SBV regularity for Burgers-Poisson equation

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Abstract

The SBV regularity of weak entropy solutions to the Burgers-Poisson equation for initial data in $\mathbf{L}^1(\mathbb{R})$ is considered. We show that the derivative of a solution consists of only the absolutely continuous part and the jump part.

Keywords: Burgers-Poisson equation, entropy weak solution, SBV regularity

1 General setting

The Burgers-Poisson equation is given by the balance law obtained from Burgers' equation by adding a nonlocal source term

$$u_t + \left(\frac{u^2}{2}\right)_x = [G * u]_x \,. \tag{1.1}$$

Here, $G(x) = -\frac{1}{2}e^{-|x|}$ is the Poisson Kernel such that

$$[G * f](x) \;=\; \int_{-\infty}^{+\infty} G(x-y)\cdot f(y)\; dy$$

solves the Poisson equation

$$\varphi_{xx} - \varphi \;=\; f\,. \tag{1.2}$$

Equation (1.1) has been derived in [16] as a simplified model of shallow water waves and admits conservation of both momentum and energy. For sufficiently regular initial data u_0, the local existence and uniqueness of solutions of (1.1) has been established in [9]. Additionally, their analysis of traveling waves showed that the equation features wave breaking in finite time. More generally, it has been demonstrated that (1.1) does not admit a global smooth solution ([12]). Hence, it is natural to consider entropy weak solutions.

Definition 1.1. A function $u \in \mathbf{L}^1_{\mathrm{loc}}([0,\infty[\times\mathbb{R}) \cap \mathbf{L}^\infty_{\mathrm{loc}}(]0,\infty[, \mathbf{L}^\infty(\mathbb{R}))$ is an *entropy weak solution* of (1.1) if u satisfies the following properties:

(i) the map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^1(\mathbb{R})$, i.e.,

$$\|u(t,\cdot) - u(s,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq L \cdot |t-s| \qquad \text{for all } \ 0 \leq s \leq t$$

for some constant $L > 0$.

(ii) For any $k \in \mathbb{R}$ and any non-negative test function $\phi \in C^1_c(]0,\infty[\times\mathbb{R},\mathbb{R})$ one has

$$\int\int \Big[|u-k|\phi_t + \text{sign}(u-k)\Big(\frac{u^2}{2} - \frac{k^2}{2}\Big)\phi_x + \text{sign}(u-k)[G_x * u(t,\cdot)](x)\phi\Big]\, dx\, dt \ \geq \ 0\,.$$

Based on the vanishing viscosity approach, the existence result for a global weak solution was provided for $u_0 \in BV(\mathbb{R})$ in [9]. However, this approach cannot be applied to the more general case with initial data in $\mathbf{L}^1(\mathbb{R})$. Moreover, there are no uniqueness or continuity results for global weak entropy solutions of (1.1) established in [9]. Recently, the existence and continuity results for global weak entropy solutions of (1.1) were established for $\mathbf{L}^1(\mathbb{R})$ initial data in [10]. The entropy weak solutions are constructed by a flux-splitting method. Relying on the decay properties of the semigroup generated by Burgers equation and the Lipschitz continuity of solutions to the Poisson equation, approximating solutions satisfy an Oleinik-type inequality for any positive time. As a consequence, the sequence of approximating solutions is precompact and converges in $\mathbf{L}^1_{\text{loc}}(\mathbb{R})$. Moreover, using an energy estimate, they show that the characteristics are Hölder continuous, which is used to achieve the continuity property of the solutions. The Oleinik-type inequality gives that the solution $u(t,\cdot)$ is in $BV_{\text{loc}}(\mathbb{R})$ for every $t > 0$. In particular, this implies that the Radon measure $Du(t,\cdot)$ is divided into three mutually singular measures

$$Du(t,\cdot) \ = \ D^a u(t,\cdot) + D^j u(t,\cdot) + D^c u(t,\cdot)$$

where $D^a u(t,\cdot)$ is the absolutely continuous measure with respect to the Lebesgue measure, $D^j u(t,\cdot)$ is the *jump part* which is a countable sum of weighted Dirac measures, and $D^c u(t,\cdot)$ is the non-atomic singular part of the measure called the *Cantor part*. For a given $w \in BV_{\text{loc}}(\mathbb{R})$, the Cantor part of Dw does not vanish in general. A typical example of $D^c w$ is the derivative of the Cantor-Vitali ternary function. If $D^c w$ vanishes then we say the function w is locally in the space of special functions of bounded variation, denoted by $SBV_{\text{loc}}(\mathbb{R})$. The space of SBV_{loc} functions was first introduced in [11] and plays important role in the theory of image segmentation and with variational problems in fracture mechanics. Motived by results on SBV regularity for hyperbolic conservation laws ([2, 15, 4, 13]), we show that

Theorem 1.2. *Let $u : [0,\infty[\times\mathbb{R} \to \mathbb{R}$ be the unique locally BV-weak entropy solution of (1.1) with initial data $u_0 \in \mathbf{L}^1(\mathbb{R})$. Then there exists a countable set $\mathcal{T} \subset \mathbb{R}^+$ such that*

$$u(t,\cdot) \ \in \ SBV_{\text{loc}}(\mathbb{R}) \qquad \text{for all } \ t \in \mathbb{R}^+ \setminus \mathcal{T}\,.$$

As a consequence, the slicing theory of BV functions and the chain rule of Vol'pert [3] implies that the weak entropy solution u is in $SBV_{\text{loc}}([0,+\infty[\times\mathbb{R})$. This is the first example of the SBV regularity for scalar conservation laws with nonlocal source term. A common theme in the proofs of recent results on SBV regularity involve an appropriate geometric functional which has certain monotonicity properties and jumps at time t if $u(t,\cdot)$ does not belong to SBV (see e.g. in [2]). More precisely, let $\mathcal{J}(t)$ be the set of jump discontinuities $\mathcal{J}(t)$ of $u(t,\cdot)$. For each $x_j \in \mathcal{J}(t)$, there are minimal and maximal backward characteristics $\xi_j^-(s)$ and $\xi_j^+(s)$ emanating from (t,x_j) which define a nonempty interval $I_j(s) :=]\xi_j^-(s), \xi_j^+(s)[$ for any $s < t$. In this case, the functional $F_s(t)$ defined as the sum of the measures of $I_j(s)$ is monotonic and bounded. Relying on a careful study of generalized characteristics, one shows that if the measure $Du(t,\cdot)$ has a non-vanishing Cantor part then the function F_s "jumps" up at time t which implies that the Cantor part is only present at countably many t. Due to the nonlocal source, $u(t,\cdot)$ does not necessarily have compact support. Thus, we approach the domain by first looking at compact sets and then "glue" the sections together to recover the full domain.

2 Preliminaries

2.1 BV and SBV functions

Let us now introduce the concept of functions of bounded variation in $\mathbb{R}$. We refer to [3] for a comprehensive analysis.

Definition 2.1. Given an open set $\Omega \subseteq \mathbb{R}$, let w be in $\mathbf{L}^1(\Omega)$. We say that w is a *function of bounded variation in* Ω (denoted by $w \in BV(\Omega)$) if the distributional derivative of w is representable by a finite Radon measure Du on Ω, i.e.,

$$-\int_\Omega w \cdot \varphi' \; dx \;=\; \int_\Omega \varphi \; dDw \qquad \text{for all } \; \varphi \in \mathcal{C}_c^\infty(\Omega)$$

with *total variation* (denoted by $\|Dw\|$) given by

$$\|Dw\|(\Omega) \;=\; \sup\left\{ \int_\Omega w \cdot \varphi' \;\; dx \;:\; \varphi \in \mathcal{C}_c^\infty(\Omega), \; \|\varphi\|_{\mathbf{L}^\infty} \le 1 \right\}.$$

Moreover, w is of *locally bounded variation* on Ω (denoted by $w \in BV_{\text{loc}}(\Omega)$) if $w \in \mathbf{L}^1_{\text{loc}}(\Omega)$ and w is in $BV(U)$ for all $U \subset\subset \Omega$.

Given $w \in BV_{loc}(\mathbb{R})$, we split Dw into the *absolutely continuous part* $D^a w$ and *singular part* $D^s w$ provided by the Radon-Nikodým theorem (see e.g. [3, Theorem 1.28]). In the 1-D case, the singular part is concentrated on the $\mathbf{L}^1$-negligible set

$$S_w \;=\; \left\{ t \in \mathbb{R} \;\middle|\; \lim_{\delta \to 0} \frac{|Dw|(t-\delta, t+\delta)}{|\delta|} \;=\; +\infty \right\}.$$

We can further decompse $D^s w$ by isolating the set of atoms $A_w = \{t \in \mathbb{R} \mid Dw(\{t\}) \neq 0\}$, contained in S_w. Hence, we can consider two mutually singular measures

$$D^j w \;:=\; D^s w \llcorner A_w \qquad \text{and} \qquad D^c w \;:=\; D^s w \llcorner (S_w \setminus A_w)$$

respectively called the *jump part* of the derivative and the *Cantor part* of the derivative. Furthermore, we have the following structure result (see e.g. [3, Theorem 3.28])

Proposition 2.2. *Let $\Omega \subseteq \mathbb{R}$ and $w \in BV(\Omega)$. Then, for any $x \in A_w$, the left and right hand limits of $w(x)$ exist and*

$$D^j w \;=\; \sum_{x\in A_w} \left(w(x+) - w(x-)\right)\delta_x$$

where $w(x\pm)$ denote the one-sided limits of w at x. Moreover, $D^c w$ vanishes on any sets which are σ-finite with respect to $\mathcal{H}^0$.

Definition 2.3. Let w be in $BV_{loc}(\mathbb{R})$ then w is a *special function of bounded variation* (denote by $w \in SBV$) if the Cantor part $D^c w$ vanishes.

We want to show that the weak entropy solutions of (1.1) belong to SBV.

2.2 Oleinik-type inequality and non-crossing of characteristics

The global existence and BV-regularity of (1.1) was studied extensively in [10]. For convenience, we recall their main results here.

Theorem 2.4. *The Cauchy problem* (1.1)-(1.2) *with initial data $u_0 = u(0,\cdot) \in \mathbf{L}^1(\mathbb{R})$ admits a unique solution $u(t,x)$ such that for all $t > 0$ the following hold:*

(i) the $\mathbf{L}^1$-norm is bounded by

$$\|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; e^t \cdot \|u_0\|_{\mathbf{L}^1(\mathbb{R})}\,; \tag{2.1}$$

(ii) the solution satisfies the following Oleinik-type inequality

$$u(t,y) - u(t,x) \;\leq\; \frac{K_t}{t}\cdot(y-x) \qquad \text{for all} \quad y > x \tag{2.2}$$

with $K_t \;=\; 1 + 2t + 2t^2 + 4t^2 e^t \cdot \|u_0\|_{\mathbf{L}^1(\mathbb{R})}$;

(iii) the $\mathbf{L}^\infty$-norm is bounded by

$$\|u(t,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R})} \;\leq\; \sqrt{\frac{2K_t}{t}\,\|u(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}} \;\leq\; \sqrt{\frac{2K_t e^t}{t}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\,. \tag{2.3}$$

In particular, this implies that that for all $t > 0$, $u(t,\cdot)$ is in $BV_{\mathrm{loc}}(\mathbb{R})$ and satisfies

$$u(t,x-) \;\geq\; u(t,x+) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{2.4}$$

We recall the definition and theory of generalized characteristic curves associated to (1.1). For a more in depth theory of generalized characteristics, we direct the readers to [7].

Definition 2.5. For any $(t,x) \in]0,+\infty[\times\mathbb{R}$, an absolutely continuous curve $\xi_{(t,x)}(\cdot)$ is called a *backward characteristic curve* starting from (t,x) if it is a solution of differential inclusion

$$\dot{\xi}_{(t,x)}(s) \;\in\; \left[u\left(s,\xi_{(x,t)}(s)+\right), u\left(s,\xi_{(t,x)}(s)-\right)\right] \qquad a.e.\ s \in [0,t] \tag{2.5}$$

with $\xi_{(t,x)}(t) = x$. If $s \in [t,+\infty[$ in (2.5) then ξ is called a *forward characteristic curve*, denoted by $\xi^{(t,x)}(\cdot)$. The characteristic curve ξ is called *genuine* if $u(t,\xi(t)-) = u(t,\xi(t)+)$ for almost every t.

The existence of backward (forward) characteristics was studied by Fillipov. As in [7] and [15], the speed of the characteristic curves are determined and genuine characteristics are essentially classical characteristics:

Proposition 2.6. *Let $\xi : [a,b] \to \mathbb{R}$ be a characteristic curve for the Burgers-Poisson equation (1.1), associated with an entropy solution u. Then for almost every time $t \in [a,b]$, it holds that*

$$\dot{\xi}(t) \;=\; \begin{cases} u(t,\xi(t)) & \text{if} \quad u(t,\xi(t)+) \;=\; u(t,\xi(t)-)\,, \\ \dfrac{u(t,\xi(t)+)+u(t,\xi(t)-)}{2} & \text{if} \quad u(t,\xi(t)+) \;<\; u(t,\xi(t)-)\,. \end{cases} \tag{2.6}$$

In addition, if ξ is genuine on $[a,b]$, then there exists $v(t) \in C^1([a,b])$ such that

$$u(t,\xi(t)-) \;=\; v(t) \;=\; u(t,\xi(t)+) \qquad \text{for all} \quad t \in]a,b[$$

and $(\xi(\cdot), v(\cdot))$ solve the system of ODEs

$$\begin{cases} \dot{\xi}(t) \;=\; v(t) \\ \dot{v}(t) \;=\; [G * u(t,\cdot)]_x(\xi(t)) \end{cases} \qquad \text{for all} \quad t \in]a,b[\,. \tag{2.7}$$

Backward characteristics $\xi_{(t,x)}(\cdot)$ are confined between a maximal and minimal backward characteristics, as defined in [7] $\big($denoted by $\xi_{(t,x+)}(\cdot)$ and $\xi_{(t,x-)}(\cdot)$ $\big)$. Relying on the above proposition and (2.4), we can obtain properties of generalized characteristics, associated with entropy solutions of the Burgers-Poisson equation, including the non-crossing property of two genuine characteristics.

Proposition 2.7. *Let u be an entropy solution to* (1.1). *Then for any $(t,x) \in]0+\infty[\times\mathbb{R}$, the following holds:*

(i) The maximal and minimal backward characteristics $\xi_{(t,x\pm)}$ are genuine and thus the function $u\left(\tau, \xi_{(t,x\pm)}(\tau)\right)$ solves (2.7) for $\tau \in]0,t[$ with initial data $u(t,\xi_{(t,x\pm)}(t))$.

(ii) [Non-crossing of genuine characteristics] Two genuine characteristics may intersect only at their endpoints.

(iii) If $u(t,\cdot)$ is discontinuous at a point x, then there is a unique forward characteristic $\xi^{(t,x)}$ which passes though (t,x) and

$$u\left(\tau, \xi^{(t,x)}(\tau)-\right) \;>\; u\left(\tau, \xi^{(t,x)}(\tau)+\right) \qquad \text{for all} \quad \tau \geq t\,.$$

Throughout this paper, we shall denote by $\mathcal{J}(t) = \{x \in \mathbb{R} : u(t,x-) > u(t,x+)\}$, the jump set of $u(t,\cdot)$ for any $t > 0$. For any $x \in \mathcal{J}(t)$, the base of the backward characteristic cone starting from (t,x) at time $s \in [0,t[$ is

$$I_{(t,x)}(s) \;:=\;]\xi_{(t,x-)}(s), \xi_{(t,x+)}(s)[. \tag{2.8}$$

By the non-crossing property, for any $T > 0$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$, the set

$$\mathcal{A}^T_{[z_1,z_2]} \;:=\; \bigcup_{s\in[0,T]} A^T_{[z_1,z_2]}(s) \qquad \text{with} \qquad A^T_{[z_1,z_2]}(s) \;:=\;]\xi_{(T,z_1)}(s), \xi_{(T,z_2)}(s)[\tag{2.9}$$

confines all backward characteristics starting from (T,x) with $x\in]z_1,z_2[$. For any $0<s<\tau\leq T$, we denote by

$$I^{\tau,T}_{[z_1,z_2]}(s) \;=\; \bigcup_{x\in A^T_{[z_1,z_2]}(\tau)\bigcap \mathcal{J}(\tau)} I_{(\tau,x)}(s). \tag{2.10}$$

Due to the no-crossing property of two genuine backward characteristics and the uniqueness of forward characteristics in Proposition 2.7, the following holds:

Corollary 2.8. *Given $T>0$ and $z_1<z_2\in\mathbb{R}\backslash\mathcal{J}(T)$, the map $\tau\mapsto I^{\tau,T}_{[z_1,z_2]}(s)$ is increasing in the interval $]s,T]$ in the following sense*

$$I^{\tau_1,T}_{[z_1,z_2]}(s) \;\subseteq\; I^{\tau_2,T}_{[z_1,z_2]}(s) \qquad \text{for all } \; 0\leq s<\tau_1\leq\tau_2\leq T. \tag{2.11}$$

Moreover, for any $x\in A^T_{[z_1,z_2]}(\tau_1)\setminus I^{\tau_2,T}_{[z_1,z_2]}(\tau_1)$ with $0<\tau_1<\tau_2<t$, the unique forward characteristic $\xi^{(\tau_1,x)}$ passing through (τ_1,x) is genuine in $[\tau_1,\tau_2]$.

Proof. Let $x\in\mathcal{J}(\tau_1)\cap A^T_{[z_1,z_2]}(\tau_1)$ and let $\chi(\cdot)$ be the unique forward characteristic emenating from (τ_1,x). By property (iii) of Proposition 2.7, for a fixed $\tau_2\in[\tau_1,T]$ we have that $\chi(\tau_2)\in\mathcal{J}(\tau_2)$ and by the non-crossing property, $\chi(\tau_2)\in A^T_{[z_1,z_2]}(\tau_2)$. Since the backward characteristics that form the base of a characteristic cone are genuine, the non-crossing property implies that

$$I_{(\tau_1,x)}(s)\subseteq I_{(\tau_2,\chi(\tau_2))}(s)\subset A^T_{[z_1,z_2]}(s) \qquad \text{for all } \; s\in[0,\tau_1]$$

yielding (2.11). The later statement follows directly. □

3 SBV-regularity

Throughout this section, let $u:[0,\infty[\times\mathbb{R}\to\mathbb{R}$ be the unique locally BV-weak entropy solution of (1.1) for some initial data $u_0\in\mathbf{L}^1(\mathbb{R})$. The section aims to prove Theorem 1.2. For simplicity, denote the jump and Cantor parts of $Du(t,\cdot)$ by

$$\nu_t \;=\; D^j u(t,\cdot) \qquad \text{and} \qquad \mu_t \;=\; D^c u(t,\cdot) \qquad \text{for any } \; t\in]0,+\infty[$$

which, by (2.2), are both non-positive. We will show that $\mu_t(\mathbb{R})<0$ for at most countable positive times $t>0$. In order to do so, let us first establish some basic bounds on backward characteristics.

Lemma 3.1. *For any given $0<t_0<t$ and $x_1\leq x_2$, let $\xi_i(\cdot)$ be a genuine backward characteristic starting from (t,x_i) and*

$$v_i(s) \;=\; u(s,\xi_i(s)) \quad \text{for all } \; s\in[0,t], \; i\in\{1,2\}.$$

Then the followings hold:

$$|v_2(s)-v_1(s)|+|\xi_2(s)-\xi_1(s)| \;\leq\; c_t(s)\cdot\left(|v_2(t)-v_1(t)|+|\xi_2(t)-\xi_1(t)|\right) \tag{3.1}$$

for all $s \in [0,t]$ *and*

$$\xi_2(t_0) - \xi_1(t_0) \;\geq\; \frac{x_2 - x_1 + (v_1(t_0) - v_2(t_0)) \cdot (t - t_0)}{\Gamma_{[t_0,t]}} \tag{3.2}$$

with

$$\begin{cases} c_t(s) &= \exp\left\{2 \cdot \left(\sqrt{2K_t e^t \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + (e^t \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})} + 1) \cdot \sqrt{t}\right) \cdot (\sqrt{t} - \sqrt{s})\right\}, \\ \Gamma_{[t_0,t]} &= 1 + \left(\sqrt{\frac{2K_t e^t}{t_0} \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + e^t \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}\right) \cdot \frac{e^{K_t} t}{t_0} \cdot (t - t_0)^2. \end{cases} \tag{3.3}$$

Proof. **1.** Let's first proof (3.1). From Proposition 2.6, it holds that

$$\begin{cases} \dot{\xi}_i(s) &= v_i(s) \\ \dot{v}_i(s) &= [G * u(s,\cdot)]_x(\xi_i(s)) \end{cases} \qquad \text{for all} \quad s \in]0,t[. \tag{3.4}$$

In particular, this implies that

$$\frac{d}{ds}\left|\xi_2(s) - \xi_1(s)\right| \;\geq\; -\left|v_2(s) - v_1(s)\right|$$

and

$$\frac{d}{ds}\left|v_2(s) - v_1(s)\right| \;\geq\; -\Big|[G * u(s,\cdot)]_x(\xi_2(s)) - [G * u(s,\cdot)]_x(\xi_1(s))\Big|.$$

Since $\xi_2(s) \geq \xi_1(s)$ for all $s \in]0,t]$, we estimate

$$\begin{aligned} \Big|[G * u(s,\cdot)]_x(\xi_2(s)) - [G * u(s,\cdot)]_x(\xi_1(s))\Big| \;&\leq\; \frac{1}{2} \cdot \int_{-\infty}^{\xi_1(s)} |u(s,z)| \cdot \left|e^{z-\xi_2(s)} - e^{z-\xi_1(s)}\right| \, dz \\ + \frac{1}{2} \cdot \int_{\xi_1(s)}^{\xi_2(s)} |u(s,z)| \cdot \left|e^{z-\xi_2(s)} + e^{\xi_1(s)-z}\right| \, dz &+ \frac{1}{2} \cdot \int_{\xi_2(s)}^{+\infty} |u(s,z)| \cdot \left|e^{\xi_1(s)-z} - e^{\xi_2(s)-z}\right| \, dz \\ \leq\; \frac{1}{2} \cdot \left(1 - e^{\xi_1(s)-\xi_2(s)}\right) &\int_{\mathbb{R}\setminus[\xi_1(s),\xi_2(s)]} |u(s,z)| \, dz + \int_{[\xi_1(s),\xi_2(s)]} |u(s,z)| \, dz \\ \leq\; \left(\frac{1}{2} \cdot \left\|u(s,\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})} + \left\|u(s,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R})}\right) &\cdot \left|\xi_2(s) - \xi_1(s)\right|. \end{aligned}$$

Hence, (2.1) and (2.3) imply that

$$\begin{aligned} &\Big|[G * u(s,\cdot)]_x(\xi_2(s)) - [G * u(s,\cdot)]_x(\xi_1(s))\Big| \\ &\qquad \leq\; \left(\sqrt{\frac{2K_t e^t}{s} \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + e^t \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}\right) \cdot \left|\xi_2(s) - \xi_1(s)\right|. \end{aligned} \tag{3.5}$$

Setting $M_t = \sqrt{2K_t e^t \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + (e^t \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})} + 1) \cdot \sqrt{t}$, we have

$$\frac{d}{ds}\left(\left|\xi_2(s) - \xi_1(s)\right| + \left|v_2(s) - v_1(s)\right|\right) \;\geq\; -\frac{M_t}{\sqrt{s}} \cdot \left(\left|\xi_2(s) - \xi_1(s)\right| + \left|v_2(s) - v_1(s)\right|\right),$$

for all $s \in]0,t]$, and Grönwall's inequality yields (3.1).

2. To prove (3.2), we first apply (2.2) to (3.4) to get

$$\dot{\xi}_2(s) - \dot{\xi}_1(s) \;=\; u(s,\xi_2(s)) - u(s,\xi_1(s)) \;\leq\; \frac{K_t}{s}\cdot(\xi_2(s)-\xi_1(s)),$$

and this implies

$$\xi_2(s)-\xi_1(s) \;\leq\; \frac{e^{K_t}s}{t_0}\cdot(\xi_2(t_0)-\xi_1(t_0)) \;\leq\; \frac{e^{K_t}t}{t_0}\cdot(\xi_2(t_0)-\xi_1(t_0)) \quad \text{for all } \; s\in[t_0,t]. \tag{3.6}$$

Therefore, from (3.4) and (3.5), it holds for $s \in [t_0, t]$ that

$$\begin{aligned} v_2(s) - v_1(s) \; &= v_2(t_0) - v_1(t_0) + \int_{t_0}^{s} [G * u(\tau,\cdot)]_x(\xi_2(\tau)) - [G * u(\tau,\cdot)]_x(\xi_1(\tau))\; d\tau \\ &\leq v_2(t_0) - v_1(t_0) + \int_{t_0}^{s} \left(\sqrt{\frac{2K_t e^t}{t_0}\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + e^t\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}\right)\cdot\big(\xi_2(\tau)-\xi_1(\tau)\big)\; d\tau \\ &\leq v_2(t_0) - v_1(t_0) + \gamma_{[t_0,t]}\cdot(\xi_2(t_0)-\xi_1(t_0)) \end{aligned}$$

with

$$\gamma_{[t_0,t]} \;=\; \left(\sqrt{\frac{2K_t e^t}{t_0}\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + e^t\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}\right)\cdot\frac{e^{K_t}t}{t_0}\cdot(t-t_0).$$

Integrating the first equation in (3.4) over $[t_0,t]$, we get

$$\begin{aligned} \xi_2(t) - \xi_1(t) \; &= \xi_2(t_0) - \xi_1(t_0) + \int_{t_0}^{t} v_2(\tau) - v_1(\tau)\; d\tau \\ &\leq\; (v_2(t_0)-v_1(t_0))\cdot(t-t_0) + \big(1+\gamma_{[t_0,t]}\cdot(t-t_0)\big)\cdot(\xi_2(t_0)-\xi_1(t_0)) \end{aligned}$$

and this yields (3.2). □

As a consequence, we obtain the following two corollaries. The first one provides an upper bound on the base of characteristic cone $C_{(t,x)}$ at time $s \in]0,t[$ for every $x \in \mathcal{J}(t)$.

Corollary 3.2. *For any $(t,x) \in]0,+\infty[\times \mathcal{J}(t)$, it holds that*

$$\left|I_{(t,x)}(s)\right| \;\leq\; -\,c_t(s)\cdot\nu_t(\{x\}) \quad \text{for all} \quad s\in[0,t[. \tag{3.7}$$

Proof. Since $x \in \mathcal{J}(t)$, the inequality (2.4) implies that

$$\nu_t\{x\} \;=\; u(t,x+) - u(t,x-) \;<\; 0.$$

Thus, recalling (3.1), we obtain

$$\left|\xi_{(t,x+)}(s) - \xi_{(t,x-)}(s)\right| \;\leq\; c_t(s)\cdot|u(t,x+)-u(t,x-)|$$

and this yields (3.7). □

In the next corollary, we show that two distinct characteristics are separated for all positive time; moreover, the distance between them is proportional to the difference in the values of the solution along the characteristics.

Corollary 3.3. *Given* $x_1 < x_2$ *and* σ *and* t*, such that* $0 < \sigma < t \leq T$*, let* $\xi_i(\cdot)$ *be a genuine backward characteristic starting from* (t, x_i) *and*

$$v_i(s) \;=\; u(s, \xi_i(s)) \quad \text{for all} \quad s \in [0, t[, \;\; i \in \{1, 2\}.$$

Then it holds that

$$\xi_2(\sigma/2) - \xi_1(\sigma/2) \;\geq\; \kappa_{[\sigma,T]} \cdot (v_1(t) - v_2(t)) \tag{3.8}$$

where

$$\kappa_{[\sigma,T]} \;=\; \frac{\sigma}{2}\left[\Gamma_{[\sigma/2,T]} + \left(\sqrt{\frac{4K_T e^T}{\sigma}\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + e^T \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}\right) \cdot e^{K_T} T \cdot (T - \sigma/2)\right]^{-1}.$$

Proof. Integrating the second equation in (2.7) over $[\sigma/2, t]$ yields

$$\begin{aligned} v_1(t) - v_2(t) \;&= v_1(\sigma/2) - v_2(\sigma/2) + \int_{\sigma/2}^{t} [G * u(\tau,\cdot)]_x(\xi_1(\tau)) - [G * u(\tau,\cdot)]_x(\xi_2(\tau))\, d\tau \\ &\leq v_1(\sigma/2) - v_2(\sigma/2) + \int_{\sigma/2}^{t} \Big|[G * u(\tau,\cdot)]_x(\xi_2(\tau)) - [G * u(\tau,\cdot)]_x(\xi_1(\tau))\Big|\, d\tau \end{aligned}$$

and by (3.5) and (3.6) it holds that

$$\begin{aligned} v_1(t) - v_2(t) \;\leq\; & v_1(\sigma/2) - v_2(\sigma/2) \\ & + \left(\sqrt{\frac{4K_T e^T}{\sigma}\left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}} + e^T \left\|u_0\right\|_{\mathbf{L}^1(\mathbb{R})}\right) \cdot \frac{2e^{K_T} T}{\sigma} \cdot (T - \sigma/2) \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)). \end{aligned} \tag{3.9}$$

On the other hand, by (3.2) we have that

$$v_1(\sigma/2) - v_2(\sigma/2) \;\leq\; \frac{\Gamma_{[\sigma/2,t]}}{t - \sigma/2} \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)) \;\leq\; \frac{2\Gamma_{[\sigma/2,T]}}{\sigma} \cdot (\xi_2(\sigma/2) - \xi_1(\sigma/2)).$$

which, when applied to (3.9), implies (3.8). □

The next lemma shows that, for a certain positive time s, if $u(s,\cdot)$ is not in SBV, then at future times $s + \varepsilon$ the Cantor part of $u(s,\cdot)$ gets transformed into jump singularities. Following the main idea in [2, 15], for any $s \in]0, T[$ and $z_1 < z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$, let us consider the set of points $E^T_{[z_1,z_2]}(s)$ in $A^T_{[z_1,z_2]}(s)$ where the Cantor part of $D_x u(s,\cdot)$ prevails, i.e.,

$$E^T_{[z_1,z_2]}(s) \;=\; \left\{x \in A^T_{[z_1,z_2]}(s) : \lim_{\eta \to 0+} \frac{\eta + |D_x u(s,\cdot) - \mu_s|([x-\eta, x+\eta])}{-\mu_s([x-\eta, x+\eta])} \;=\; 0\right\}. \tag{3.10}$$

Besicovitch differentiation theorem [3] gives that $\mu_s\left(A^T_{[z_1,z_2]}(s) \setminus E^T_{[z_1,z_2]}(s)\right) = 0$ and

$$\lim_{\eta \to 0^+} \frac{u^-(s, x-\eta) - u^+(s, x+\eta)}{-\mu_s([x-\eta, x+\eta])} \;=\; 1 \qquad \text{for all} \;\; x \in E^T_{[z_1,z_2]}(s). \tag{3.11}$$

Moreover, for μ_s-a.e. x in $E^T_{[z_1,z_2]}(s)$, it holds that

$$\lim_{\eta\to 0}\frac{u(s,x+\eta)-u(s,x)}{\eta} \;=\; -\infty. \tag{3.12}$$

Lemma 3.4. *Let $0<s<t\le T$ and $z_1<z_2\in\mathbb{R}\setminus\mathcal{J}(T)$ be fixed. Then, it holds for μ_s-a.e. $x\in A^T_{[z_1,z_2]}(s)$ that*

$$]x-\eta_x,x+\eta_x[\;\subset\; I^{t,T}_{[z_1,z_2]}(s)\qquad\text{for some } \eta_x>0.$$

Proof. Since $I^{t,T}_{[z_1,z_2]}(s)$ is open, it is sufficient to prove that every point $x\in E^T_{[z_1,z_2]}(s)\setminus\mathcal{J}(s)$ satisfying (3.12) is in $I^{t,T}_{[z_1,z_2]}(s)$. Assume by a contradiction that

$$x\;\in\; A^T_{[z_1,z_2]}(s)\setminus\overline{I^{t,T}_{[z_1,z_2]}(s)}\bigcup\partial(\overline{I^{t,T}_{[z_1,z_2]}(s)}).$$

1. If $x\in A^T_{[z_1,z_2]}(s)\setminus\overline{I^{t,T}_{[z_1,z_2]}(s)}$ then

$$]x-\eta_0,x+\eta_0[\;\bigcap\;\overline{I^{t,T}_{[z_1,z_2]}(s)}\;=\;\emptyset\qquad\text{for some } \eta_0>0. \tag{3.13}$$

Given any $\eta\in[0,\eta_0[$, let $\xi^\eta_1(\cdot)$ and $\xi^\eta_2(\cdot)$ be the unique forward characteristics emanating from $x-\eta$ and $x+\eta$ at time τ_0. From Corollary 2.8, both $\xi^\eta_1(\cdot)$ and $\xi^\eta_2(\cdot)$ are genuine in $[t_0,t]$ and

$$\xi^\eta_2(\tau)-\xi^\eta_1(\tau)\;\ge\;0\qquad\text{for all }\;\tau\in[s,t] \tag{3.14}$$

Thus, (3.2) in Lemma 3.1 implies

$$\begin{aligned}2\eta\;&=\xi^\eta_2(s)-\xi^\eta_1(s)\;\ge\;\frac{\xi^\eta_2(t)-\xi^\eta_1(t)+(u(s,x-\eta)-u(s,x+\eta))\cdot(t-s)}{\Gamma_{[s,t]}}\\ &\ge\;-\frac{(u(s,x+\eta)-u(s,x-\eta))\cdot(t-s)}{\Gamma_{[s,t]}}\end{aligned}$$

which yields a contradiction to (3.12) when η is sufficiently small.

2. Suppose that $x\in\partial(\overline{I^{t,T}_{[z_1,z_2]}(s)})$. In this case, $\xi_{(s,x)}(\cdot)$ is either a minimal or maximal backward characteristic in $[s,t]$. Moreover, for every $\eta>0$ there exists $x_\eta\in\;]x-\eta,x[\bigcup]x,x+\eta[$ such that $x_\eta\notin\overline{I^{t,T}_{[z_1,z_2]}(s)}$ and the unique forward characteristics $\xi^{(s,x_\eta)}(\cdot)$ emenating from x_η at time s is genuine and does not cross $\xi_{(s,x)}(\cdot)$ in the time interval $[s,t]$. With the same computation in the previous step, we get

$$\frac{u(s,x_\eta)-u(s,x)}{x_\eta-x}\;\ge\;-\frac{\Gamma_{[s,t]}}{t-s}$$

and this also yields a contradiction to (3.12) when η is sufficiently small. $\square$

We are now ready to prove our first main theorem.

Proof of Theorem 1.2. The proof is divided into two steps:

Step 1. Fix $T > 0$ and $z_1, z_2 \in \mathbb{R} \setminus \mathcal{J}(T)$ with $z_1 < z_2$ and, recalling (2.9) and (2.10) let

$$\mathcal{A} \;=\; \mathcal{A}^T_{[z_1,z_2]}, \qquad A_t \;= A^T_{[z_1,z_2]}(t) \qquad \text{and} \qquad I^t(s) \;=\; I^{t,T}_{[z_1,z_2]}(s) \tag{3.15}$$

for all $0 < s < t \leq T$. We claim that the set

$$\mathcal{T}_{[z_1,z_2]} \;:=\; \{t \in [0,T] \;:\; \mu_t\,(A_t) \text{ does not vanish}\} \tag{3.16}$$

is at most countable.

(i). Fix $\sigma \in]0,T[$. By Proposition 2.6 and (2.3), one has

$$|A_t| \;\leq\; |z_2 - z_1| + 2\sqrt{\frac{2K_Te^T}{\sigma}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\cdot T \qquad \text{for all } \; t \in [\sigma,T],$$

and the Oleinik-type inequality (2.2) yields

$$|Du(t,\cdot)|\big(A_t\big) \;\leq\; M^T_\sigma \qquad \text{for all } \; t \in [\sigma,T]$$

with

$$M^T_\sigma \;=\; 2\sqrt{\frac{2K_Te^T}{\sigma}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}} + \frac{2K_T}{\sigma}\cdot\left(|z_2 - z_1| + 2\sqrt{\frac{2K_Te^T}{\sigma}\,\|u_0\|_{\mathbf{L}^1(\mathbb{R})}}\cdot T\right).$$

Let the geometric functional $F_\sigma : [\sigma,T] \to [0,\infty[$ be defined by

$$F_\sigma(t) \;=\; \left|\bigcup_{x\in\mathcal{J}(t)\bigcap A_t} I_{(t,x)}\,(\sigma/2)\right| \;=\; \sum_{x\in\mathcal{J}(t)\bigcap A_t} \left|I_{(t,x)}\,(\sigma/2)\right| \qquad \text{for all } \; t \in [\sigma,T]$$

where the second equality follows by the non-crossing property. By Corollaries 2.8 and 3.2, the map $t \mapsto F_\sigma(t)$ is non-decreasing in $[\sigma,T]$ and uniformly bounded

$$\sup_{t\in[\sigma,T]} F_\sigma(t) \;\leq\; c_T(\sigma/2)\cdot \sup_{t\in\sigma,T}\,(|\nu_t|(A_t)) \;\leq\; c_T(\sigma/2)\cdot M^T_\sigma \tag{3.17}$$

with $c_T(\sigma/2)$ defined in (3.3).

(ii). Assume that a Cantor part is present in $\mathcal{A}$ at time $t \in]\sigma,T[$, i.e.,

$$\mu_t\,(A_t) \;\leq\; -\,\alpha \qquad \text{for some } \alpha > 0, \tag{3.18}$$

which by (3.10) is concentrated on $E_t := E^T_{[z_1,z_2]}(t)$. We will show that

$$F_\sigma(t+) - F_\sigma(t) \;\geq\; \frac{\kappa_{[\sigma,T]}}{2}\cdot\alpha \tag{3.19}$$

where $\kappa_{[\sigma,T]}$ is defined in Corollary 3.3. It is sufficient to prove that

$$F_\sigma(t+\varepsilon) - F_\sigma(t) \;=\; \left|I^{t+\varepsilon}(\sigma/2)\setminus I^t(\sigma/2)\right| \;\geq\; \frac{\kappa_{[\sigma,T]}}{2}\cdot\alpha$$

for any given $\varepsilon \in]0, T-t[$. By Lemma 3.4, for μ_t-a.e. $x \in E_t$ there exists $\eta_x > 0$ such that

$$]x-\eta_x, x+\eta_x[\ \subset \ I^{t+\varepsilon}(t). \tag{3.20}$$

On the other hand, given $x \in E_t$ and $\eta > 0$, we denote the interval

$$J^{\sigma/2}_{x,\eta} \ = \ ]\xi_{(t,x-\eta)}(\sigma/2), \xi_{(t,x+\eta)}(\sigma/2)[,$$

and Corollaries 3.2 and 3.3 imply that

$$\begin{aligned}\left|J^{\sigma/2}_{x,\eta} \setminus I^t(\sigma/2)\right| \ &= \xi_{(t,x+\eta)}(\sigma/2) - \xi_{(t,x-\eta)}(\sigma/2) - \left|J^{\sigma/2}_{x,\eta} \cap I^t(\sigma/2)\right| \\ &\geq \kappa_{[\sigma,T]} \cdot (u(t,x-\eta) - u(t,x+\eta)) + c_T(\sigma/2)\nu_t(]x-\eta, x+\eta[).\end{aligned}$$

Furthermore, by (3.11) and the definition of E_t, there exists $\eta_0 > 0$ such that

$$\left|J^{\sigma/2}_{x,\eta} \setminus I^t(\sigma/2)\right| \ \geq \ -\frac{\kappa_{[\sigma,T]}}{2}\mu_t(]x-\eta, x+\eta[) \qquad \text{for all} \ \ \eta \in]0, \eta_0]. \tag{3.21}$$

By the Besicovitch covering lemma, we can cover μ_t-a.e. E_t with countably many pairwise disjoint intervals $[x_j - \eta_j, x_j + \eta_j]$ where η_j is chosen such that both (3.20)and (3.21) hold. Proposition 2.7 (ii) implies that the intervals $J^{\sigma/2}_{x_j,\eta_j}$ are pairwise disjoint and by (3.20) we have that $J^{\sigma/2}_{x_j,\eta_j}$ is contained in $A_{\sigma/2}$. Therefore, it holds that

$$F_\sigma(t+\varepsilon) - F_\sigma(t) \ = \ \left|I^{t+\varepsilon}(\sigma/2) \setminus I^t(\sigma/2)\right| \ \geq \ \sum_j \left|J^{\sigma/2}_{x_j,\eta_j} \setminus I^t(\sigma/2)\right|.$$

Applying (3.21) and then (3.18) to the above inequality yields

$$F_\sigma(t+\varepsilon) - F_\sigma(t) \ \geq \ -\frac{\kappa_{[\sigma,T]}}{2}\sum_j \mu_t([x_j-\eta_j, x_j+\eta_j]) \ \geq \ -\frac{\kappa_{[\sigma,T]}}{2}\mu_t(E_t) \ \geq \ \frac{\kappa_{[\sigma,T]}}{2}\alpha,$$

and therefore (3.19) holds.

(iii). By the monotonicity of F_σ and (3.17), F_σ has at most countable many discontinuities on $[\sigma, T]$. Thus, for any given $\sigma \in]0, T[$, (3.18)-(3.19) imply that the set

$$\bigcup_{n\in\mathbb{N}} \left\{t \in [\sigma, T] \ : \ \mu_t(A_t) \leq -2^{-n}\right\} \ = \ \{t \in [\sigma, T] \ : \ \mu_t(A_t) < 0\}$$

is at most countable and therefore,

$$\bigcup_{n\in\mathbb{N}} \left\{t \in [2^{-n}, T] \ : \ \mu_t(A_t) < 0\right\} \ = \ \mathcal{T}_{[z_1,z_2]} \ \text{ is countable.}$$

Step 2. To complete the proof, it is sufficient to show that for any given $T > 0$, there exists an at most countable subset $\mathcal{T}_T$ of $[0, T]$ such that

$$u(t,\cdot) \ \in \ SBV_{\mathrm{loc}}(\mathbb{R}) \qquad \text{for all} \ \ t \in [0,T] \setminus \mathcal{T}_T. \tag{3.22}$$

For any $k \in \mathbb{Z}$, we pick a point $\bar{z}_k \in]k, k+1[\backslash \mathcal{J}(T)$. Let $\xi_k(\cdot)$ be the unique genuine backward characteristic starting at point $(T, \bar{z}_k)$ for every $k \in \mathbb{Z}$ and define

$$\mathcal{A}_k^T = \mathcal{A}^T_{[\bar{z}_k, \bar{z}_{k+1}]} \bigcup \{(\xi_k(t), t) : t \in [0,T]\} \quad \text{and} \quad A_k^T(t) = A^T_{[\bar{z}_k, \bar{z}_{k+1}]}(t) \bigcup \{\xi_k(t)\}.$$

Due to the no-crossing property of two genuine backward characteristics in Proposition 2.7, it holds that

$$\bigcup_{k \in \mathbb{Z}} \mathcal{A}_k^T = [0,T] \times \mathbb{R} \qquad \text{and} \qquad \bigcup_{k \in \mathbb{Z}} A_k^T(t) = \mathbb{R} \qquad \text{for all } \ t \in [0,T].$$

From Step 1, it holds that, for every $k \in \mathbb{Z}$, the set

$$\{t \in [0,T] : \mu_t(A_k^T(t)) \neq 0\} \ \text{ is countable.}$$

Hence,

$$\mathcal{T}_T = \{t \in [0,T] : \mu_t(A_k^T(t)) \neq 0 \ \text{ for some } k \in \mathbb{Z}\} \ \text{ is also countable.}$$

and this yields (3.22). □

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