$$

The proof is complete. □

Theorem 7.15 *Let $\Omega \subseteq \mathbb{R}^d$ is open and bounded, with $\mu(\Omega) < \infty$ and $1 < p < \infty$. Assume that $(f_n)_{n\geq 1}$ converges to f almost everywhere in Ω and*

$$\sup_{n\geq 1} \|f_n\|_p < M \qquad \text{for some } M > 0.$$

Then, f_n converges weakly to f in $\mathbf{L}^p(\Omega)$.

Proof. Using Egoroff's theorem, for every $\varepsilon > 0$ there exists $E_\varepsilon \subseteq \Omega$ such that

$$\mu(E_\varepsilon) \leq \varepsilon \qquad \text{and} \qquad \lim_{n\to\infty} \left(\sup_{x\in\Omega\setminus E_\varepsilon} |f_n(x) - f(x)|\right) = 0.$$

For any $g \in \mathbf{L}^q$, we have

$$\int_\Omega (f_n - f) g dx \leq \int_{E_\varepsilon} |f_n - f||g| dx + \int_{\Omega\setminus E_\varepsilon} |f_n - f||g| dx.$$

Since f_n converges uniformly to f in $\Omega\setminus E_\varepsilon$, we have that

$$\sup_{x\in\Omega\setminus E_\varepsilon} |f_n(x) - f(x)| \leq \varepsilon \qquad \text{for all } n \geq N_\varepsilon.$$

Using the Hölder's inequality, we get

$$\begin{aligned}\int_{\Omega\setminus E_\varepsilon} |f_n - f|\cdot|g|d\mu &\le \left(\int_{\Omega\setminus E_\varepsilon} |f_n - f|^p d\mu\right)^{\frac{1}{p}}\cdot\|g\|_q \\ &\le \mu(\Omega)^{1/p}\cdot\|g\|_q\cdot\varepsilon \qquad \text{for all } n\ge N_\varepsilon.\end{aligned}$$

On the other hand, using the Fatou's lemma, we have

$$\|f\|_p \le \liminf_{n\to\infty}\|f_n\|_p \le M$$

and this implies that $\|f_n - f\|_p \le 2M$. Thus,

$$\int_{E_\varepsilon} |f - f_n|\cdot|g|d\mu \le 2M\cdot\left(\int_{E_\varepsilon}|g|^q d\mu\right)^{\frac{1}{q}}.$$

By the continuity property of Lebesgue integral, we have

$$\lim_{\varepsilon\to 0}\int_{E_\varepsilon}|g|^q d\mu = 0.$$

and this yields

$$\lim_{n\to\infty}\int_\Omega (f - f_n)\cdot g d\mu = 0 \qquad \text{for all } g\in\mathbf{L}^q.$$

Therefore, f_n converges weakly to g. □

Problem 59: Suppose that $u_n \in C^1([0,1])$ satisfy $u_n \rightharpoonup u$ and $u_n' \rightharpoonup u'$, both in L^2, and $u\in C^1$. Argue that u_n converges strongly to u in $\mathbf{L}^2$.

Problem 60: Show that if f_n converges strongly to f in $\mathbf{L}^p$ and g_n converges weakly to g in $\mathbf{L}^q$, with p and q Holder conjugate, then

$$\lim_{n\to\infty}\int f_n g_n\, d\mu = \int fg\, d\mu$$

8 Product measures

8.1 Product measures and Fubini's theorem

Given $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ measurable spaces, denote by

$$X\times Y = \{(x,y) : x\in X, y\in Y\}.$$

For any $A\in\mathcal{M}$ and $B\in\mathcal{N}$, a set

$$A\times B \ \textit{is a measurable rectangle.}$$

Let $\mathcal{R}$ be a collection of $E \subset X \times Y$ such that

$$E \;=\; \bigcup_{n=1}^{M} A_n \times B_n, \qquad \begin{cases} A_n \in \mathcal{M}, & B_n \in \mathcal{N}, \\ \\ A_n \times B_n \bigcap A_m \times B_m = \varnothing. \end{cases}$$

One can easily check that $\mathcal{R}$ is an algebra.

Definition 8.1 (Product σ-algebra) *The collection*

$$\mathcal{M} \times \mathcal{N} \;:=\; \sigma(\mathcal{R})$$

is called the product σ-algebra of $\mathcal{M}$ and $\mathcal{N}$

Proposition 8.1.1 *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $E \in \mathcal{M} \times \mathcal{N}$ then*

(i). For any $(x, y) \in X \times Y$, it holds

$$E_x \;=\; \{y \in Y : (x, y) \in E\} \qquad \text{and} \qquad E_y \;=\; \{x \in X : (x, y) \in E\}$$

are measurable.

(ii). The following functions

$$x \;\mapsto \nu(E_x) \qquad \text{and} \qquad y \;\mapsto \mu(E_y)$$

are Borel. Moreover,

$$\int_X \nu(E_x) d\mu \;=\; \int_Y \mu(E_y)\; d\nu.$$

Proof. It is divided into several steps:

(i). Let's consider

$$E \;=\; \bigcup_{i=1}^{N} A_i \times B_i \;\in\; \mathcal{R}\,.$$

For any $(x, y) \in X \times Y$, it holds

$$E_x \;=\; \bigcup_{i=1}^{N} (A_i \times B_i)_x \qquad \text{and} \qquad E_y \;=\; \bigcup_{i=1}^{N} (A_i \times B_i)_y$$

where

$$(A_i \times B_i)_x \;=\; \begin{cases} B_i & \text{if} \quad x \in A_i \\ \\ \varnothing & \text{if} \quad x \notin A_i \end{cases} \qquad \text{and} \quad (A_i \times B_i)_y \;=\; \begin{cases} A_i & \text{if} \quad y \in B_i \\ \\ \varnothing & \text{if} \quad y \notin B_i\,. \end{cases}$$

This implies that $E_x \in \mathcal{N}$ and $E_y \in \mathcal{M}$ and

$$\mathcal{R} \subseteq \mathcal{F} = \{E \in \mathcal{M} \times \mathcal{N} : E_x \in \mathcal{N}, E_y \in \mathcal{M} \quad \text{for all } (x,y) \in X \times Y\}.$$

Since $\mathcal{F}$ is a σ-algebra, one has that $\mathcal{F} = \mathcal{M} \times \mathcal{N}$.

(ii). Assume that μ and ν are finite measure. Set

$$\mathcal{G} = \{E \in \mathcal{M} \times \mathcal{N} : E \text{ satisfies } (ii)\}.$$

we need show that

$$\mathcal{G} = \mathcal{M} \times \mathcal{N}.$$

We first claim that

$$\mathcal{R} \subseteq \mathcal{G}.$$

Indeed, for any set $E = \bigcup_{i=1}^{N} A_i \times B_i \in \mathcal{R}$, it holds that the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E_y)$ defined by

$$\nu(E_x) = \nu\left(\bigcup_{i=1}^{N} (A_i \times B_i)_x\right) = \sum_{i=1}^{N} \nu(B_i) \cdot \chi_{A_i}(x)$$

and

$$\mu(E_y) = \mu\left(\bigcup_{i=1}^{N} (A_i \times B_i)_y\right) = \sum_{i=1}^{N} \mu(A_i) \cdot \chi_{B_i}(y)$$

are Borel and satisfies

$$\int_X \nu(E_x) d\mu = \sum_{i=1}^{N} \nu(B_i) \cdot \mu(A_i) = \int_Y \mu(E_y) d\nu.$$

By Halmos's theorem, if $\mathcal{G}$ is monotone, i.e., for any monotone sequence of sets $(E_n)_{n\geq 1} \subseteq \mathcal{G}$ converges to E, it holds that $E \in \mathcal{G}$. Then,

$$\sigma(\mathcal{R}) \subseteq \mathcal{G} \qquad \Longrightarrow \qquad \mathcal{M} \times \mathcal{N} = \mathcal{G}.$$

To complete this part, we show that $\mathcal{G}$ is monotone. Two cases are considered:

- If $(E^n)_{n\geq 1} \subseteq \mathcal{G}$ is increasing and converges to E then both $(E^n_x)_{n\geq 1}$ and $(E^n_y)_{n\geq 1}$ are increasing and converge to E_x and E_y respectively. This implies that $\mu(E^n_y)$ and $\nu(E^n_x)$ are also increasing and

$$\lim_{n\to\infty} \mu(E^n_y) = \mu(E_y), \qquad \lim_{n\to\infty} \nu(E^n_x) = \nu(E_x). \tag{8.1}$$

  In particular, the functions

$$x \mapsto \nu(E_x) \qquad \text{and} \qquad y \mapsto \mu(E_y)$$

are Borel. Using the monotone convergent theorem, we obtain that

$$\int_X \nu(E_x)\, d\mu \;=\; \lim_{n\to\infty} \int_X \nu(E^n_x) d\mu \;=\; \lim_{n\to\infty} \int_Y \nu(E^n_y) d\nu \;=\; \int_Y E_y d\nu$$

and E is in $\mathcal{G}$.

- With a similar argument, we can show that if $(E^n)_{n\geq 1} \subseteq \mathcal{G}$ is decreasing and converges to E. In this case, we need to use the finite property of μ and ν to verify (8.1).

(iii). Assume that μ and ν are σ-finite. Then there exists increasing sequence of sets $(X_n)_n \geq 1$ and $(Y_n)_{n\geq 1}$ such that

$$X \;=\; \bigcup_{n=1}^{\infty} X_n, \quad Y \;=\; \bigcup_{n=1}^{\infty} Y_n \quad \text{and} \quad \mu(X_n), \nu(Y_n) \;<\; +\infty \quad \text{for all } n \geq 1.$$

Denote by

$$\mu_n \;=\; \mu_{|X_n} \qquad \text{and} \qquad \nu_n \;=\; \nu_{|Y_n}.$$

For any $E \in \mathcal{M} \times \mathcal{N}$, we have that

$$\nu_n(E_x) \;=\; \nu(E_x \cap Y_n) \;\;\uparrow\to\;\; \nu(E_x)$$

and

$$\mu_n(E_y) \;=\; \mu(E_y \cap X_n) \;\;\uparrow\to\;\; \mu(E_y).$$

Thus, the functions

$$x \;\mapsto \nu(E_x) \qquad \text{and} \qquad y \;\mapsto \mu(E_y)$$

are Borel. Again, using the monotone convergent theorem, we get

$$\begin{aligned} \int_X \nu(E_x) d\mu \;&=\; \lim_{n\to\infty} \int_X \nu_n(E_x)\, d\mu \;=\; \lim_{n\to\infty} \int_{X_n} \nu_n(E_x) d\mu_n \\ &=\; \lim_{n\to\infty} \int_{Y_n} \mu_n(E_y) d\nu_n \;=\; \lim_{n\to\infty} \int_X \mu_n(E_y)\, d\nu \;=\; \int_Y \mu(E_y) d\nu \end{aligned}$$

and the proof is complete. □

Theorem 8.2 *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. The map $\mu \otimes \nu : \mathcal{M} \times \mathcal{N} \to [0, +\infty]$ such that*

$$(\mu \otimes \nu)(E) \;=\; \int_X \nu(E_x) d\mu \;=\; \int_Y \mu(E_y) d\nu \quad \textit{for all } E \in \mathcal{M} \times \mathcal{N}$$

is an σ-finite measure. Moreover, $\mu \otimes \nu$ is a unique measure satisfied

$$(\mu \otimes \nu)(A \times B) \;=\; \mu(A) \cdot \nu(B)$$

for all $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

Proof. Let's show that $\mu \otimes \nu$ is σ-additive. For any mutually disjoint sequence of sets $(E^n)_{n\geq 1} \subseteq \mathcal{M} \times \mathcal{N}$, we have

$$\begin{aligned}(\mu \otimes \nu)\left(\bigcup_{n\geq 1} E^n\right) &= \int_X \nu\left(\left[\bigcup_{n=1}^{\infty} E^n\right]_x\right) d\mu = \int_X \sum_{n=1}^{\infty} \nu((E^n_x) d\mu \\ &= \sum_{n=1}^{\infty} \int_X \nu(E^n_x) d\mu = \sum_{n=1}^{\infty} (\mu \otimes \nu)(E_n) .\end{aligned}$$

Thus, $\mu \otimes \nu$ is a measure.

To show that $\mu \otimes \nu$ is σ-finite. Let $(X_n)_n \geq 1$ and $(Y_n)_{n\geq 1}$ be such that

$$X = \bigcup_{n=1}^{\infty} X_n, \qquad Y = \bigcup_{n=1}^{\infty} Y_n \qquad \text{and} \qquad X_n, Y_n < +\infty \qquad \text{for all } n \geq 1.$$

Denote by $Z_n := X_n \times Y_n$, we have

$$X \times Y = \bigcup_{n=1}^{\infty} Z_n \qquad \text{and} \qquad \mu \otimes \nu(Z_n) = \mu(X_n) \times \nu(Y_n) < +\infty$$

and the proof is complete. □

Corollary 8.3 *Let $E \in \mathcal{M} \times \mathcal{N}$ be such that*

$$(\mu \otimes \nu)(E) = 0.$$

Then it holds

$$\begin{cases} \mu(E_y) = 0 & \nu \text{ a.e. } \; y \in Y \\ \nu(E_x) = 0 & \mu \text{ a.e. } \; x \in X. \end{cases}$$

Theorem 8.4 (Tonelli's theorem) *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to [0, \infty]$ be measurable (on the product space). Then all of the following integrals are well-defined and the following equalities hold:*

$$\int_{X\times Y} f(x, y) d(\mu \otimes \nu) = \int_X \int_Y f(x, y) d\nu d\mu = \int_Y \int_X f(x, y) d\mu d\nu. \tag{8.2}$$

Proof. Assume that $f = \chi_E$ for $E \in \mathcal{M} \times \mathcal{N}$. By the previous theorem, we have

$$\begin{aligned}\int_{X\times Y} f(x, y) d(\mu \otimes \nu) &= (\mu \otimes \nu)(E) = \int_X \nu(E_x) d\mu = \int_Y \mu(E_y) d\nu \\ &= \int_X \int_Y f(x, y) d\nu d\mu = \int_Y \int_X f(x, y) d\mu d\nu.\end{aligned}$$

Thus, (8.2) holds for simple functions. For a general f, we can approximate f by an increasing of simple sequence $f_n : X \times Y \to [0. + \infty]$ such that

$$\lim_{n\to\infty} f_n(x,y) = f(x,y) \qquad \text{for all } (x,y) \in X \times Y.$$

By the monotone converging theorem, it holds that

$$\int_{X\times Y} f(x,y)d(\mu\otimes\nu) = \lim_{n\to\infty}\int_{X\times Y} f_n(x,y)d(\mu\otimes\nu).$$

On the other hand, for any $x \in X$, the sequence $f_n(x,\cdot)$ is increasing and converges pointwise to $f(x,\cdot)$. Thus, $y \mapsto f(x,y)$ is Borel and $x \mapsto \int_Y f_n(x,y)d\nu$ is an increasing sequence of Borel functions such that

$$\lim_{n\to\infty}\int_Y f_n(x,y)d\nu = \int_Y f(x,y)d\nu \qquad \text{for all } x \in X.$$

Again, the monotone converging theorem, one has

$$\lim_{n\to\infty}\int_X\int_Y f_n(x,y)d\nu d\mu = \int_X\int_Y f(x,y)d\nu d\mu$$

and this yields

$$\int_{X\times Y} f(x,y)d(\mu\otimes\nu) = \int_X\int_Y f(x,y)d\nu d\mu.$$

Similarly, we can also show that

$$\int_{X\times Y} f(x,y)d(\mu\otimes\nu) = \int_Y\int_X f(x,y)d\mu d\nu.$$

The proof is complete. □

Using the above theorem and decompose the function $f = f^+ - f^-$ with $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$, we can prove the following theorem.

Theorem 8.5 (Fubini's theorem) *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. A measurable function*

$$f : X \times Y \to \overline{\mathbb{R}}$$

is integrable if

$$\min\left\{\int_Y\left(\int_X |f^y|d\mu\right)d\nu, \int_X\left(\int_Y |f^x|d\nu\right)d\mu\right\} < \infty$$

where $f^y(x) = f^x(y) = f(x,y)$. In this case,

$$\int_{X\times Y} fd\,(\mu\otimes\nu) = \int_Y\int_X f^y d\mu d\nu = \int_X\int_Y f^x d\nu d\mu.$$

Proof. I would leave it to students to read. □

Example 8.1 *Consider the space $X = Y = [0,1]$, where X is equipped with the Lebesgue measure m and Y is equipped with the counting measure μ. Consider the function $f(x,y) = \chi_{x=y}$. Then*

$$\int_X \int_Y f(x,y)\, d\mu(y)\, dm(x) = \int_X 1\, dm(x) = 1 \neq 0 = \int_Y 0\, d\mu(y) = \int_Y \int_X f(x,y)\, dm(x)\, d\mu(y).$$

This demonstrates why one needs σ-finite measure spaces in order to have a Fubini theorem (indeed μ is not σ-finite).

Example 8.2 *Consider the function*

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2},$$

on the set $E = [0,1] \times [0,1]$. The integrand is precisely $\partial_x \partial_y tan^{-1}(y/x)$. Hence one can integrate explictly, and one gets that

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = -\frac{\pi}{4} \neq \frac{\pi}{4} = \int_0^1 \int_0^1 f(x,y)\, dy\, dx.$$

Hence the assumption on integrability is essential in Fubini's theorem.

Example 8.3 *Consider $X = Y = \mathbb{N}$ both equipped with the counting measure μ. Consider the function $f(i,j) = \chi_{i=j} - \chi_{i=j+1}$. Then*

$$\int_X \int_Y f d\mu d\mu = \sum_i \sum_j f(i,j) = 1 \neq 0 = \sum_j \sum_i f(i,j) = \int_Y \int_X f d\mu d\mu.$$

This example again reiterates the need for absolute integrability in Fubini's theorem.

8.2 Convolution and Approximation

Let f and g be Borel function from $\mathbb{R}^d$ to $\overline{\mathbb{R}}$. Assume that the map $y \mapsto f(x-y)\cdot g(y)$ is in $\mathbf{L}^1(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$. The convolution product $f * g$ is defined by

$$(f*g)(x) \;=\; \int_{\mathbb{R}^d} f(x-y)\cdot g(y)\; dy \qquad \text{for all } x \in \mathbb{R}^d.$$

The following holds:

Lemma 8.6 *Let p and q be conjugated. For any $f \in \mathbf{L}^p(\mathbb{R}^d)$ and $g \in \mathbf{L}^q(\mathbb{R}^d)$, it holds that $f * g$ is continuous and*

$$\|f*g\|_\infty \;\leq\; \|f\|_p \cdot \|g\|_q.$$

Proof. For any $x \in \mathbb{R}^d$, we have that

$$|f * g(x)| \ = \ \left|\int_{\mathbb{R}^d} f(x-y)\cdot g(y)\ dy\right| \ \le\ \|f(x-\cdot)\|_p \cdot \|g\|_q \ = \ \|f\|_p \cdot \|g\|_q$$

and it yields

$$\|f * g\|_\infty \ \le\ \|f\|_p \cdot \|g\|_q < \infty.$$

To show that $f * g$ is continuous, we estimate

$$\begin{aligned} |(f*g)(x+h) - (f*g)(x)| \ &\le\ \int_{\mathbb{R}^d} |f(x+h-y) - f(x-y)| \cdot g(y)\ dy \\ &\le\ \|f(x+h-\cdot) - f(x-\cdot)\|_p \cdot \|g\|_q \end{aligned}$$

The translation continuity in $\mathbf{L}^p$ implies that

$$\limsup_{|h|\to 0} |(f*g)(x+h) - (f*g)(x)| \ \le\ \|g\|_q \cdot \lim_{|h|\to 0} \|f(x+h-\cdot) - f(x-\cdot)\|_p \ = \ 0$$

and it yields the continuity of $f * g$. □

Some basic properties:

(a) $(f * g)(x) = (g * f)(x)$ for all $x \in \mathbb{R}^d$.

(b) Let $1 \le p, q, r \le +\infty$ be such that

$$\frac{1}{p} + \frac{1}{q} \ = \ 1 + \frac{1}{r}.$$

For any $f \in \mathbf{L}^p$ and $g \in \mathbf{L}^q$, one has that

$$\|f * g\|_r \ \le\ \|f\|_p \cdot \|g\|_q.$$

In particular, if $f \in L^1$ and $g \in \mathbf{L}^q$ then

$$\|f * g\|_q \ \le\ \|f\|_1 \cdot \|g\|_q\,.$$

(c) If f, g and h are in $\mathbf{L}^1$ then (by Fubini's theorem) it holds

$$(f * g) * h \ = \ f * (g * h)\,.$$

Proposition 8.6.1 (Smoothness) *Let $f \in \mathbf{L}^p(\mathbb{R}^d)$ and $K \in C^m_c(\mathbb{R}^d)$. Then the function $f * K$ is in $C^m(\mathbb{R}^d)$ and*

$$D^\alpha(f * K) \ = \ f * D^\alpha K$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ with $\sum_{i=1}^{d} \alpha_i \le m$ and

$$D^\alpha g \ = \ \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial^{\alpha_d} x_d}.$$

Idea of proof. We have that

$$(f * K)(x) \;=\; \int_{\mathbb{R}^d} f(x-y)\cdot K(y)dy \;=\; \int_{\mathbb{R}^d} f(y)\cdot K(x-y)dy.$$

For any $i \in \overline{1,d}$, one has

$$\frac{\partial (f*K)}{\partial x_i}(x) \;=\; \int_{\mathbb{R}^d} f(y)\cdot \frac{\partial K(x-y)}{\partial x_i}\, dy.$$

Since $f \in \mathbf{L}^p$ and $\dfrac{\partial K(x-y)}{\partial x_i} \in \mathbf{L}^q$, the map $\dfrac{\partial (f*K)}{\partial x_i}(x)$ is continuous and this implies that $f * K$ is in C^1. By induction, one can show that $f*K$ is in $C^m(\mathbb{R}^d)$ $\square$

Approximation by smooth functions. Consider $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq \mathbf{L}^1(\mathbb{R}^d)$ such that

(i). $\varphi_\varepsilon \geq 0$ and $\displaystyle\int_{\mathbb{R}^d} \varphi_\varepsilon(x)dx = 1$ for all $\varepsilon > 0$;

(ii) For every $\delta > 0$, it holds

$$\lim_{\varepsilon\to 0} \int_{\|x\|\geq\delta} \varphi_\varepsilon(x)dx \;=\; 0.$$

Proposition 8.6.2 *The followings holds*

(i) If $f \in \mathbf{L}^\infty$ is continuous at x_0 then

$$\lim_{\varepsilon\to 0}\; (f*\varphi_\varepsilon)(x_0) \;=\; f(x_0).$$

In addition, if $f \in \mathbf{L}^\infty$ is uniformly continuous then

$$\lim_{\varepsilon\to 0} \|f*\varphi_\varepsilon - f\|_\infty \;=\; 0.$$

(ii) If $f \in \mathbf{L}^p$ then

$$\lim_{\varepsilon\to 0} \|f*\varphi_\varepsilon - f\|_p \;=\; 0\,.$$

Proposition 8.6.3 *Let $\Omega \subseteq \mathbb{R}^d$ be open. Then $C_c^\infty(\Omega)$ is dense in $\mathbf{L}^p(\Omega)$ for every $1 \leq p < +\infty$.*

Proof. Since $C_c(\Omega)$ is dense in $\mathbf{L}^p$, we only need to prove that

$$C_c(\Omega) \;\subseteq\; \overline{C_c^\infty(\Omega)}.$$

Given any $f \in C_c(\Omega)$, we need to construct $\{f_m\}_{m\geq 1} \subseteq C_c^\infty(\Omega)$ such that f_m converges to f in $\mathbf{L}^p$.

In order to do so, we extend f to $\mathbb{R}^d$ such that

$$f(x) \;=\; \begin{cases} f(x) & x \in \Omega \\ \\ 0 & x \in \mathbb{R}^d \backslash \Omega. \end{cases}$$

Introduce a standard modifier

$$\rho_\varepsilon(x) \;=\; \begin{cases} C \cdot \varepsilon^{-d} \cdot e^{\frac{\varepsilon^2}{|x|^2-\varepsilon^2}} & |x| < \varepsilon \\ \\ 0 & |x| \geq \varepsilon \end{cases}$$

with

$$C \;=\; \frac{1}{\displaystyle\int_{|x|\leq 1} e^{\frac{1}{|x|^2-1}} dx}.$$

The function ρ_ε satisfies the following properties

(i) $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ $\quad$ $supp(\rho_\varepsilon) \;\subseteq\; \overline{B}(0,\varepsilon)$;

(ii) $\rho_\varepsilon \geq 0$ and $\displaystyle\int_{\mathbb{R}^d} \rho_\varepsilon(x) dx = 1$;

(iii) For every $\varepsilon > 0$, one has

$$\lim_{\varepsilon \to \infty} \int_{\|x\| \geq \delta} \rho_\varepsilon(x) dx \;=\; 0.$$

For any $m \geq 1$, denote by

$$f_m(x) \;:=\; f * \rho_{\frac{1}{m}}(x) \;=\; \int_{\mathbb{R}^d} f(x-y) \cdot \rho_{\frac{1}{m}}(y) dy$$

From the previous proposition, it holds

$$f_m \;\in\; C^\infty(\Omega) \qquad \text{and} \qquad supp(f_m) \;\subseteq\; supp(f) + supp\left(\rho_{\frac{1}{m}}\right) \;\subseteq\; \tilde{K} \subseteq \Omega$$

for some compact set $\tilde{K}$ and $m \geq 1$ sufficiently large. On the other hand, since f is uniformly continuous, we have that

$$\lim_{m\to\infty} \|f_m - f\|_\infty \;=\; \lim_{m\to\infty} \|f * \rho_{1/m} - f\|_\infty \;=\; 0.$$

This implies that for $m \geq 1$ sufficiently large it holds

$$\int_\Omega |f_m - f|^p dx \;=\; \int_{\tilde{K}} |f_m - f|^p dx \;\leq\; |m(\tilde{K})| \cdot \|f_m - f\|_\infty^p.$$

Thus, f_m converges to f in $\mathbf{L}^p$. $\square$

Recalling the Weierstrass theorem, for all $f \in C_c(\mathbb{R}^d)$, there exists a sequence of polynomials $(P_k)_{k\geq 1}$ which converges to f uniformly on all compact subsets of $\mathbb{R}^d$, we obtain the following.

Corollary 8.7 *Given any $X \in \mathcal{B}(\mathbb{R}^d)$ bounded, define*

$$P_X = \{P : X \to \mathbb{R} : P \text{ is a polynomial}\}.$$

Then, the set P_X is dense in $\mathbf{L}^p$ for all $1 \leq p < \infty$.

9 A tour of Calculus topics for measure theory

Here we're going to review a few ideas from calculus in the context of measure theory. This is meant to expose us to ideas and techniques, rather than to develop anything fully. The proofs are small modifications of Giovanni Leoni's lecture notes on measure theory.

The first goal will be to create an analog of the formula

$$\frac{d}{dt}\int_0^t f(s)\,ds = f(t),$$

which holds in some measure theoretic sense. The appropriate version is as follows:

Theorem 9.1 (Lebesgue differentiation) *Let μ be a Radon measure on $\mathbb{R}^d$ (over the Borel sets), and let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function, i.e.,*

$$\int_K |f(x)| d\mu(x) < +\infty \qquad \text{for all } K \text{ compact.}$$

Then there exists a Borel set E with $\mu(E) = 0$ so that for any $x \in E^c$ we have

$$\lim_{r\to 0^+} \frac{1}{\mu(\overline{B(x,r)})}\int_{\overline{B(x,r)}} |f(y) - f(x)|\,d\mu(y) = 0.$$

Any point where the above limit is zero is known (when μ is the Lebesgue measure) as a *Lebesgue point* of f. Essentially f is locally well-behaved, in a very controlled sense, near Lebesgue points. The previous theorem is sometimes stated as "almost every point is a Lebesgue point of f". Any Lebesgue point will satisfy

$$f(x) = \lim_{r\to 0^+} \frac{1}{\mu(\overline{B(x,r)})}\int_{\overline{B(x,r)}} f(y)\,d\mu(y).$$

This can be interpreted as saying that locally integrable functions actually are equal to local averages at most places.

The proof of this theorem requires some machinery. In particular, it uses the (Hardy-Littlewood) maximal function:

Definition 9.2 (Maximal function) *Given a locally integrable function f, the maximal function $M(f)$ of f is given by*

$$M(f)(x) := \begin{cases} 0 & \text{if } \mu(B(x,r)) = 0 \text{ for some } r > 0, \\ \sup_{r>0} \frac{1}{\mu(\overline{B(x,r)})} \int_{\overline{B(x,r)}} |f| \, d\mu & \text{otherwise.} \end{cases}$$

Maximal functions are important technical tools in several topics, in particular in the study of Fourier analysis and singular integrals (which are important for classical solution formulas for differential equations).

Theorem 9.3 *Let μ be a Radon measure over the Borel sets in $\mathbb{R}^d$. Then*

1. *If $f \in \mathbf{L}^p$, $1 < p \leq \infty$, then*

$$\|M(f)\|_p \leq C(d,p)\|f\|_p$$

2. *If $f \in \mathbf{L}^1$ then for $t > 0$,*

$$\mu(\{x : M(f) > t\}) \leq \frac{C(d)}{t} \int_{\mathbb{R}^d} |f| \, d\mu.$$

Note that $M(f)$ will not necessarily be in $\mathbf{L}^1$ if $f \in \mathbf{L}^1$. The inequality that the theorem gives is known as a *weak* $\mathbf{L}^1$ *inequality.*

Proof of Lebesgue differentiation theorem ($\mathbf{L}^1$ case): By the density theorem, we can approximate f be a function $g_\varepsilon \in \mathcal{C}_c(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |f(x) - g_\varepsilon(x)| d\mu \ \leq \ \varepsilon.$$

Since it is continuous on a compact set, g_ε is uniformly continuous, i.e., $\eta > 0$ there exists a $\delta > 0$ so that

$$|g_\varepsilon(x) - g_\varepsilon(y)| \ \leq \ \eta \qquad \text{for all } |x - y| < \delta.$$

Hence, for any $0 < r < \delta$, we have

$$\frac{1}{\mu(\overline{B(x,r)})} \int_{\overline{B(x,r)}} |g_\varepsilon(x) - g_\varepsilon(y)| \, d\mu(y) \leq \frac{1}{\mu(\overline{B(x,r)})} \int_{\overline{B(x,r)}} \eta \, d\mu(y) = \eta.$$

and this implies that

$$\lim_{r \to 0^+} \frac{1}{\mu(\overline{B(x,r)})} \int_{\overline{B(x,r)}} |g_\varepsilon(x) - g_\varepsilon(y)| \, d\mu(y) = 0.$$

In turn, we may write

$$\begin{aligned}\limsup_{r\to 0^+}\frac{1}{\mu(\overline{B(x,r)})}\int_{\overline{B(x,r)}}|f(y)-f(x)|\,d\mu(y)\\ \leq\ \limsup_{r\to 0^+}\frac{1}{\mu(\overline{B(x,r)})}\int_{\overline{B(x,r)}}|f(y)-g_\varepsilon(y)|\,d\mu(y)+|g_\varepsilon(x)-f(x)|\\ \leq\ M(f-g_\varepsilon)(x)+|g_\varepsilon(x)-f(x)|.\end{aligned}$$

Thus, if we set

$$G_t\ :=\ \left\{x\in\mathbb{R}^d:\limsup_{r\to 0^+}\frac{1}{\mu(\overline{B(x,r)})}\int_{\overline{B(x,r)}}|f(y)-f(x)|\,d\mu(y)>t\right\}$$

and

$$E_{t,\varepsilon}\ :=\ \{x:M(f-g_\varepsilon)(x)>t\}\,,\quad F_{t,\varepsilon}\ :=\ \{x:|g_\varepsilon(x)-f(x)|>t\}\,,$$

then

$$G_{2t}\ \subset\ E_{t,\varepsilon}\cup F_{t,\varepsilon}\qquad\text{for all } t>0.$$

Using the weak $\mathbf{L}^1$ inequality for maximal functions and Markov's inequality, we get

$$\mu(E_{t,\varepsilon})\leq\frac{C(n)\varepsilon}{t},\qquad \mu(F_{t,\varepsilon})\leq\frac{\varepsilon}{t}.$$

and this implies that

$$\mu(G_{2t})\leq\frac{C(d)+1}{t}\cdot\varepsilon.$$

Taking $\varepsilon\to 0$ gives that $\mu(G_{2t})=0$. Finally, let $E=\bigcup_n G_{1/n}$. Then $\mu(E)=0$, and furthermore for any $x\in E^c$ and any n we have

$$\limsup_{r\to 0^+}\frac{1}{\mu(\overline{B(x,r)})}\int_{\overline{B(x,r)}}|f(y)-f(x)|\,d\mu(y)\leq\frac{1}{n}.$$

This concludes the proof. □

Corollary 9.4 *For a continuous function, every point is a Lebesgue point.*

Example 9.1 *Consider the space $X=[-1,1]$ equipped with the Lebesgue measure, and let $f=\chi_{x>0}$. Then any point except $x=0$ is a Lebesgue point (as f is continuous at all other points). However, f is not a Lebesgue point. Indeed, we may compute*

$$\frac{1}{2t}\int_{-t}^{t}|\chi_{x>0}-a|\,dx=|a|+|1-a|\neq 0.$$

This means that even if we redefine f however we like at the point zero, it cannot be a Lebesgue point of f.

Example 9.2 *Consider a measure $\nu \ll \mu$, with μ being a Radon measure on $\mathbb{R}^n$. Then by the Radon-Nikodym theorem, letting $f = \frac{d\nu}{d\mu}$*

$$\nu(E) = \int_E f(x) d\mu(x).$$

Now if we let $E = \overline{B(x,r)}$, and take a limit as $r \to 0^+$, by the Lebesgue differentiation theorem for μ almost every x

$$f(x) = \lim_{r \to 0^+} \frac{1}{\mu(\overline{B(x,r)})} \int_{\overline{B(x,r)}} f(y)\, d\mu(y) = \lim_{r \to 0^+} \frac{\nu(\overline{B(x,r)})}{\mu(\overline{B(x,r)})}$$

This gives a formula for computing the Radon-Nikodym derivative.

One crucial concept in multi-variable calculus is the change of variables formula. This formula continues to hold under Lebesgue integration. We begin by stating (without proof) a lemma:

Lemma 9.5 *Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map. Then for every Lebesgue measurable set E, we have that $L(E)$ is Lebesgue measurable and that*

$$\mathcal{L}^d(L(E)) = |det(L)| \mathcal{L}^d(E).$$

Proof. The proof relies on reducing to the case of elementary matrix transformations, and using the fact that the Lebesgue measure is (up to a constant) the only translation invariant measure on $\mathbb{R}^d$. □

We now state the change of variables formula:

Theorem 9.6 (The change of variables) *Let $U, V \subset \mathbb{R}^d$ be open sets, and $\Psi : U \to V$ be invertible such that both Ψ and Ψ^{-1} are differentiable. Then for any integrable function f*

$$\int_V f(y)\, dy = \int_U f(\Psi(x)) |\det \nabla \Psi(x)|\, dx.$$

Proof. We define a measure $\mu : \mathcal{M} \to [0, +\infty]$ such that

$$\mu(E) \;=\; \mathcal{L}^d(\Psi(E)) \qquad \text{for all } E \in \mathcal{M}.$$

Using the smoothness of Ψ, one can show that μ is a Radon measure, which is absolutely continuous with respect to the Lebesgue measure. Thus, we can write

$$\mu(E) = \int_E \frac{d\mu}{d\mathcal{L}^d}(x)\, dx \qquad \text{for all } E \in \mathcal{M}.$$

Now, given a Lebesgue measurable set H, the set $E = \Psi^{-1}(H)$ is also Lebesgue measurable, and so we can write

$$\int_{\Psi(U)} \chi_H d\mathcal{L}^d = \mathcal{L}^d(H) = \mathcal{L}^d(\Psi(E)) = \int_E \frac{d\mu}{d\mathcal{L}^d}(x)\,dx = \int_U \chi_H(\Psi(x))\frac{d\mu}{d\mathcal{L}^d}(x)\,dx.$$

By the linearity, the identity will also hold for simple functions, and hence using the monotone convergence theorem it will also hold for non-negative functions. In turn for any integrable function we have

$$\int_V f(y)\,dy = \int_U f(\Psi(x))\frac{d\mu}{d\mathcal{L}^d}(x)\,dx.$$

By the Lebesgue differentiation theorem, we have that

$$\frac{d\mu}{d\mathcal{L}^d}(x) = \lim_{r\to 0^+} \frac{\mu(\overline{B(x,r)})}{\mathcal{L}^d(\overline{B(x,r)})} \qquad \mathcal{L}^d \text{ a.e. } x.$$

On the other hand, by the differentiability of Ψ, for any given $\varepsilon > 0$, there exists $r > 0$ small enough such that

$$-\Psi(x) + \Psi(\overline{B(x,r)}) \;\subset\; \nabla\Psi(x)(\overline{B(0, r(1+\varepsilon))}).$$

In turn

$$\mu(\overline{B(x,r)}) \;=\; \mathcal{L}^d(\Psi(\overline{B(x,r)})) \;\leq\; \mathcal{L}^d(\nabla\Psi(x)(\overline{B(0, r(1+\varepsilon))})).$$

Thus, from the previous Lemma, we get

$$\mu(\overline{B(x,r)}) \leq |\det(\nabla\Psi(x))|(1+\varepsilon)^d \mathcal{L}^d(\overline{B(x,r)})$$

Taking r to zero and the ε to zero, we find that

$$\frac{d\mu}{d\mathcal{L}^d}(x) \leq |\det(\nabla\Psi(x))|.$$

This establishes the change of variables formula as an inequality. Switching U, V and Ψ, Ψ^{-1}, one can then prove the opposite inequality, which concludes the proof. □

The coarea formula can be seen as a specialized sort of change of variables:

Theorem 9.7 *Let u be Lipschitz and $g \in \mathbf{L}^1$, both on $\Omega \subset \mathbb{R}^d$ open. Then*

$$\int_\Omega g(x)|\nabla u(x)|\,dx = \int_{\mathbb{R}} \int_{u^{-1}(t)} g(x)\,d\mathcal{H}^{d-1}(x)\,dt.$$

9.1 Weak derivatives

For a smooth function f and a function $\phi \in C_c^\infty$, we have the identity

$$\int_{\mathbb{R}^d} f \partial_{x_i} \phi \, dx = -\int \partial_{x_i} f \phi \, dx.$$

The first of these integrals is well-defined for any integrable function f. Hence we can define a *weak derivative* of a function f in the following way:

Definition 9.8 *We say that an integrable function v is the weak derivative of f (in the x_i direction) if*

$$\int_{\mathbb{R}^d} f \partial_{x_i} \phi \, dx = -\int v \phi \, dx$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Often, we will just write $v = \partial_{x_i} f$. This function is *not* a classical derivative, it's only a derivative in an integrated sense. However, it can be associated with the classical derivative in an almost everywhere sense. Similarly, we can say that

- a measure μ_{x_i} is *a derivative of f in the sense of distributions* if

  $$\int_{\mathbb{R}^n} f \partial_{x_i} \phi \, dx = -\int \phi \, d\mu_{x_i}$$

  for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Using these, we can then define Sobolev spaces and BV spaces:

Definition 9.9 *The Sobolev space $\mathbf{W}^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, is the space such that f and its weak first partial derivatives (with respect to all of the inputs x_i) are all $\mathbf{L}^p$ functions. $\mathbf{W}^{1,p}$ is a Banach space (after considering equivalence classes) under the norm*

$$\|f\|_{1,p} = \|f\|_p + \sum_{i=1}^d \|\partial_i f\|_p.$$

Definition: The space $BV(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, of functions of bounded variation is the space of $\mathbf{L}^1$ functions which whose first partial derivatives, in the sense of distributions, all have finite total variation. The BV norm is given by

$$\|f\|_{BV} = \|f\|_1 + \sum_{i=1}^d |\mu_{x_i}|(\Omega).$$

These spaces show up in many applications. One very convenient property about these spaces is their compactness:

Theorem 9.10 *Suppose, for $1 < p < \infty$, that u_n is bounded in $W^{1,p}$. Then it is compact in $\mathbf{L}^p$, and the limit point lies in $\mathbf{W}^{1,p}$. Similarly, if u_n is bounded in BV then it is compact in $\mathbf{L}^1$ with limit point in BV.*

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