An Equilibrium Model of Debt and Bankruptcy

Alberto Bressan and Khai T. Nguyen

Department of Mathematics, Penn State University.
University Park, PA 16802, USA.

e-mails: bressan@math.psu.edu , ktn2@psu.edu

January 13, 2016

Abstract

We study optimal strategies for a borrower, who services a debt in an infinite time horizon, taking into account the risk of possible bankruptcy. In a first model, the interest rate as well as the instantaneous bankruptcy risk are given, increasing functions of the total amount of debt. In a second model only the bankruptcy risk is given, while the interest rate is determined from a Nash equilibrium, in a game between the borrower and a pool of risk-neutral lenders. This yields a non-standard optimal control problem for the borrower, where the dynamics involves all future values of the control. For this model, optimal repayment strategies are constructed, in open-loop form. In addition, for optimal strategies in feedback form, our analysis shows that the value function should satisfy a new kind of nonlinear, degenerate elliptic equation.

1 Introduction

This paper is concerned with problems of optimal debt management, in an infinite time horizon. Denoting by $x(t)$ the total amount of debt at time t, we seek optimal repayment strategies which minimize a total cost, exponentially discounted in time. A key feature of our models is the possibility of bankruptcy. When this happens, the borrower incurs in a very large cost B, while the lenders lose part of the money they invested.

As it is well known, there is no way to predict with certainty the time T_b when bankruptcy occurs [11]. We thus regard T_b as a random variable. Its probability distribution is described in terms of a instantaneous bankruptcy risk $\rho(x)$, depending on the total amount of debt. Roughly speaking, the probability that the borrower goes bankrupt within the time interval $[t,\, t+\varepsilon]$ is measured by $\varepsilon\,\rho(x(t)) + o(\varepsilon)$, where $o(\varepsilon)$ denotes a higher order infinitesimal as $\varepsilon \to 0$.

In most of the literature [3, 5, 6, 7, 9], the interest rate is either given a priori, or modeled as a random variable. A distinguished feature of our model is that the interest rate on new loans is determined by the competition among a a pool of risk-neutral lenders. In this game-theoretical setting, the current rate is thus affected by the anticipation of the bankruptcy risk over the entire future. This yields a highly non-standard optimal control problem, where the dynamics in the state space depends not only on the instantaneous value of the control, but also on all of its future values.

In Section 2 we first consider a model where both the bankruptcy risk and the interest rate are given functions of the total amount of debt. As the debt increases (measured as a fraction of the yearly income of the borrower), so does the bankruptcy risk. In turn, this forces an increase of the interest rate charged by lenders, to offset the possible loss of their capital. For this model, the problem of minimizing the expected cost to the borrower can be formulated as a deterministic optimal control problem in infinite time horizon, exponentially discounted in time.

A detailed description of the optimal feedback control is provided in Section 3. Toward this goal we analyze the Hamilton-Jacobi differential equation for the value function, constructing its unique viscosity solution, with the appropriate boundary conditions.

In Section 4 we introduce a further model, where only the bankruptcy risk is exogenously given. Here the two state variables are the total amount of debt X and the average interest rate A payed on the various portions of this debt. We assume that the borrower announces in advance his entire repayment strategy $t \mapsto u(t)$. Here $u(t) \in [0,1]$ describes the portion of the borrower's income which is allocated to servicing the debt, at time $t \in [0, +\infty[$. In turn, the interest rate $I(\tau)$ charged on new loans initiated at time τ is determined by the perfect competition within a pool of risk-neutral lenders. This will depend on the bankruptcy risk $\rho(X(t))$ at all future times $t \in [\tau, +\infty[$. As a consequence, the time derivatives $\dot{X}(\tau)$, $\dot{A}(\tau)$ depend not only on $X(\tau), A(\tau)$ and $u(\tau)$, but also on all future values of the debt, i.e. on the function $t \mapsto X(t)$, for $t \geq \tau$.

Given initial conditions

$$X(0) \;=\; X_0\,, \qquad\qquad A(0) \;=\; A_0\,, \tag{1.1}$$

our analysis shows that, contrary to what happens for standard control systems, in general the control function $u(\cdot)$ does not uniquely determine the future evolution of the state. Indeed, for the same repayment strategy $u(\cdot)$, the competition among lenders may produce infinitely many Nash equilibrium solutions. In connection with the financial model, this fact can be interpreted as follows. If lenders regard their investment as being risky, they will charge a high interest rate. In turn, this will push up the size of the debt, and hence the chances of bankruptcy. In other words, the widespread belief that the investment is risky makes high risk a reality. On the other hand, given the same initial amount of debt, a widespread belief that the investment is safe will result in a lower interest rate, thus reducing the actual chances of bankruptcy.

In Section 5 we prove that, for any given initial data (1.1), there exists a control $u^*(\cdot)$ and a corresponding solution $(X(\cdot), A(\cdot))$ which minimizes the expected total cost to the borrower.

Section 6 contains a more detailed analysis of the set of solutions (X, A) corresponding to a given control function $u(\cdot)$. Thanks to a monotonicity property of the ODEs for the variables X and A, we show that a unique *minimal solution* (X_*, A_*) can be singled out. For every time $t \geq 0$, $X_*(t)$ provides the pointwise minimum among all debt sizes $X(t)$ that can be achieved for the same control and initial data. By selecting this minimal solution we obtain a deterministic optimal control problem for the borrower.

As proved in Theorem 3, under natural conditions, this minimal solution provides the smallest expected cost to the borrower. However, in some exceptional cases where the bankruptcy cost is small compared with the cost of servicing the debt, we show that the minimal solution is not the most advantageous for the borrower.

In Section 7 we observe that the optimal strategy for the borrower may not always be time-consistent. More precisely, let $t \mapsto u^*(t)$ be an optimal control and let $t \mapsto (X^*(t), A^*(t))$ be the corresponding optimal trajectory. For any $\tau > 0$ one can consider the new optimization problem with initial data

$$X(0) = X^*(\tau), \qquad A(0) = A^*(\tau).$$

A dynamic programming principle would imply that the control $t \mapsto u^*(t-\tau)$ is optimal for this new problem. However, this is not true in general. Indeed, in a game-theoretical setting, it is well known that optimal Stackelberg solutions may not be time-consistent [2].

Modeling the evolution of the debt as a game between the borrower and a pool of lenders, in Section 7 we seek Nash equilibrium solutions in feedback form. Formal computations show that, if a smooth value function $V(X,A)$ exists, it satisfies a second order, highly nonlinear, degenerate elliptic equation. Unfortunately, PDEs of this kind do not fall within the known theory. Their properties will be the topic of a future investigation.

For the basic theory of optimal control and viscosity solutions we refer to [1, 4].

2 The infinite horizon optimal control problem

The model includes the following constants:

- r = discount rate,
- B = bankruptcy cost to the borrower,

and the variables

- t = time, measured in years,
- x = total debt, measured as a fraction of the yearly income of the borrower,
- $u \in [0,1]$ = payment rate, as a fraction of the income,
- $\alpha(x)$ = interest rate payed on debt at a given time,
- $\rho(x)$ = instantaneous risk of bankruptcy,
- $L(u)$ = cost to the borrower for implementing the control u.

We regard u as the control variable for the borrower. Observe that, in this model, both α and ρ are given functions of x. If the total debt is large, so is the bankruptcy risk and thus the interest rate charged by lenders. The following assumptions will be used.

(A1) *The discount rate is a constant $r > 0$. The functions α, ρ are continuously differentiable for $x \in [0, M[$ and satisfy*

$$\alpha\,,\ \alpha' \ \geq\ 0\,, \qquad \rho\,,\rho' \ \geq\ 0\,, \qquad \textit{for all } x \in [0, M[\,, \tag{2.1}$$

$$\lim_{x\to M-} \alpha(x) \ =\ \lim_{x\to M-} \rho(x) \ =\ +\infty. \tag{2.2}$$

Moreover, L is twice continuously differentiable for $u \in [0, 1[$ and satisfies

$$L(0) = 0, \qquad L' > 0, \quad L'' > 0, \qquad \lim_{u\to 1-} L(u) \ =\ +\infty. \tag{2.3}$$

In (2.2) the large constant M denotes a maximum size of the debt, beyond which bankruptcy immediately occurs. Calling T_b the random time at which bankruptcy occurs, its distribution is determined as follows. If at time τ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then the probability that bankruptcy will occur shortly after time τ is measured by

$$\text{Prob.}\Big\{T_b \in [\tau,\, \tau + \varepsilon] \ \Big|\ T_b > \tau\,,\ x(\tau) = y\Big\} \ =\ \rho(y) \cdot \varepsilon + o(\varepsilon), \tag{2.4}$$

where $o(\varepsilon)$ denotes a higher order infinitesimal as $\varepsilon \to 0$. As long as the total debt remains strictly smaller that M, the probability that the borrower is not yet bankrupt at time $t > 0$ is thus computed as

$$\textit{Prob.}\{T_b > t\} \ =\ \exp\left\{-\int_0^t \rho(x(\tau))\, d\tau\right\}. \tag{2.5}$$

Notice that this depends on the function $\tau \mapsto x(\tau)$. If debt is maintained at a higher level, then the probability of bankruptcy increases. This reflects lack of confidence in the ability of the borrower to eventually repay the debt. In turn, a higher bankruptcy risk determines a higher interest rate α asked by the lenders, to offset the possible loss of capital. This motivates the assumptions in (2.1).

The last assumption in (2.3) is motivated by the fact that a borrower needs at least part of his income to survive. Therefore, the cost $L(u)$ approaches infinity as $u \to 1-$.

For a given initial value of the debt

$$x(0) \ =\ y \ \geq\ 0\,, \tag{2.6}$$

the optimal control problem for the borrower can be formulated as follows.

(OCP) Optimal control problem with bankruptcy risk. *Given the initial size y of the debt, find a control $t \mapsto u(t) \in [0, 1]$ which minimizes the expected cost*

$$J \ \doteq\ E\left[\int_0^{T_b} e^{-rt} L(u(t))\, dt + B\, e^{-rT_b}\right], \tag{2.7}$$

subject to

$$\dot{x}(t) \ =\ \alpha(x(t))\, x(t) - u(t)\,, \qquad x(0) \ =\ y\,, \tag{2.8}$$

and with the constraint

$$x(t) \geq 0 \qquad \text{for all} \qquad t \geq 0\,.$$

Here and in the sequel, an upper dot denotes a derivative w.r.t. time. The ODE in (2.8) accounts for the increase of the total debt, depending on the interest rate α, minus the rate u at which it is payed back.

Remark 1. One can modify the model, assuming that the income (say, a gross national product) is not constant, but grows at an exponential rate γ. To handle this more general case, we introduce the variables

$$d(t) \;=\; \text{total debt}, \qquad g(t) \;=\; \text{yearly income}, \qquad x(t) \;\doteq\; \frac{d(t)}{g(t)}, \qquad U(t) \;\doteq\; \frac{u(t)}{g(t)}.$$

It is reasonable to assume that the bankruptcy risk ρ and hence the interest rate α depend on the ration $x = d/g$, while the cost function L depends on $U = u/g$. The evolution equation for the total debt (as a fraction of the yearly income) is then expressed by

$$\dot x(t) \;=\; \frac{\dot d g - d\dot g}{g^2} \;=\; \frac{(\alpha(x)d - u)\cdot g - d\cdot\gamma g}{g^2} \;=\; (\alpha(x(t)) - \gamma))x(t) - U(t).$$

We thus obtain an optimization problem having exactly the same form as (2.7)-(2.8), simply by replacing $\alpha(x)$ with $\alpha(x) - \gamma$.

We now study the optimal control problem **(OCP)**. Given an initial condition $y < M$ and a control $t \mapsto u(t) \in [0,1]$, by (2.8) one obtains a unique trajectory $t \mapsto x(t)$. Call

$$T^M \;\doteq\; \sup\,\big\{t \geq 0\,;\;\; x(t) < M\big\} \tag{2.9}$$

One can think of T^M as the first time when the debt reaches M. In this case the interest rate becomes infinite and bankruptcy occurs instantly. Of course, one may well have $T^M = +\infty$.

Remark 2. The time $T^M \in\,]0, +\infty]$ is uniquely determined by the initial condition and the control $u(\cdot)$. On the other hand the time T_b when bankruptcy occurs is random. By (2.4), the probability that at time t the borrower is not yet bankrupt is computed by

$$P(t) \;\doteq\; Prob.\{T_b > t\} \;=\; \begin{cases} \exp\left\{-\displaystyle\int_0^t \rho(x(\tau))\,d\tau\right\} & \text{if} \quad t < T^M, \\ \qquad 0 & \text{if} \quad t \geq T^M. \end{cases} \tag{2.10}$$

Notice that, if $T^M < \infty$ and $\int_0^{T^M} \rho(x(t))dt < \infty$, then there is a positive probability that bankruptcy occurs exactly at time T^M. However, if $T^M < \infty$ and

$$\int_*^M \frac{\rho(x)}{\alpha(x)}\,dx \;=\; +\infty, \tag{2.11}$$

then with probability one the bankruptcy must occur before time T^M. To see this, choose $x_1 \in\,]y, M[$ so large that $\alpha(x_1)\,x_1 > 1$. In this case, when $x \geq x_1$, by (2.8) the debt is monotonically increasing in time, hence the inverse function $t = t(x)$ is well defined. Calling $t_1 = t(x_1)$ the time when the debt reaches the value x_1, we have

$$\int_0^{T^M} \rho(x(t))\,dt \;\geq\; \int_{t_1}^{T^M} \rho(x(t))\,dt \;=\; \int_{x_1}^M \frac{\rho(x)}{\alpha(x)\,x - u(t(x))}\,dx \;\geq\; \int_{x_1}^M \frac{\rho(x)}{\alpha(x)\,M}\,dx \;=\; +\infty.$$

Hence $\exp\Big\{-\int_0^{T^M}\rho(x(t))\,dt\Big\} = 0$.

We also remark that, if $T^M = +\infty$, then one may have $\int_0^\infty \rho(x(\tau))\,d\tau < \infty$. In this case there is a positive probability that bankruptcy never happens.

Assuming that (2.11) holds, the expected cost in (2.7) can now be written as

$$\begin{aligned} J &= \int_0^\infty \rho(x(t))\exp\Big\{-\int_0^t \rho(x(s))\,ds\Big\}\Big\{e^{-rt}B + \int_0^t e^{-rs}L(u(s))\,ds\Big\}\,dt \\ &= \int_0^{T^M} \exp\Big\{-rt - \int_0^t \rho(x(s))\,ds\Big\}\Big\{\rho(x(t))B + L(u(t))\Big\}\,dt\,. \end{aligned} \tag{2.12}$$

Notice that the term $\rho(x(t))\exp\Big\{-\int_0^t \rho(x(s))\,ds\Big\}\,dt$ yields the probability that bankruptcy occurs within the time interval $[t,\, t+dt]$. Moreover

$$\begin{aligned} e^{-rt}B &= \text{discounted bankruptcy cost, at time } t, \\ \int_0^t e^{-rs}L(u(s))\,ds &= \text{total cost of repayments, up to time } t. \end{aligned}$$

Using (2.12), the optimization problem (2.7)-(2.8) can now be reformulated as a deterministic optimal control problem in infinite time horizon, with exponential discount.

$$\text{minimize:} \quad J(u) \doteq \int_0^{T^M} e^{-rt}z(t)\cdot\Big\{\rho(x(t))B + L(u(t))\Big\}\,dt\,, \tag{2.13}$$

for the control system with dynamics and initial conditions

$$\begin{cases} \dot{x}(t) = \alpha(x(t))x(t) - u(t)\,, \\ \dot{z}(t) = -\rho(x(t))\,z(t) \end{cases} \qquad \begin{cases} x(0) = x_0\,, \\ z(0) = z_0 = 1\,. \end{cases} \tag{2.14}$$

In (2.13), the integral ranges up to the time $T^M = T^M(x_0, u(\cdot))$ defined at (2.9).

3 Structure of the optimal feedback control

Aim of this section is to give a detailed description of the value function and of the optimal feedback control, for the infinite horizon optimization problem **(OCP)**. The main result is summarized in Fig. 1. Calling $W(y)$ the expected cost of the strategy that keeps the debt constantly equal to y, we prove that the value function admits the representation

$$V(x) = \min_{y\in\mathcal{A}} Z(x,y)\,,$$

where $\mathcal{A}$ is a suitable subset of $[0, M]$ and $Z(\cdot, y)$ is the cost of a suitable strategy that steers the debt asymptotically to the value y. Each $Z(\cdot, y)$ can be found by solving a specific ODE, with boundary values $Z(y, y) = W(y)$ if $y < M$ and $Z(M, y) = B$ if $y = M$. In a generic

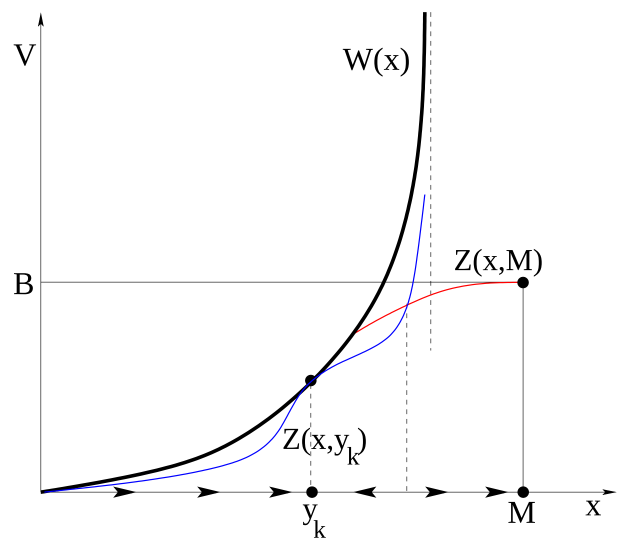

Figure 1: The value function V for the optimization problem (2.7)-(2.8) is obtained as the minimum of solutions $Z(\cdot, y)$ to suitable ODEs. Here $Z(\cdot, y)$ is the cost of a strategy steering the debt asymptotically to the value y (or producing bankruptcy in finite time, if $y = M$).

situation, the set $\mathcal{A}$ contains finitely many points. Hence the problem is reduced to the solution of a small number of ODEs.

Call $\widetilde{V}(x_0, z_0)$ the value function for the problem (2.13)–(2.14). Then $\widetilde{V}$ provides a viscosity solution to the Hamilton-Jacobi equation

$$r\,\widetilde{V}(x,z) \;=\; \min_{u\in[0,1]} \Big\{ \widetilde{V}_x(x,z)\;\cdot (\alpha(x)\,x-u) - \widetilde{V}_z(x,z)\;\rho(x)z + z\Big(\rho(x)B + L(u)\Big)\Big\}\,. \qquad (3.1)$$

Observing that the above equations are linear homogeneous w.r.t. the variable z, we can introduce the reduced value function

$$V(x) \;\doteq\; \widetilde{V}(x,1)\,. \qquad (3.2)$$

Using the identities

$$\widetilde{V}(x,z) \;=\; z\,\widetilde{V}(x,1) \;=\; z\,V(x), \qquad\qquad \begin{cases} \widetilde{V}_x(x,z) &= \; z\,V_x(x)\,, \\[2mm] \widetilde{V}_z(x,z) &= \; V(x)\,, \end{cases}$$

from (3.1) we obtain

$$V(x) \;=\; \frac{1}{r+\rho(x)}\cdot H(x, V'(x)), \qquad (3.3)$$

where

$$H(x,\xi) \;=\; \min_{\omega\in[0,1]} \Big\{ L(\omega) - \xi\cdot\omega + \xi\cdot\alpha(x)x + \rho(x)B\Big\}\,. \qquad (3.4)$$

The corresponding optimal feedback control is

$$U(\xi) \;=\; \arg\min_{\omega\in[0,1]} \;\{L(\omega) - \xi\cdot\omega\}. \qquad (3.5)$$

Observe that, for each $x > 0$, the map $\xi \mapsto H(x,\xi)$ is the minimum of a family of linear functions, hence it is concave down. Moreover, if $\alpha(x)\,x < 1$, then

$$\lim_{\xi\to\infty} H(x,\xi) \;=\; -\infty.$$

Differentiating w.r.t. ξ one obtains

$$H_\xi(x,\xi) \;=\; \frac{\partial}{\partial\xi}\Big(L(U(\xi)) - \xi\cdot U(\xi) + \xi\cdot\alpha(x)x + \rho(x)\,B\Big) \;=\; \alpha(x)\,x - U(\xi)\,. \tag{3.6}$$

Therefore, the map $\xi \mapsto H(x,\xi)$ attains its maximum at a point $\xi^\sharp(x)$ satisfying

$$0 \;=\; H_\xi(x,\xi^\sharp(x)) = \;\alpha(x)\,x - U(\xi^\sharp(x))\,. \tag{3.7}$$

By the optimality condition $L'(U(\xi)) = \xi$, this yields

$$\xi^\sharp(x) \;=\; L'\big(U(\xi^\sharp(x))\big) \;=\; L'(\alpha(x)x). \tag{3.8}$$

To compute the value function $V(\cdot)$, we first consider the functions

$$H^{max}(y) \;\doteq\; \max_\xi H(y,\xi)\,, \qquad\qquad W(y) \;\doteq\; \frac{1}{r+\rho(y)}\cdot H^{max}(y).$$

Notice that $W(y)$ is expected cost achieved by the (constant in time) control $u^\sharp(y) = \alpha(y)\,y$ which keeps the debt constant: $x(t)\equiv y$ for all $t \geq 0$. Using (2.12), we obtain

$$W(y) \;=\; \int_0^\infty e^{-(r+\rho(y))t}\Big[\rho(y)\,B + L(\alpha(y)y)\Big]dt \;=\; \frac{1}{r+\rho(y)}\cdot\Big[\rho(y)\,B + L\big(\alpha(y)y\big)\Big]\,. \tag{3.9}$$

The function W is defined on a maximal interval $[0, x^*[$, where $x^* \in\,]0, M[$ is implicitly defined by the identity $\alpha(x^*)x^* = 1$. Moreover, $W(y) \to \infty$ as $x \to x^*-$. For convenience, we also define $W(y) \doteq +\infty$ for $y \geq x^*$.

We regard (3.3) as an implicit ODE for the function V. For every $x \in [0, x^*[$, when $V > W(x)$, this equation does not determine any value of V'. On the other hand, when

$$\rho(x)B \;\leq\; (r+\rho(x))V \;\leq\; H^{max}(x)$$

the implicit ODE is equivalent to the differential inclusion

$$V' \;\in\; \Big\{F^-(x,V)\,,\;\; F^+(x,V)\Big\}\,, \tag{3.10}$$

where $V' = F^-$ and $V' = F^+$ are the two solutions of the equation (3.3).

- The value $V' = F^+(x,V) \geq \xi^\sharp(x)$ corresponds to the choice of an optimal control such that $\dot{x} < 0$.
- The value $V' = F^-(x,V) \leq \xi^\sharp(x)$ corresponds to the choice of an optimal control such that $\dot{x} > 0$.

For any $y_0 \in [0, x^*[$, we consider the Cauchy problem

$$Z'(x) \;=\; \begin{cases} F^+(x,Z) & \text{if} \quad y_0 < x < x^*, \\ \\ F^-(x,Z) & \text{if} \quad 0 \le x < y_0\,, \end{cases} \qquad\qquad Z(y_0) \;=\; W(y_0). \tag{3.11}$$

We shall denote by $x \mapsto Z(x, y_0)$ the solution to this Cauchy problem. Since the functions $(x, V) \mapsto F^{\pm}(x, V)$ are smooth for $V < W(x)$ but only Hölder continuous w.r.t. V near the curve where $V = W(x)$, and not defined for $V > W(x)$, the definition of $Z(\cdot, y_0)$ requires some care.

For $\varepsilon > 0$, we denote by Z_ε the solution to the ODE in (3.11) with initial datum

$$Z_\varepsilon(y_0) \;=\; W(y_0) - \varepsilon\,.$$

This solution is uniquely defined on a maximal interval $[a_\varepsilon(y_0),\, b_\varepsilon(y_0)]$, where $a_\varepsilon, b_\varepsilon$ are points where $Z_\varepsilon = W$. By a comparison argument, for any $0 < \varepsilon < \varepsilon'$ we have

$$Z_{\varepsilon'}(\cdot, y_0) \;\le\; Z_\varepsilon(\cdot, y_0), \qquad\qquad a_{\varepsilon'}(y_0) \;\le\; a_\varepsilon(y_0) \;\le\; y_0 \le\; b_\varepsilon(y_0) \;\le\; b_{\varepsilon'}(y_0)\,. \tag{3.12}$$

We can thus define the lower solution to the Cauchy problem (3.11) by setting

$$Z(x, y_0) \;=\; \lim_{\varepsilon \to 0+} Z_\varepsilon(x, y_0)\,, \qquad\qquad x \in \left[\sup_{\varepsilon > 0} a_\varepsilon(y_0)\;,\quad \inf_{\varepsilon > 0} b_\varepsilon(y_0)\right]\,.$$

If the initial size of the debt is $\bar{x} \in [a(y_0), b(y_0)]$, we think of $Z(\bar{x}; y_0)$ as the expected cost achieved by the feedback strategy

$$u^{y_0}(x) \;\doteq\; \arg\min_{\omega \in [0,1]} \left\{ L(\omega) - \frac{\partial Z(x, y_0)}{\partial x} \cdot \omega \right\}. \tag{3.13}$$

With this strategy, the debt has the asymptotic behavior $x(t) \to y_0$ as $t \to \infty$.

Adopting the convention that

$$Z(x, y_0) \;\doteq\; W(x) \qquad \text{if} \qquad x \notin [a(y_0), b(y_0)],$$

the map $(x, y_0) \mapsto Z(x, y_0) \in [0, +\infty]$ is then lower semicontinuous.

Next, we consider the equation (3.3) for $x \in [x^*, M[$. Notice that in this case the borrower has no way to reduce the debt, and bankruptcy must occur within a finite time.

When $x \in [x^*, M[$, the map $\xi \;\mapsto\; H(x, \xi)$ is increasing and satisfies

$$H(x, 0) \;=\; \rho(x) B\,, \qquad\qquad \lim_{\xi \to \infty} H(x, \xi) \;=\; +\infty.$$

Therefore, for every $V \ge \frac{\rho(x)\,B}{r + \rho(x)}$, there exists a unique value $F^-(x, V)$ such that

$$V \;=\; \frac{1}{r + \rho(x)} \cdot H(x, F^-(x, V)).$$

We observe that

$$\lim_{x \to M-} F^-(x, B) \;=\; 0.$$

For $x \in [x^*, M[$, the implicit equation (3.3) is equivalent to the ODE

$$V'(x) \;=\; F^-(x, V(x)). \tag{3.14}$$

Observing that the function $V \mapsto F^-(x, V)$ is uniformly Lipschitz continuous, the Cauchy problem

$$V'(x) \;=\; F^-(x, V(x))\,, \qquad\qquad V(M) \;=\; B \tag{3.15}$$

has a unique backward solution, defined on some maximal interval $[a(M), M[$. This solution will be denoted by $x \mapsto Z(x; M)$.

Concatenations of F^+-solutions and F^--solutions provide admissible viscosity solutions as long as, at each point y where V' is discontinuous, one has

$$V'(y-) \;\geq\; V'(y+). \tag{3.16}$$

In other words, trajectories should not tend to the point y with positive speed from both sides.

For any initial datum x_0 (and $z_0 = 1$), it is clear that the minimum cost in (2.13)-2.14) cannot be greater than B. Indeed, the trivial control $u(t) \equiv 0$ achieves a cost $\leq B$.

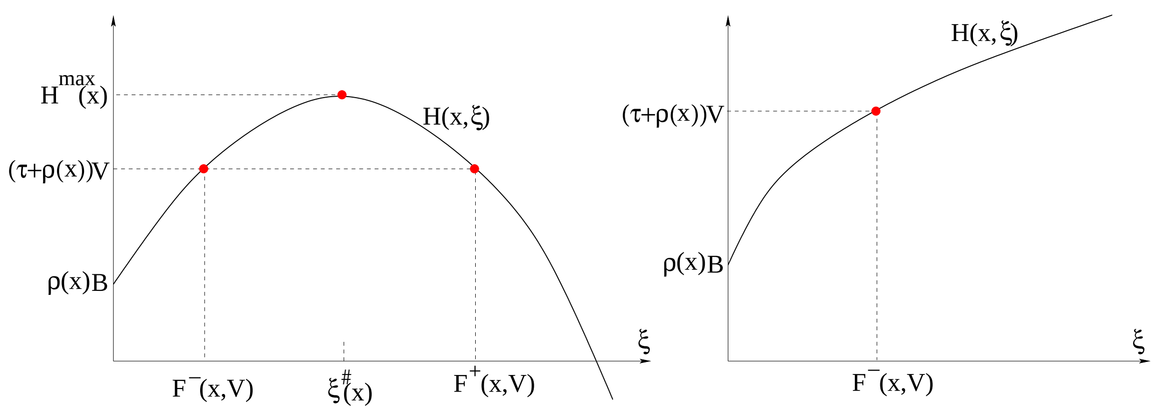

Figure 2: Left: the case where $x < x^*$. For $(r + \rho(x))V > H^{max}(x)$ the equation (3.3) has no solution. At a point where $(r + \rho(x))V < H^{max}(x)$, it determines two distinct values F^-, F^+ for V'. Right: the case where $x > x^*$. For any $(r + \rho(x))V > \rho(x)B$, the equation (3.3) determines a unique solution $V' = F^-(x, V)$.

The main result in this section characterizes the value function for the optimization problem (OCP). We recall that $Z(\cdot; y_0)$ is the solution of (3.11), defined on a maximal interval $[a(y_0),\, b(y_0)[$.

Theorem 1. *Let the functions α, ρ, L satisfy the assumptions (A1) together with (2.11). Let W be the function in (3.9) and consider the set*

$$\mathcal{A} \;\doteq\; \Big\{ y \in [0, M]\,;\;\; W'(y) \;=\; L'(\alpha(y)\, y) \Big\} \cup \{0, M\}. \tag{3.17}$$

Then the value function for the optimization problem (2.7)-(2.8) is given by

$$V(x) \;=\; \min_{y_0 \in \mathcal{A}} Z(x; y_0). \tag{3.18}$$

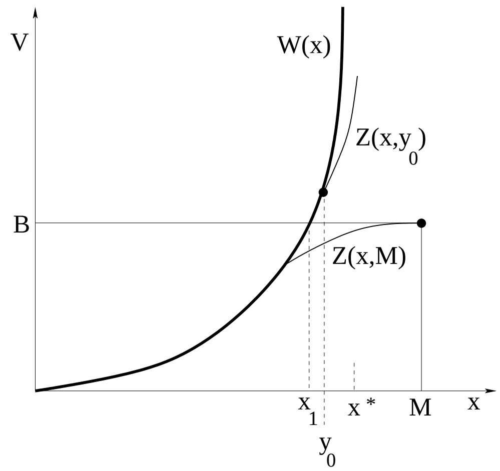

Figure 3: For $y_0 > x_1$ and every $x \in [y_0, M]$ we have $Z(x, y_0) \geq Z(x, M)$.

Proof. 1. As in Fig. 3, let $x_1 \in]0, x^*[$ be the point where

$$W(x_1) \;=\; B\,.$$

Consider the function

$$Z_*(x) \;\doteq\; \inf_{y_0 \in [0, x^*[\, \cup \{M\}} Z(x; y_0)\,, \tag{3.19}$$

we claim that

$$Z_*(x) \;=\; \min_{y_0 \in [0, x_1] \cup \{M\}} Z(x; y_0)\,. \tag{3.20}$$

Indeed, by the lower semicontinuity of the map $(x, y_0) \mapsto Z(x, y_0)$, for every $x \in [0, M]$ the minimum in (3.20) exists.

If $Z_*(x) = Z(x, y_0)$ for some $y_0 > x_1$, then two cases can arise

- If $x \geq y_0 > x_1$ then

$$Z(x, y_0) \;\geq\; Z(y_0, y_0) \;=\; W(y_0) \;> B \;\geq Z(x, M).$$

- If $x < y_0$, then again $Z(x, y_0) \geq Z(x, M)$ because the maps $Z(\cdot, y_0)$ and $Z(\cdot, M)$ are solutions to the same ODE, namely $\dot{Z} = F^-(x, Z)$. Moreover, at $x = y_0$ one has $Z(y_0, y_0) = W(y_0) > M \geq Z(y_0, M)$.

This proves (3.20).

2. Fix $x_0 \in [0, M]$, we claim that

$$Z_*(x_0) \;=\; V(x_0)\,. \tag{3.21}$$

For any $y_0 \in [0, x_1] \cup M$ such that $x_0 \in [a(y_0), b_(y_0)]$, $Z(x_0, y_0)$ is the expected cost achieved by a feedback strategy for which the debt has the asymptotic behavior $x(t) \to y_0$ as $t \to \infty$, with initial data $x(0) = x_0$. Hence $Z(x_0, y_0) \geq V(x_0)$. This implies

$$Z_*(x_0) \;=\; \min_{y_0 \in [0,x_1]\cup\{M\}} Z(x; y_0) \;\geq\; V(x_0)\,.$$

It now remains to show that $Z_*(x_0) \;\leq\; V(x_0)$. For any control $t \mapsto u(t) \in [0,1]$, let $x(\cdot)$ be the solution of (2.8) associated with control $u(t)$ and initial debt x_0, and let $T^M \in [0, +\infty]$ be the first time when the debt reaches the value M, as in (2.9). We claim that

$$Z_*(x_0) \;\leq\; \int_0^{T^M} \exp\left\{-rt - \int_0^t \rho(x(s))\,ds\right\}\left\{\rho(x(t))B + L(u(t))\right\} dt\,. \tag{3.22}$$

For every $\tau \in [0, T^M]$, define

$$\begin{aligned}\Phi^u(\tau) \;\doteq\; & \int_0^\tau \exp\left\{-rt - \int_0^t \rho(x(s))\,ds\right\}\left\{\rho(x(t))B + L(u(t))\right\} dt \\ & + \exp\left\{-r\tau - \int_0^\tau \rho(x(s))\,ds\right\} \cdot Z_*(x(\tau))\,.\end{aligned}$$

Observe that Z_* is uniformly Lipschitz in $[0, M]$, while the map $\tau \mapsto \Phi^u(\tau)$ is absolutely continuous.

At any point $x \in [0, M]$ where Z_* is differentiable, one has

$$(r + \rho(x)) \cdot Z_*(x) \;=\; H(x, Z_*'(x))\,.$$

By (3.4) this implies

$$(r + \rho(x)) \cdot Z_*(x) \;\leq\; L(u) + \rho(x)B + Z_*'(x) \cdot (\alpha(x)x - u) \qquad \text{for all } u \in [0,1]\,. \tag{3.23}$$

We can now find a set $\mathcal{N} \subset [0, T^M]$ of measure zero such that, for every $\tau \notin \mathcal{N}$,

(i) τ is a Lebesgue point of $u(\cdot)$,

(ii) either $\dot{x}(\tau) = 0$ or else Z_* is differentiable at $x(\tau)$.

For $\tau \in [0, T^M] \setminus \mathcal{N}$, we compute

$$\begin{aligned}\frac{d}{d\tau}\,\Phi^u(\tau) \;=\; & \exp\left\{-r\tau - \int_0^\tau \rho(x(s))\,ds\right\} \cdot \Big\{\rho(x(\tau))B + L(u(\tau)) \\ & + Z_*'(x(\tau)) \cdot (\alpha(x(\tau))x(\tau) - u(\tau)) - (r + \rho(x(\tau)) \cdot Z_*(x(\tau))\Big\} \;\geq\; 0\,.\end{aligned}$$

Therefore the map $\tau \mapsto \Phi^u(\tau)$ is non-decreasing on $[0, T^M]$. In particular,

$$\begin{aligned}Z_*(x_0) \;=\; \Phi^u(0) \;\leq\; & \lim_{\tau \to T^M-} \Phi^u(\tau) \\ =\; & \int_0^{T^M} \exp\left\{-rt - \int_0^t \rho(x(s))\,ds\right\}\left\{\rho(x(t))B + L(u(t))\right\} dt\,.\end{aligned} \tag{3.24}$$

Indeed, since Z_* is uniformly bounded, in both cases $T^M < +\infty$ or $T^M = +\infty$ by the second assumption in (2.2) together with (2.11) one has

$$\lim_{\tau \to T^M -} \exp\left\{-r\tau - \int_0^\tau \rho(x(s))\,ds\right\} \cdot Z_*(x(\tau)) \;=\; 0\,.$$

This proves our claim (3.22).

3. To complete the proof, we show that the minimum in (3.20) is attained for some $y_0 \in \mathcal{A}$. Denote by $V_\mathcal{A}$ the right hand side of (3.18). By the definitions of $Z_*, V_\mathcal{A}$ and by the previous analysis, it is clear that

$$Z_*(x) \;\le\; V_\mathcal{A}(x) \qquad\qquad \text{for all } x \in [0, M],$$

$$Z_*(x) \;=\; V_\mathcal{A}(x) \;=\; Z(x, M) \qquad\qquad \text{for all } x \in [x_1, M]\,.$$

To prove the identity $Z_* = V_\mathcal{A}$, it thus remains to show that

$$Z_*(x) \;\ge\; V_\mathcal{A}(x) \quad \text{for all } x \in [0, x_1]. \tag{3.25}$$

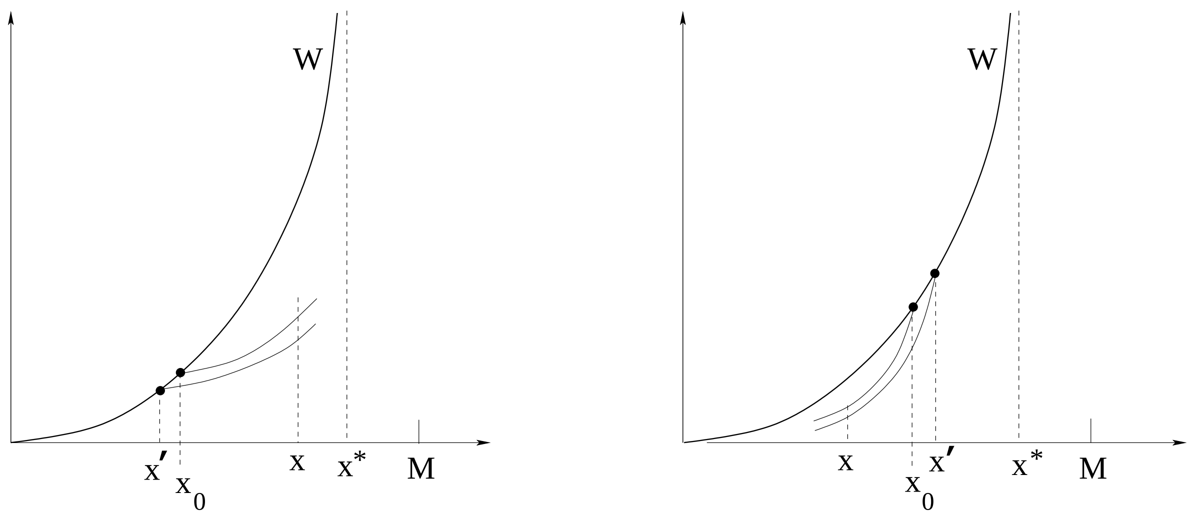

Figure 4: Proving that the minimum in (3.20) cannot be attained when $y_0 \notin \mathcal{A}$. Left: the case $W'(y_0) > \xi^\sharp(y_0)$. Right: the case $W'(y_0) < \xi^\sharp(y_0)$.

Assume $Z_*(\bar{x}) = Z(\bar{x}; y_0)$ for some $y_0 \in [0, x_1]$, $y_0 \notin \mathcal{A}$. We claim that this leads to a contradiction. Two cases will be considered.

CASE 1: $W'(y_0) \;>\; \xi^\sharp(y_0) \;=\; L'(\alpha(y_0)\,y_0)$. Then $Z(\cdot, y_0)$ is defined only for $x \ge y_0$. By continuity, we can find $x' < y_0$ sufficiently close to y_0 such that $Z(\cdot; x')$ is defined on $[x', \bar{x}]$ and

$$Z(y_0; x') \;<\; Z(y_0, y_0) \;=\; W(y_0).$$

Since $Z(\cdot; y_0)$ and $Z(\cdot; x')$ satisfy the same ODE, namely $Z' = F^+(x, Z)$, with different initial data, this implies

$$Z(x; x') \;<\; Z(x; y_0)., \quad \text{for all } x \in [y_0, \bar{x}]\,.$$

In particular, $Z(\bar{x}; x') < Z(\bar{x}; y_0) = V(\bar{x})$, reaching a contradiction.

CASE 2: $W'(y_0) \; < \; \xi^\sharp(y_0) \; = \; L'(\alpha(y_0)\, y_0)$. Then $Z(\cdot; y_0)$ is defined only for $x \le y_0$. By continuity, we can find $x' > y_0$ sufficiently close to y_0 such that $Z(\cdot; x')$ is defined on $[\bar x,\, x',]$ and

$$Z(y_0; x') \; < \; Z(y_0; y_0) \; = \; W(y_0).$$

Since $Z(\cdot; y_0)$ and $Z(\cdot; x')$ satisfy the same ODE, namely $Z' = F^-(x, Z)$, with different initial data, this implies

$$Z(x; x') \; < \; Z(x; y_0) \quad \text{for all} \quad x \in [\bar x, y_0]\,.$$

In particular, $Z(\bar x; x') < Z(\bar x; y_0) = V(\bar x)$, reaching a contradiction.

The previous arguments show that (3.25) holds, completing the proof. □

4 A game-theoretical model

In the previous model, both the instantaneous bankruptcy risk $\rho(x)$ and the interest rate $\alpha(x)$ were functions of the total debt x, given a priori. We now consider a model where only the bankruptcy risk is assigned a priori, while the interest rate is determined endogenously, a result of the competition among several lenders.

In an accurate description of a debt resulting from several different loans, one should keep track of (i) the size, (ii) the interest rate, and (iii) the expiration date of the each loan. In a continuum model, the state of the system at any time t should thus be described by a density function $\phi = \phi(I, \tau)$, where

$$\int_{I_1}^{I_2} \int_{\tau_1}^{\tau_2} \phi(I, \tau)\, dI d\tau$$

yields the total amount of loans at interest rate $I \in [I_1\,,\, I_2]$, expiring during the time interval $[\tau_1\,, \tau_2]$. This would lead to a very complicated optimization problem, where the current state of the system is described by a density function depending on two variables: $\phi = \phi(I, \tau)$.

We consider here a highly simplified model, where at any given time the "state" of the system is described by two scalar numbers: the total amount of debt X and the average interest rate A payed on this debt.

To achieve a consistent model, we assume that in every loan the principal is payed back at a fixed exponential rate λ. In other words, if at time t_0 the borrower receives a loan in the amount K, by time $t > t_0$ the remaining size of the loan will be $e^{-\lambda(t-t_0)}K$. At all times $t > t_0$, the borrower pays a running interest I on the remaining part $e^{-\lambda(t-t_0)}K$ of the loan. This interest rate $I = I(t_0)$ is negotiated at the initial time and never changes.

We claim that this model yields a closed system of ODEs for the two state variables X, A, independent of the detailed structure ϕ of the debt. Indeed, let $X(t)$ is the total amount of debt at time t and let $u(t)$ the instantaneous payment rate, regarded as the control function chosen by the borrower. Then the total amount of debt satisfies

$$\dot X(t) \; = \; A(t)X(t) - u(t)\,. \tag{4.1}$$

If $u(t) = (A(t) + \lambda)X(t)$, then no new loans are initiated at time t, and the average rate payed on the overall debt remains constant. On the other hand, if $u(t) < (A(t) + \lambda)X(t)$, then the difference must be covered by new loans, at the current interest rate $I(t)$. The average interest rate payed on the entire debt thus evolves according to

$$\frac{d}{dt}A(t) \;=\; \lim_{\varepsilon\to 0}\,\frac{1}{\varepsilon}\cdot\left[\frac{A(X-\varepsilon\lambda X)+\varepsilon I\big[(A+\lambda)X-u\big]}{X+\varepsilon(AX-u)} - A\right].$$

This yields

$$\dot A \;=\; (I-A)\cdot\Big(A+\lambda-\frac{u}{X}\Big)\,. \tag{4.2}$$

Notice that, if $u = AX$ so that the total debt remains constant in time, then the average interest payed on the loan changes only as $\dot A = \lambda(I-A)$. This reflects the fact that the old loans at interest A are replaced with new loans at rate I. However, if $u \neq AX$, then the additional terms on the right hand side of (4.2) account for the change in the average interest, due to the change in the total size of the debt. Throughout the sequel, we assume that

$$u(t) \;\in\; \big[0\,,\;(A(t)+\lambda)X(t)\big]\cap[0,1]\,. \tag{4.3}$$

In other words, the borrower cannot extinguish the existing loans at a rate larger than λ. The maximum value $(\lambda+A)X$ of the control corresponds to the case where no new loans are initiated. On the other hand, it is quite possible to have $u(t) < \lambda X(t)$, or even $u(t) = 0$. In this case, the borrower will always pay the rate $(\lambda + A(t))X(t)$ on previous loans. To make up for the difference $(\lambda + A(t))X(t) - u(t)$ he will simply ask for new loans, which are available at the current interest rate $I(t)$.

As a first step, we assume that the instantaneous interest rate $I = I(t)$ charged on new loans is a given function of time. As in (2.4), we denote by $\rho(X)$ the instantaneous bankruptcy risk, when the total debt is X. For simplicity, we assume that ρ does not depend on the average interest A payed on the loan. In the more general case where $\rho = \rho(X, A)$ the analysis is similar.

The borrower seeks to minimize the expected value of the cost (2.7). As in (2.13)-(2.14) this can be reformulated as a deterministic optimal control problem in infinite horizon, namely

$$\text{minimize:}\quad J(u)\;\doteq\;\int_0^{T^M}\gamma(t)\Big\{\rho(X(t))\,B+L(u(t))\Big\}\,dt\;+\;\gamma(T^M)\,B\,, \tag{4.4}$$

subject to

$$\left\{\begin{array}{rcl}\dot X &=& AX-u\,,\\[2mm] \dot A &=& (I-A)\cdot\big(\lambda+A-\frac{u}{X}\big)\,,\end{array}\right.\qquad\qquad \left\{\begin{array}{rcl}X(0)&=&X_0\,,\\[2mm] A(0)&=&A_0\,.\end{array}\right. \tag{4.5}$$

Here

$$\gamma(t)\;\doteq\;e^{-rt}\exp\left\{-\int_0^t\rho(X(s))\,ds\right\}\,,\qquad\qquad t\in[0,T^M]\,, \tag{4.6}$$

while

$$T^M\;\doteq\;\sup\big\{t\geq 0\,;\;\; X(t)<M\big\} \tag{4.7}$$

is the time where the debt reaches the maximum value M and bankruptcy occurs instantly.

Remark 3. If either $T^M = \infty$ or else $T^M < \infty$ and $\int_0^{T^M}\rho(X(s))\,ds = \infty$, then $\gamma(T^M) = 0$ and the last term in (4.4) vanishes. However, one can also consider that case where $T^M < +\infty$

and $\int_0^{T^M} \rho(X(s))\,ds < \infty$. In this case there is a positive probability that the borrower goes bankrupt exactly at the time T^M when the debt reaches the maximum size M. If this happens, the second term in (4.4) accounts for this bankruptcy cost, exponentially discounted in time. It is important to keep in mind that the bankruptcy time T_b is random, but the time T^M in (4.7) is not random. Indeed, T^M is uniquely determined by the solution to the Cauchy problem (4.5).

Our main concern is how to model the instantaneous interest rate $I(t)$, reflecting the confidence of risk-neutral investors. This rate should be determined by the the future risk of insolvency.

Consider a pool of risk-neutral lenders. Let the constant $\sigma \in [0,1]$ denote the salvage rate. In other words, at the time when bankruptcy occurs, only a fraction σ of the currently invested capital will be returned to lenders. The expected payoff for an agent who at time $t = 0$ lends a unit amount of money at interest $I \geq r$ is then computed by

$$J^\sharp \;=\; E\left[\int_0^{T_b} e^{-rt}(\lambda + I)e^{-\lambda t}dt \;+\; e^{-rT_b}\,\sigma\, e^{-\lambda T_b}\right]. \tag{4.8}$$

Here T_b is the random bankruptcy time. We observe that

- At any time $t \in [0, T_b]$, the remaining size of the loan is $e^{-\lambda t}$. The payment due on this loan is the sum of two terms: $\lambda e^{-\lambda t}$ models the repayment of the principal, while $Ie^{-\lambda t}$ accounts for the interest. As usual, future earnings are discounted by the factor e^{-rt}.
- At the bankruptcy time T_b, the lender receives only a fraction $\sigma \in [0,1]$ of the outstanding loan. Quite possibly $\sigma = 0$. Again, the expression for his payoff contains the discount factor e^{-rT_b}.

As before, we assume that T_b is a random variable, with distribution function

$$P(t) \;\doteq\; Prob.\{T_b > t\} \;=\; \begin{cases} \exp\left\{-\displaystyle\int_0^t \rho(X(\tau))\,d\tau\right\} & \text{if} \quad t < T^M, \\ 0 & \text{if} \quad t \geq T^M. \end{cases} \tag{4.9}$$

Here $t \mapsto (X(t), A(t))$ denotes the solution to the Cauchy problem (4.5). Of course, this also depends on the control $u(\cdot)$ implemented by the borrower.

For a lender who invests a unit amount of money at time $t_0 = 0$ at interest rate I, the expected return (exponentially discounted in time) is computed by

$$\mathcal{R}(I) \;=\; \frac{I+\lambda}{r+\lambda}\,P(+\infty) - \int_0^\infty \left[\int_0^t (I+\lambda)e^{-(r+\lambda)s}\,ds \;+\; \sigma e^{-(r+\lambda)t}\right] dP(t)\,. \tag{4.10}$$

The above expression deserves a few explanations.

- If bankruptcy never occurs, then the (exponentially discounted) payoff for the lender is
$$\int_0^\infty (I+\lambda)e^{-(r+\lambda)t}\,dt \;=\; \frac{I+\lambda}{r+\lambda}\,.$$
Moreover, the probability that bankruptcy never occurs is $P(+\infty) = \lim_{t\to+\infty} P(t)$. This accounts for the first term in (4.10).

- If bankruptcy occurs at time $T_b = t$, the payoff for the lender is computed as

$$\int_0^t (I+\lambda)e^{-(r+\lambda)s}\,ds \,+\, \sigma e^{-(r+\lambda)t} \;=\; \frac{I+\lambda}{r+\lambda} + \left(\sigma - \frac{I+\lambda}{r+\lambda}\right) e^{-(r+\lambda)t}. \tag{4.11}$$

  Moreover, for any $0 \le t_1 < t_2$,

$$\text{Prob.}\bigl\{T_b \in\,]t_1,\, t_2]\bigr\} \;=\; P(t_1) - P(t_2).$$

  This accounts for the Stiltjes integral in (4.10).

The assumption of risk-neutrality of the lenders implies

$$\mathcal{R}(I) \;=\; \frac{I+\lambda}{r+\lambda} + \left(\sigma - \frac{I+\lambda}{r+\lambda}\right)\cdot \int_0^\infty -e^{-(r+\lambda)t}\,dP(t) \;=\; 1\,. \tag{4.12}$$

If the evolution of the system (4.5) is known, this determines the interest rate I to be charged by risk-neutral lenders. Indeed, if the salvage rate is $\sigma = 1$, then by (4.12) the interest rate is simply $I = r$. On the other hand, when $0 \le \sigma < 1$ we must have $I > r$. Setting

$$\beta \;\doteq\; \int_0^\infty -e^{-(r+\lambda)t}\,dP(t) \;=\; 1 - (r+\lambda)\int_0^\infty e^{-(r+\lambda)t}\,P(t)\,dt\,, \tag{4.13}$$

an explicit computation yields

$$\frac{I+\lambda}{r+\lambda}(1-\beta) \;=\; 1 - \sigma\beta\,,$$

$$I \;=\; r + (1-\sigma)(r+\lambda)\frac{\beta}{1-\beta}\,. \tag{4.14}$$

Remark 4. The value of β in (4.13) can be interpreted as follows. Consider an investor lending a unit amount of money. This will be paid back in time at an exponential rate. At the random time T_b when bankruptcy occurs, the expected value of the remaining portion of this loan (exponentially discounted) is then measured by β.

Assuming $0 \le \sigma < 1$, from (4.14) the following limit behavior can be deduced.

- If the expected bankruptcy time approaches infinity, then the lenders feel that their investment is safe, and the interest rate approaches the discount rate:

$$\beta \,\to\, 0 \qquad \Longrightarrow \qquad I \,\to\, r\,.$$

- If the expected bankruptcy time approaches zero, then the lenders regard their investment as highly risky, and the interest rate approaches infinity:

$$\beta \,\to\, 1 \qquad \Longrightarrow \qquad I \,\to\, +\infty\,.$$

According to the previous analysis, by (4.13)-(4.14) at time t the instantaneous interest rate $I(t)$ is computed by

$$I(t) \;=\; r + (1-\sigma)(r+\lambda)\frac{\beta(t)}{1-\beta(t)}\,, \tag{4.15}$$

where

$$\beta(t) \;=\; 1-(r+\lambda)\int_t^{T^M} e^{-(r+\lambda)(\tau-t)}\exp\left\{-\int_t^\tau \rho(X(s))\,ds\right\}\,d\tau\,. \tag{4.16}$$

As in (4.7), $T^M \in]0,+\infty]$ denotes the first time when $X(t)=M$, and bankruptcy instantly occurs.

Differentiating (4.16) w.r.t. time we obtain

$$\begin{aligned}\dot\beta \;&=\; (r+\lambda)-(r+\lambda)\int_t^{T^M} e^{-(r+\lambda)(\tau-t)}\exp\left\{-\int_t^\tau \rho(X(s))\,ds\right\}\cdot\Big(r+\lambda+\rho(X(t))\Big)d\tau\\ &=\; (r+\lambda)+(r+\lambda+\rho(X))\,(\beta-1).\end{aligned} \tag{4.17}$$

5 The optimization problem for the borrower

Aim of this section is to prove the existence of an optimal strategy for the borrower, in open-loop form. Introducing the auxiliary variable

$$Y \;\doteq\; AX\,,$$

the Optimization Problem for the Borrower can be reformulated as follows.

(OPB) *Given an initial size of the debt X_0 and an initial average interest rate A_0, find a control $t\mapsto u(t)$ and a map $t\mapsto (X,Y,I,\beta)(t)$ which minimize the expected cost*

$$J(u,X) \;\doteq\; \int_0^{T^M}\gamma(t)\Big\{\rho(X(t))\,B + L(u(t))\Big\}\,dt \;+\; \gamma(T^M)\,B\,, \tag{5.1}$$

subject to the dynamics

$$\left\{\begin{aligned}\dot X \;&=\; Y-u\,,\\ \dot Y \;&=\; Y(I-\lambda)+I(\lambda X-u)\,,\end{aligned}\right. \qquad \left\{\begin{aligned}X(0) \;&=\; X_0\,,\\ Y(0) \;&=\; A_0X_0\end{aligned}\right. \tag{5.2}$$

and the constraints

$$u(t)\;\in\;[0,1]\,, \qquad\qquad u(t)\;\le\; Y(t)+\lambda X(t)\,, \tag{5.3}$$

$$\beta(t) \;=\; 1-(r+\lambda)\int_t^{T^M} e^{-(r+\lambda)(\tau-t)}\exp\left(-\int_t^\tau \rho(X(s))\,ds\right)\,d\tau\,, \tag{5.4}$$

$$I \;=\; I(\beta) \;\doteq\; r+(1-\sigma)(r+\lambda)\frac{\beta}{1-\beta}\,. \tag{5.5}$$

Here

$$T^M \;\doteq\; \sup\big\{t\ge 0\,;\quad X(t)<M\big\} \;\in\; [0\,,+\infty] \tag{5.6}$$

is the first time where $X(t)=M$ and hence bankruptcy instantly occurs.

The existence of an optimal solution will be proved under the following assumptions.

(A2) *The function ρ is continuously differentiable on $[0, M[$ and satisfies*

$$\rho\,,\rho' \;\geq\; 0 \qquad\qquad \textit{for all } X \in [0, M[\,, \tag{5.7}$$

$$\lim_{X\to M-} \rho(X) \;=\; +\infty. \tag{5.8}$$

The cost function L is twice continuously differentiable for $u \in [0,1[$ and satisfies

$$L(0) = 0, \qquad L' > 0, \quad L'' > 0, \qquad\qquad L(1) \;=\; \lim_{u\to 1-} L(u) \;\in\; I\!R \cup \{+\infty\}. \tag{5.9}$$

Theorem 2. *Let the assumptions* **(A2)** *hold. Then the optimization problem* **(OPB)** *admits an optimal solution.*

Proof. 1. For every initial data (X_0, A_0), the trivial control $u(t) \equiv 0$ yields a total cost $J(u, X) \leq B$, regardless of the trajectory $X(\cdot)$. Indeed, this cannot be worse than the cost of immediate bankruptcy. It thus suffices to prove the theorem assuming that

$$0 \;\leq\; J_{min} \;\doteq\; \inf_{u,X} \; J(u, X) \;<\; B\,. \tag{5.10}$$

Following the direct method in the Calculus of Variations, we consider a sequence of functions $(u_n, X_n, Y_n, \gamma_n, I_n, \beta_n)$ satisfying (5.2)–(5.5), such that

$$J(u_n, X_n) \;\to\; J_{min} \qquad\qquad \text{as} \qquad n \to \infty. \tag{5.11}$$

We will show that, by taking a subsequence, one can achieve the weak convergence $u_n \rightharpoonup u^*$ together with the strong convergence $(X_n, Y_n, \gamma_n, I_n, \beta_n) \to (X^*, Y^*, \gamma^*, I^*, \beta^*)$, uniformly for t in bounded sets. Furthermore, using the convexity of the cost function L and the fact that u enters linearly in the equations (4.5), we will prove that these limit functions provide an optimal solution.

2. We first consider the case where the corresponding times T_n^M in (5.6) satisfy

$$\liminf_{n\to\infty} \; T_n^M \;=\; \infty. \tag{5.12}$$

For every bounded interval $[0, T]$, there exist $\delta, N_0 > 0$ such that

$$T_n^M \;\geq\; T{+}\delta \quad , \quad X_n(t) \;\leq\; M{-}\delta \quad \text{and} \quad \rho(X_n(t)) \;<\; \rho(M{-}\delta)\,, \qquad \text{for all } t \in [0, T], n \geq N_0\,.$$

Hence, from (5.4) and (5.5), one obtains that for all $n \geq N_0$

$$0 \;\leq\; \beta_n(t) \;\leq\; \hat{\beta} < 1 \quad , \quad I_n(t) \;<\; \widehat{I}, \qquad \text{for all } t \in [0, T]\,,$$

for some constants $\hat{\beta}, \widehat{I}$. Hence and β_n, I_n are Lipschitz continuous with some Lipschitz constant C_1 for some constants $\beta_1, L_1 > 0$. Since u_n and I_n are uniformly bounded on $[0, T]$ for all $n \geq N_0$, the ODE (5.2) implies that for all $n \geq N_0$

$$X_n(t),\, Y_n(t) \;\leq\; C_2, \qquad \text{for all } t \in [0, T]\,,$$

and thus X_n and Y_n are Lipschitz continuous with a Lipschitz constant C_3 on $[0,T]$, for some constants $C_2, C_3 > 0$. Thanks to this Lipschitz continuity, by possibly taking a subsequence, we can assume the convergence

$$(X_n, Y_n, \gamma_n, I_n, \beta_n)(t) \;\to\; (X^*, Y^*, \gamma^*, I^*, \beta^*)(t) \qquad \text{as } \; n \to \infty \tag{5.13}$$

uniformly on every bounded interval $[0,\, T]$.

Since u enters linearly in the equations (5.2), by (5.13) and the weak convergence $u_n \rightharpoonup u^*$ it is clear that the functions $t \mapsto (u^*, X^*, Y^*, \gamma^*, I^*, \beta^*)(t)$ satisfy the corresponding equations and constraints in (5.2)–(5.5).

If $J(u^*, X^*) > J_{min}$, then there exists a bounded interval $[0, T]$ such that

$$\int_0^T \gamma^*(t) \Big\{ \rho(X^*(t))\, B + L(u^*(t)) \Big\}\, dt \;>\; J_{min}\,. \tag{5.14}$$

Using the convexity of the cost function L we obtain

$$\begin{aligned} J_{min} \;&<\; \int_0^T \gamma^*(t) \Big\{ \rho(X^*(t))\, B + L(u^*(t)) \Big\}\, dt \\ &\leq\; \liminf_{n\to\infty} \int_0^T \gamma_n(t) \Big\{ \rho(X_n(t))\, B + L(u_n(t)) \Big\}\, dt \\ &\leq\; J_{min} \end{aligned}$$

reaching a contradiction. Hence $J(u^*, X^*) \leq J_{min}$, proving the optimality of u^*.

3. Next, consider the case where

$$T^M \;\doteq\; \liminf_{n\to\infty} T_n^M \;<\; \infty. \tag{5.15}$$

By possibly taking a subsequence, we can assume that $T_n^M \to T^M$, and that the convergence (5.13) holds, uniformly on every subinterval of the form $[0,\, T^M - \delta]$, with $\delta > 0$. As before, one checks that the functions $t \mapsto (u^*, X^*, Y^*, \gamma^*, I^*, \beta^*)(t)$ satisfy the corresponding equations and constraints in (5.2)–(5.5).

If $J(u^*, X^*) \;>\; J_{min}$, then there exists $\delta, \varepsilon > 0$ such that

$$\int_0^{T^M - \delta} \gamma(t) \Big\{ \rho(X(t))\, B + L(u^*(t)) \Big\}\, dt + \gamma(T^M) B \;>\; J_{min} + \varepsilon\,,$$

and $\big(1 - e^{-r\delta}\big) B \;<\; \varepsilon$. Using again the convexity of the cost function L and recalling that

$$\gamma_n(t) \;=\; e^{-rt} \cdot \exp\left\{ - \int_0^t \rho(X_n(s))\, ds \right\},$$

we obtain

$$
\begin{aligned}
J_{min} \;&=\; \liminf_{n\to\infty} \left\{ \int_0^{T_n^M} \gamma_n(t) \Big(\rho(X_n(t))\, B + L(u_n(t)) \Big)\, dt + \gamma_n(T_n^M) B \right\} \\
&\geq\; \int_0^{T^M-\delta} \gamma^*(t) \Big\{ \rho(X^*(t))\, B + L(u^*(t)) \Big\}\, dt + \gamma(T^M-\delta) B \\
&\qquad\qquad + \liminf_{n\to\infty} \left[\int_{T^M-\delta}^{T_n^M} \gamma_n(t)\rho(X_n(t))\, B\, dt + \gamma_n(T_n^M)B - \gamma_n(T^M-\delta)B \right] \\
&\geq\; \int_0^{T^M-\delta} \gamma^*(t) \Big\{ \rho(X^*(t))\, B + L(u^*(t)) \Big\}\, dt + \gamma(T^M-\delta) B \qquad\qquad (5.16) \\
&\qquad\qquad + \liminf_{n\to\infty} \Bigg[- e^{-rT_n^M} \int_{T^M-\delta}^{T_n^M} \frac{d}{dt} \exp\left\{ - \int_0^t \rho(X_n(s))\, ds \right\} B\, dt \\
&\qquad\qquad\qquad\qquad\qquad + \gamma_n(T_n^M)B - \gamma_n(T^M-\delta)B \Bigg] \\
&>\; J_{min} + \varepsilon + \liminf_{n\to\infty} \Big(\gamma_n(T^M-\delta) \cdot \Big[e^{-r\cdot(\delta + T_n^M - T^M)} - 1 \Big] \Big) \\
&\geq\; J_{min} + \varepsilon - (1 - e^{-r\delta}) B \;>\; J_{min}\,.
\end{aligned}
$$

This contradiction shows that $J(u^*, X^*) \leq J_{min}$, completing the proof. □

6 Minimal solutions

In general, for a given initial data X_0 and an open-loop control $t \mapsto u(t)$, the system (5.2)-(5.5) may have several solutions $X(\cdot), Y(\cdot)$. To remove this nonuniqueness, it is natural to look for a "minimal" solution, where for each $t > 0$ the pointwise value $X(t)$ yields the minimum among all admissible solutions. This will make the model deterministic. Toward this goal, we introduce

Definition 1. *Let any open loop control $t \mapsto u(t) \in [0,1]$ be given. We say that $t \mapsto (X(t), Y(t), \beta(t))$ is a* **supersolution** *to (5.2)–(5.5) if there exists a time $T^M \geq 0$ such that the following holds.*

(i) The initial data satisfy $X(0) = X_0$, $Y(0) = A_0 X_0$.

(ii) The functions X, Y, β are absolutely continuous on every compact subinterval of $[0, T^M[$. For a.e. $t \in [0, T^M[$ they satisfy

$$
\begin{cases} \dot X \;\geq\; Y - \hat u\,, \\ \dot Y \;\geq\; -\lambda Y + I(Y + \lambda X - \hat u)\,, \end{cases} \qquad \hat u(t) \;\doteq\; \min\big\{ u(t), Y(t) + \lambda X(t) \big\}\,, \quad (6.1)
$$

$$\beta(t) \;\geq\; 1-(r+\lambda)\int_t^{T^M} e^{-(r+\lambda)(\tau-t)}\exp\left(-\int_t^\tau \rho(X(s))\,ds\right)\,d\tau\,. \tag{6.2}$$

(iii) If $T^M<+\infty$, then

$$\lim_{t\to T^M-}\beta(t) \;=\; 1, \qquad \lim_{t\to T^M-} X(t) \;=\; M \qquad\qquad \left\{\begin{array}{rcl} X(t) &=& +\infty\,,\\ Y(t) &=& +\infty\,,\\ \beta(t) &=& 1\,,\end{array}\right. \qquad \textit{for all}\, t\geq T^M. \tag{6.3}$$

Lemma 1. *Under the assumptions (A2), fix a control $t\mapsto u(t)$ and let (X_1,Y_1,β_1), (X_2,Y_2,β_2) be two supersolutions, satisfying (i)-(iii) with $T^M=T_i^M$, $i=1,2$.*

Then the map $t\mapsto (X(t),Y(t),\beta(t))$ with

$$\left\{\begin{array}{rcl} X(t) &=& \min\,\big\{X_1(t),\,X_2(t)\big\},\\[2mm] Y(t) &=& \min\,\big\{Y_1(t),\,Y_2(t)\big\},\\[2mm] \beta(t) &=& \min\,\big\{\beta_1(t),\,\beta_2(t)\big\},\end{array}\right. \tag{6.4}$$

is also a supersolution.

Proof. 1. For $i=1,2$, let (X_i,Y_i,β_i) be a supersolution, and let (X,Y,β) be as in (6.4).

It is clear that the initial data $X(0)$ and $Y(0)$ satisfy (i). If $T^M=\max\{T_1^M,\,T_2^M\}<+\infty$, it is straightforward to check that the properties (6.3) also hold.

2. To prove (ii), we first compute the derivatives of the right hand sides of (6.1).

$$\frac{\partial}{\partial Y}\Big(Y-\min\big\{u,\,\lambda X+Y,\big\}\Big) \;=\; \left\{\begin{array}{ll} 1 & \text{if}\quad u<\lambda X+Y,\\ 0 & \text{if}\quad u>\lambda X+Y,\end{array}\right. \tag{6.5}$$

$$\frac{\partial}{\partial X}\bigg(Y(I-\lambda)+I\Big(\lambda X-\min\big\{u,\,\lambda X+Y\big\}\Big)\bigg) \;=\; \left\{\begin{array}{ll} \lambda I & \text{if}\quad u<\lambda X+Y,\\ 0 & \text{if}\quad u>\lambda X+Y\,.\end{array}\right. \tag{6.6}$$

In all cases, the right hand sides of (6.5) and (6.6) are non-negative. This quasi-monotonicity property is the key ingredient in the proof of (ii).

3. For $i=1,2$ define

$$\hat u_i(t) \;\doteq\; \min\,\Big\{u(t)\,,\;\lambda X_i(t)+Y_i(t)\Big\}\,.$$

We first check that, for every time t,

$$\begin{aligned}
\beta(t) &= 1-(r+\lambda)\cdot \max_{i=1,2}\left\{\int_t^{T_i^M} e^{-(r+\lambda)(\tau-t)}\exp\left(-\int_t^\tau \rho(X_i(s))\,ds\right)\,d\tau\right\}\\
&\geq \ 1-(r+\lambda)\cdot \int_t^{T^M} e^{-(r+\lambda)(\tau-t)}\exp\left(-\int_t^\tau \min\bigl\{\rho(X_1(s))\,,\ \rho(X_2(s))\bigr\}\,ds\right)\,d\tau\\
&= 1-(r+\lambda)\cdot \int_t^{T^M} e^{-(r+\lambda)(\tau-t)}\exp\left(-\int_t^\tau \rho(X(s))\,ds\right)\,d\tau\,.
\end{aligned} \tag{6.7}$$

Hence (6.2) holds.

4. Finally, we prove the inequalities (6.1). To fix the ideas, assume that $X(t)=X_1(t)<X_2(t)$. Two cases need to be considered.

CASE 1: $Y_1(t)<Y_2(t)$. In this case, observing that $I(t)\leq I_1(t)$, we have

$$\dot X(t) \ = \ \dot X_1(t) \ \geq \ Y_1(t)-\hat u_1(t) \ = \ Y(t)-\hat u(t),$$

$$\begin{aligned}
\dot Y(t) &= \dot Y_1(t) \ \geq \ Y_1(t)(I_1(t)-\lambda)+I_1(t)(\lambda X_1(t)-\hat u(t))\\
&\geq \ Y(t)(I(t)-\lambda)+I(t)(\lambda X(t)-\hat u(t))
\end{aligned}$$

CASE 2: $Y_1(t)>Y_2(t)$. In this case, using (6.5) we obtain

$$\dot X(t) \ = \ \dot X_1(t) \ \geq \ Y_1(t)-\hat u_1(t) \ > \ Y_2(t)-\hat u(t) \ = \ Y(t)-\hat u(t)\,.$$

Next, observing that $I(t)\leq I_2(t)$ and using (6.6), we obtain

$$\begin{aligned}
\dot Y(t) &= \dot Y_2(t) \ \geq \ Y_2(t)(I_2(t)-\lambda)+I_2(t)(\lambda X_2(t)-\hat u(t))\\
&\geq \ Y_2(t)(I_2(t)-\lambda)+I_2(t)(\lambda X_1(t)-\hat u(t)) \ \geq \ Y(t)(I(t)-\lambda)+I(t)(\lambda X(t)-\hat u(t)).
\end{aligned}$$

The borderline cases where $X_1(t)=X_2(t)$ or $Y_1(t)=Y_2(t)$ can be handled by a limiting argument, valid for all times t outside a set of measure zero. This completes the proof. □

The above monotonicity result motivates

Definition 2. *Let an initial data (X_0, A_0) and a control function $t\mapsto u(t)$ be given. The* **minimal solution** $t\mapsto (X_*(t), Y_*(t), \beta_*(t))$ *of the system (5.2)–(5.5) is defined as the pointwise infimum:*

$$X_*(t) \ \doteq \ \inf\ X(t), \qquad Y_*(t) \ \doteq \ \inf Y(t), \qquad \beta_*(t) \ \doteq \ \inf \beta(t)\,, \tag{6.8}$$

where the infimum is taken over all supersolutions $t\mapsto (X(t), Y(t), \beta(t))$.

Using Lemma 1, we now show that the above definition yields a well defined solution to the Cauchy problem (5.2)–(5.5).

Lemma 2. *Let an initial data* (X_0, A_0) *and a control function* $t \mapsto u(t)$ *be given. Then the* **minimal solution** $t \mapsto (X_*(t), Y_*(t), \beta_*(t))$ *is well defined and satisfies all equations in (5.2)–(5.5).*

Proof. 1. Let $T_* = \inf T^M$, where T^M is the supremum of the set of times t where $X(t) < \infty$, for some supersolution $(X(t), Y(t), \beta(t))$. In the first part of the proof we show that there exists a countable sequence of supersolutions (X_n, Y_n, β_n) such that

$$X_*(t) \;=\; \inf_n X_n(t), \qquad Y_*(t) \;=\; \inf_n Y_n(t), \qquad \beta_*(t) \;=\; \inf_n \beta_n(t) \tag{6.9}$$

for all $t \in [0, T_*[\,$.

2. Fix a time $\tau \in [0, T_*[$ and consider a sequence of supersolutions (X_n, Y_n, β_n) such that (6.9) holds at $t = \tau$. By Lemma 1, by possibly replacing each triple of functions (X_n, Y_n, β_n) with

$$(\widetilde{X}_n, \widetilde{Y}_n, \tilde{\beta}_n) \;\doteq\; \left(\min_{1\leq i\leq n} X_n\,, \quad \min_{1\leq i\leq n} Y_n\,, \quad \min_{1\leq i\leq n} \beta_n\right)$$

we can assume that each component the above sequence is monotone decreasing as $n \to \infty$.

In addition, it is not restrictive to assume that, for every $t \in [0, T_n^M[\,$,

$$\beta_n(t) \;=\; 1 - (r+\lambda) \int_t^{T^M} e^{-(r+\lambda)(\tau - t)} \exp\left(-\int_t^\tau \rho(X_n(s))\,ds\right) d\tau\,, \tag{6.10}$$

while the components X_n, Y_n satisfy

$$\left\{\begin{array}{rcl} \dot{X}_n & \in & [Y_n - 1\,,\; Y_n]\,, \\[2mm] \dot{Y}_n & \geq & -\lambda Y_n\,, \end{array}\right. \qquad\qquad \left\{\begin{array}{rcl} X_n(0) & = & X_0\,, \\[2mm] Y_n(0) & = & A_0 X_0\,, \end{array}\right. \tag{6.11}$$

with

$$\hat{u}_n(t) \;=\; \min\left\{u(t)\,,\; \lambda X_n(t) + Y_n(t)\right\}\,. \tag{6.12}$$

Otherwise, calling $\tilde{\beta}_n$ the right hand side of (6.10) we can simply replace (X_n, Y_n, β_n) with the smaller supersolution $(X_n, Y_n, \tilde{\beta}_n)$.

From (6.11) it follows

$$Y_n(t) \;\leq\; e^{(\tau - t)\lambda} Y_n(\tau) \qquad\qquad t \leq \tau.$$

Letting $n \to \infty$, this yields

$$Y_*(t) \;\leq\; e^{(\tau - t)\lambda} Y_*(\tau) \qquad\qquad t \leq \tau. \tag{6.13}$$

Since the above is true for every $\tau < T_*$, we conclude that the functions Y_n, Y_* are uniformly bounded, with bounded variation on every compact subinterval $[0, T] \subset [0, T_*[\,$.

Next, observing that

$$Y_n - 1 \;\leq\; \dot{X}_n \;\leq\; Y_n\,, \qquad\qquad \dot{\beta}_n \;=\; (r + \lambda + \rho(x))\beta_n - \rho(X_n),$$

we conclude that the functions $X_n(\cdot)$ and $\beta_n(\cdot)$ are uniformly Lipschitz continuous on every compact subinterval $[0,T]\subset[0,T_*[$. This implies

$$\begin{aligned} X_*(t) &\geq X_*(\tau)+L\,|t-\tau|,\\ \beta_*(t) &\geq \beta_*(\tau)+L\,|t-\tau|, \end{aligned} \tag{6.14}$$

for some Lipschitz constant L.

Given $T<T_*$, a constant L can be chosen which is uniformly valid for all $\tau,t\in[0,T]$. This implies the Lipschitz continuity of the limit functions X_*,β_* restricted to any subinterval $[0,T]$:

$$\begin{aligned} |X_*(t)-X_*(\tau)| &\leq L\,|t-\tau|,\\ |\beta_*(t)-\beta_*(\tau)| &\leq L\,|t-\tau|, \end{aligned} \tag{6.15}$$

for all $\tau,t\in[0,T]$.

3. We now consider a decreasing sequence of supersolutions $(X_n,Y_n,\beta_n)_{n\geq1}$ which converges to (X_*,Y_*,β_*) at every rational time and at every time τ where the BV function Y_* is discontinuous.

We claim that the identities in (6.9) hold for every $t\in[0,T_*[$. Indeed, assume on the contrary that

$$Y_*(\tau)\ <\ \inf_n\ Y_n(\tau)-\varepsilon \tag{6.16}$$

for some irrational time τ where Y_* is continuous and some $\varepsilon>0$. Choose a rational time $t>\tau$ such that

$$Y_*(t)\ <\ Y_*(\tau)+\frac{\varepsilon}{2}\,,\qquad\qquad e^{(t-\tau)\lambda}\Big[Y_*(\tau)+\frac{\varepsilon}{2}\Big]\ <\ Y_*(\tau)+\varepsilon\,. \tag{6.17}$$

We then have

$$\lim_{n\to\infty}\ Y_n(\tau)\ \leq\ \lim_{n\to\infty}e^{(t-\tau)\lambda}Y_n(t)\ =\ e^{(t-\tau)\lambda}Y_*(t)\ <\ e^{(t-\tau)\lambda}\Big[Y_*(\tau)+\frac{\varepsilon}{2}\Big]\ <\ Y_*(\tau)+\varepsilon\,,$$

in contradiction with (6.16).

By assumption, the limits

$$X_*(t)\ =\ \lim_{n\to\infty}\ X_n(t),\qquad\qquad \beta_*(t)\ =\ \lim_{n\to\infty}\ \beta_n(t)$$

hold at every rational time t. By the uniform Lipschitz continuity of $X_n,\beta_n,\ X_*,\beta_*$ on every compact subinterval $[0,T]\subset[0,T_*[$, these limits remain valid for every $t\in[0,T_*[$.

4. Concerning $X_*(\cdot)$, we already know that this function is locally Lipschitz continuous, hence absolutely continuous on every compact subinterval $[0,T]\subset[0,T_*[$. Moreover, the function $Y_*(\cdot)$, satisfies (6.13) and has bounded variation of every compact subinterval $[0,T]\subset[0,T_*[$. We claim that Y_* is locally Lipschitz continuous. If not, we could find sequences of times $a_k<b_k$ with

$$\lim_{k\to\infty}\ a_k\ =\ \lim_{n\to\infty}\ b_k\ =\ \bar t\ \in[0,T_*[$$

such that

$$Y_*(b_k)-Y_*(a_k)\ >\ k(b_k-a_k)$$

for all $k \geq 1$. Recalling (6.9), for any fixed $k \geq 1$ there exists $n_k \geq 1$ such that

$$\max\left\{\left|Y_{n_k}(a_k) - Y_*(a_k)\right|,\ \left|Y_{n_k}(b_k) - Y_*(b_k)\right|\right\} \ \leq\ \frac{k}{4}\cdot(b_k - a_k)\,. \tag{6.18}$$

We let $t \mapsto (X^{a_k}(t), Y^{a_k}(t))$ be absolutely continuous functions such that

$$\begin{cases} X^{a_k}_{n_k}(t) &= X_{n_k}(t), \\[2mm] Y^{a_k}_{n_k}(t) &= Y_{n_k}(t), \end{cases} \qquad \text{for}\quad t \in [0, a_k]\,,$$

while, for $t \geq a_k$, the functions X^{a_k}, Y^{a_k} provide solutions to the system of ODEs

$$\begin{cases} \dot{X} &= Y - \hat{u}\,, \\[2mm] \dot{Y} &= -\lambda Y + I(\beta_{n_k})(Y + \lambda X - \hat{u})\,. \end{cases}$$

The comparison argument yields

$$Y^{a_k}_{n_k}(t) \ \leq\ Y_{n_k}(t) \quad \text{and} \quad X^{a_k}_{n_k}(t) \ \leq\ X^{a_k}_{n_k}(t), \qquad \text{for all } t \geq 0\,.$$

and thus

$$\beta_k(t) \ \geq\ 1 - (r+\lambda)\int_t^{T^M_k} e^{-(r+\lambda)(\tau - t)} \exp\left(-\int_t^{\tau} \rho(X^{a_k}_{n_k}(s))\, ds\right) d\tau\,.$$

where

$$T^M_k \ =\ \inf\ \left\{t > 0 \ \middle|\ X^{a_k}_{n_k}(t) \ =\ M\right\}.$$

Therefore, the triple $\left(X^{a_k}_{n_k}, Y^{a_k}_{n_k}, \beta_k\right)$ is a supersolution of (5.2)–(5.5).

On the other hand, since Y_n is uniformly bounded on every compact subinterval $[0,T] \in [0, T_*[$, it holds

$$\dot{X}^{a_k}_{n_k}(t) \ \leq\ C, \quad \text{for all } k, t > 0$$

for some constant $C > 0$. Hence, by (6.18), for k large enough we obtain

$$Y^{a_k}_{n_k}(b_k) \ <\ Y_*(b_k),$$

contradicting the minimality of Y_*. Together with (6.13), this proves the Lipschitz continuity of Y_*.

5. Using the fact that every (X_n, Y_n, β_n) is a supersolution of (5.2)–(5.5), we now prove that (X_*, Y_*, β_*) is a supersolution as well. Indeed, for every $0 \leq t_1 < t_2 < T_*$ we have

$$\begin{aligned} X_*(t_2) - X_*(t_1) \ &=\ \lim_{n\to\infty}\big(X_n(t_2) - X_n(t_1)\big) \\ &\geq\ \limsup_{n\to\infty} \int_{t_1}^{t_2} \Big(Y_n(s) - \min\big\{u(s),\ \lambda X_n(s) + Y_n(s)\big\}\Big)\, ds \\ &=\ \int_{t_1}^{t_2} \Big(Y_*(s) - \min\big\{u(s),\ \lambda X_*(s) + Y_*(s)\big\}\Big)\, ds\,. \end{aligned} \tag{6.19}$$

Similarly,

$$\begin{aligned}
Y_*(t_2) - Y_*(t_1) &= \lim_{n\to\infty}\big(Y_n(t_2) - Y_n(t_1)\big) \\
&\geq \limsup_{n\to\infty} \int_{t_1}^{t_2} \Big(Y_n(s)(I_n(s)-\lambda) + I_n(s)\Big(\lambda X_n(s) - \min\big\{u(s),\, \lambda X_n(s)+Y_n(s)\big\}\Big)\Big)\, ds \\
&= \int_{t_1}^{t_2} \Big(Y_*(s)(I_*(s)-\lambda) + I_*(s)\Big(\lambda X_*(s) - \min\big\{u(s),\, \lambda X_*(s)+Y_*(s)\big\}\Big)\Big)\, ds.
\end{aligned} \tag{6.20}$$

Finally, for $t < T_*$,

$$\begin{aligned}
\beta_*(t) &= \lim_{n\to\infty} \beta_n(t) \;\geq\; 1 - (r+\lambda) \liminf_{n\to\infty} \int_t^{T_n^M} e^{-(r+\lambda)(\tau - t)} \exp\left(-\int_t^\tau \rho(X_n(s))\, ds\right) d\tau \\
&= 1 - (r+\lambda) \int_t^{T^M} e^{-(r+\lambda)(\tau-t)} \exp\left(-\int_t^\tau \rho(X_*(s))\, ds\right) d\tau \;\doteq\; \beta^\dagger(t)\,.
\end{aligned} \tag{6.21}$$

Together, (6.19)–(6.21) show that (X_*, Y_*, β_*) is a supersolution.

6. Finally, we prove that (X_*, Y_*, β_*) is indeed a solution.

Let $\beta^\dagger$ be the right hand side of (6.21). Then the triple $(X_*, Y_*, \beta^\dagger)$ is still a supersolution. By minimality, this implies $\beta_*(t) = \beta^\dagger(t)$ for all $t < T_*$.

As in step 4, for every $\tau \in [0, T_*[$ we let $t \mapsto (X^\tau(t), Y^\tau(t))$ be absolutely continuous functions such that

$$\left\{\begin{aligned} X^\tau(t) &= X_*(t), \\ Y^\tau(t) &= Y_*(t), \end{aligned}\right. \qquad \text{for} \quad t \in [0, \tau]\,, \tag{6.22}$$

while, for $t \geq \tau$, the functions X^τ, Y^τ provide solutions to the system of ODEs

$$\left\{\begin{aligned} \dot X &= Y - \hat u\,, \\ \dot Y &= -\lambda Y + I(\beta^\dagger)(Y + \lambda X - \hat u)\,. \end{aligned}\right. \tag{6.23}$$

The triple $(X^\tau(t), Y^\tau(t), \beta^\dagger(t))$ is a supersolution of (5.2)–(5.5).

Assume that there exists a time τ which is a Lebesgue point for the function $u(\cdot)$, and such that

$$\liminf_{h\to 0+} \frac{X_*(\tau+h) - X_*(\tau)}{h} \;>\; Y_*(\tau) - \hat u(\tau) + \varepsilon\,,$$

for some $\varepsilon > 0$. Then for $t > \tau$ with $t - \tau$ sufficiently small we have $X^\tau(t) < X_*(t)$, against the minimality of X_*.

Finally, assume that there exists a time τ which is a Lebesgue point for the function $u(\cdot)$, and such that

$$\liminf_{h\to 0+} \frac{Y_*(\tau+h) - Y_*(\tau)}{h} \;>\; -\lambda Y_*(\tau) + I(\beta_*(\tau))(Y_*\tau) + \lambda X_*(\tau) - \hat u(\tau)) + \varepsilon\,,$$

for some $\varepsilon > 0$. Then for $t > \tau$ with $t - \tau$ sufficiently small we have $Y^\tau(t) < Y_*(t)$, against the minimality of Y_*. Again, we reached a contradiction. This completes the proof of the lemma. $\square$

Remark 5. Given initial data (X_0, A_0), the above minimal solution can be constructed in the following alternative way. Consider the Cauchy problem for the system of three equations

$$\begin{cases} \dot X &= Y - \hat u\,, \\ \dot Y &= -\lambda Y + I(Y + \lambda X - \hat u)\,, \\ \dot\beta &= \big(r + \lambda + \rho(X)\big)\beta - \rho(X)\,, \end{cases} \qquad \begin{cases} X(0) &= X_0\,, \\ Y(0) &= A_0 X_0\,, \\ \beta(0) &= \theta\,. \end{cases} \tag{6.24}$$

where $I = I(\beta)$ is the function in (5.5). Notice that here the initial value $\theta = \beta(0)$ is regarded as a free parameter. We say that a solution $(X, Y, \beta) : [0, T^M[\ \mapsto I\!R^3$ is **admissible** if

- $X(t) \in [0, M[$ and $\beta(t) \in [0, 1[$ for all $t \in [0, T^M[$,
- either $T^M = +\infty$, or else $X(t) \to M$ and $\beta(t) \to 1$ as $t \to T^M-$.

Observe that for any $0 \le \theta < 1$ and $X_0 \in [0, M[$, there exists $\tau > 0$ such that (6.24) has a unique solution on $[0, \tau[$. We now show that the minimal solution $(X_*(t), Y_*(t), \beta_*(t))$ of (5.2)–(5.5) is the solution of (6.24) corresponding to the smallest value of the parameter θ that renders this solution admissible.

From Lemma 2, $(X_*(t), Y_*(t))$ solve the first two ODEs in (6.24) for a.e. $t \in [0, T^*_M[$. Moreover,

$$\beta_*(t) \;=\; 1 - (r + \lambda) \int_t^{T^M_*} e^{-(r+\lambda)(\tau - t)} \exp\left(- \int_t^\tau \rho(X_*(s))\, ds \right) d\tau\,, \tag{6.25}$$

where

$$T^M_* \;=\; \inf\big\{ t > 0 \mid X_*(t) < M \big\}\,.$$

Using (6.25) to compute the derivative of β_*, we obtain

$$\begin{aligned} \dot\beta_*(t) &= -(r+\lambda) - (r + \lambda + \rho(X_*(t)))(r+\lambda) \int_t^{T^M_*} e^{-(r+\lambda)(\tau - t)} \exp\left(- \int_t^\tau \rho(X_*(s))\, ds \right) d\tau \\ &= -(r+\lambda) - (r + \lambda + \rho(X_*(t))) \cdot (1 - \beta_*(t)) \end{aligned}$$

for a.e. $t \in [0, T^M_*[$. Hence, $\beta_*(t)$ solves the last ODE in (6.24).

On the other hand, for any $t \in [0, T^M_*[$ we have $\beta_*(t) < 1$ and

$$\beta_*(t) \;\ge\; 1 - (r+\lambda) \cdot \int_t^{T^M_*} e^{-(r+\lambda)(\tau - t)}\, d\tau \;=\; \lim_{\tau \to T^M_*} e^{-(r+\lambda)(\tau - t)}\,.$$

Hence $\beta_*(t) \in [0, 1[$ for all $t \in [0, T^M_*[$. If $T^M_* < \infty$ then

$$\lim_{t \to T^M_*} \lim_{\tau \to T^M_*} \beta_*(t) \;\ge\; e^{-(r+\lambda)(\tau - t)} \;=\; 1\,,$$

which yields $\lim_{t\to T_*^M} \beta_*(t) = 1$. Therefore, $(X_*(t), Y_*(t), \beta_*(t))$ is an admissible solution of (6.24).

To complete the proof, we claim that there does not exist a $\theta \in [0, \beta_*(0)[$ such that (6.24) has an admissible solution. Indeed, let $(X_0(t), Y_0(t), \beta_0(t)) : [0, T_M^0[\ \to \mathbb{R}^3$ be the solution of (6.24) with $\beta_0(0) = \theta_0 < \beta^*(0)$. Two cases will be considered:

- CASE 1: If $T_M^0 = +\infty$ then one can solve the ODE for β to obtain that

$$\beta(t) \ = \ 1 - (r+\lambda) \int_t^{\infty} e^{-(r+\lambda)(\tau - t)} \exp\left(-\int_t^{\tau} \rho(X(s))\,ds\right) d\tau, \quad \text{for all } t \in [0, \infty[\,.$$

  Thus, $(X_0(t), Y_0(t), \beta_0(t))$ is a supersolution of (5.2)–(5.5).

- CASE 2: If $T_M^0 < +\infty$ then $\lim_{t\to T_M^0} \ \beta(t) = 1$. Again, solving the ODE for β, we have

$$\beta(t) \ = \ 1 - (r+\lambda) \int_t^{T_M^0} e^{-(r+\lambda)(\tau - t)} \exp\left(-\int_t^{\tau} \rho(X(s))\,ds\right) d\tau, \quad \text{for all } t \in [0, T_M^0[\,.$$

  We extend $(X_0(t), Y_0(t), \beta(t))$ on $[T_M^0, \infty[$ such that

$$\begin{cases} X(t) &= \ +\infty\,, \\ Y(t) &= \ +\infty\,, \\ \beta(t) &= \ 1\,, \end{cases} \qquad \text{for all } t \geq T_M^0\,.$$

  and $(X_0(t), Y_0(t), \beta_0(t))$ is a supersolution of (5.2)–(5.5).

Since $(X_*(t), Y_*(t), \beta_*(t))$ is the minimal solution of (5.2)–(5.5), we have $\beta_*(0) < \beta_0(0) = \beta_0$ and it yields a contradiction.

Theorem 3. *Let $u : [0, +\infty[\mapsto [0,1]$ be a control such that*

$$L(u(t)) \ \leq \ rB \qquad \text{for all } t \geq 0. \tag{6.26}$$

Then the corresponding minimal solution $t \mapsto (X_*(t), Y_*(t), \beta_*(t))$ yields the lowest cost among all solutions of (5.2)–(5.5).

Proof. Let $(X(t), Y(t), \beta(t))$ be a solution associated to the control u. The corresponding cost is computed as

$$\begin{aligned} J^{X,Y,\beta}(u) \ &= \ \int_0^{T^M} \gamma(t)\Big\{\rho(X(t))\,B + L(u(t))\Big\}\,dt \ + \ \gamma(T^M)\,B \\ &= \ \int_0^{T^M} e^{-rt} \exp\left\{-\int_0^t \rho(X(s))\,ds\right\} \rho(X(t))\ dt + \int_0^{T^M} \gamma(t) L(u(t))\,dt \ + \ \gamma(T^M)\,B \\ &= \ B - \gamma(T^M)B - rB\int_0^{T_M} \gamma(t)\ dt + \int_0^{T^M} \gamma(t) L(u(t))\,dt \ + \ \gamma(T^M)\,B \\ &= \ B - \int_0^{T_M} e^{-rt} \exp\left\{-\int_0^t \rho(X(s))\,ds\right\} \cdot [rB - L(u(t))]\ dt\,. \end{aligned}$$

Similarly, the corresponding cost for (X_*, Y_*, β_*) is

$$J^{X_*,Y_*,\beta_*}(u) \;=\; B - \int_0^{T_M^*} e^{-rt} \,\exp\left\{-\int_0^t \rho(X_*(s))\,ds\right\} \cdot [rB - L(u(t))]\;dt\,.$$

Since

$$X(t) \;\geq\; X_*(t) \qquad \text{for all } t \geq 0\,,$$

we have $T_M \leq T_M^*$ and

$$\int_0^t \rho(X(s))ds \;\geq\; \int_0^t \rho(X_*(s))ds \qquad \text{for all } t \in [0, T_M]\,.$$

Using (6.26) we thus obtain

$$\begin{aligned}
&\int_0^{T_M} e^{-rt} \,\exp\left\{-\int_0^t \rho(X(s))\,ds\right\} \cdot [rB - L(u(t))]\;dt \\
&\qquad\qquad \leq\; \int_0^{T_M} e^{-rt} \,\exp\left\{-\int_0^t \rho(X^*(s))\,ds\right\} \cdot [rB - L(u(t))]\;dt\,.
\end{aligned}$$

Therefore

$$J^{X,Y,\beta}(u) \;\geq\; J^{X_*,Y_*,\beta_*}(u)\,.$$

□

Corollary 1. *If $L(1) \leq rB$, then for every given control function $u : [0,\infty[\,\mapsto [0,1]$ the corresponding minimal solution is the one yielding the minimum expected cost.*

Remark 6. At first sight one might guess that, for a given repayment strategy $u(\cdot)$, a lower interest rate should yield a smaller expected cost to the borrower. However, this is not always true. A higher interest rate means that bankruptcy occurs earlier. In some cases, paying the cost of bankruptcy may be preferable, compared with all subsequent costs of servicing the debt. For example, assume that the borrower chooses the control

$$u(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 \leq t \leq 1, \\ 1 & \text{if} \quad t > 1. \end{cases} \tag{6.27}$$

Consider a solution where the lenders offer a small interest rate, so that $T^M > 2$ and the probability of not being bankrupt at time $t = 2$ is

$$P(2) \;=\; \text{Prob.}\{T_b > 2\} \;>\; 0.$$

Then the expected cost to the borrower is

$$\begin{aligned}
J(u, X) \;&\doteq\; \int_0^{T^M} \gamma(t)\Big\{\rho(X(t))\,B + L(u(t))\Big\}\,dt + \gamma(T^M)\,B \\
&\geq\; \int_1^2 e^{-rt} P(t) L(u(t))\,dt \;\geq\; e^{-2r} P(2) L(1).
\end{aligned} \tag{6.28}$$

On the other hand, consider a second solution $t \mapsto \widetilde{X}(t)$ where the lenders ask for a high interest rate, pushing up the total size of the debt so that $T^M < 1$. In this case, bankruptcy occurs with probability one before time $t = 1$. Hence the expected cost to the borrower is

$$J(u, \widetilde{X}) \;=\; \int_0^{T^M} \tilde{\gamma}(t)\rho(\widetilde{X}(t))\, B\, dt + \tilde{\gamma}(T^M)\, B \;\leq\; B. \tag{6.29}$$

If the bankruptcy cost B is small compared with the cost $L(1)$, then the right hand side of (6.29) will be smaller than (6.29). For the particular control function $u(\cdot)$ in (6.27), the minimal solution is not the most advantageous for the borrower.

7 Feedback strategies

As it often happens for Stackelberg games, the optimal open-loop strategy $u^*(\cdot)$ considered in Section 5 is usually not "time consistent". In other words, in the middle of the game it may be advantageous to the borrower to deviate from his announced strategy, if he could do so without changing the interest rates obtained in the past. However, this is not possible, because these interest rates are globally determined by his past and future controls.

In this section we shall address this issue, and study strategies in feedback form.

As a first step, assume that both the bankruptcy risk ρ and the interest rate I are given a priori, as functions of X, A. This leads to a standard problem of optimal control in infinite time horizon [1, 4, 8]. The value function $V = V(X, A)$ satisfies the Hamilton-Jacobi equation

$$\begin{aligned}\Big(r+\rho\Big)V \;&=\; \min_{\omega\in[0\,,\,(\lambda+A)X\wedge 1]} \left\{L(\omega) - \Big(V_X + \frac{I-A}{X}V_A\Big)\cdot\omega\right\}\\ &\qquad +AX\,V_X + (\lambda+A)(I-A)\,V_A + \rho\, B\,.\end{aligned} \tag{7.1}$$

The optimal feedback control u^* is computed by

$$u^*(X, A, V_X, V_A) \;=\; \arg\min_{\omega\in[0\,,\,(\lambda+A)X\wedge 1]} \left\{L(\omega) - \Big(V_X + \frac{I(X,A)-A}{X}V_A\Big)\cdot\omega\right\}. \tag{7.2}$$

Assuming that the optimal feedback control takes values in the interior of the admissible set, so that $0 < u^* < (\lambda + A)X$, we can write the characteristic equations for the first order PDE (7.1), namely (see for ex. [10])

$$\left\{\begin{aligned}\dot X \;&=\; XA - u^*,\\ \dot A \;&=\; (I-A)\left(\lambda + A - \frac{u^*}{X}\right),\\ \dot V \;&=\; (XA - u^*)\,V_X + (I-A)\left(\lambda + A - \tfrac{u^*}{X}\right)\,V_A\,,\\ \dot V_X \;&=\; \rho_X(V-B) + (r+\rho-A)V_X - [I_X(\lambda + A - \tfrac{u^*}{X}) + \tfrac{(I-A)u^*}{X^2}]V_A\,,\\ \dot V_A \;&=\; \rho_A(V-B) - XV_X + [r+\rho-I+A-(I_A-1)(\lambda+A-\tfrac{u^*}{X})]V_A\,.\end{aligned}\right. \tag{7.3}$$

By the previous analysis, one has:

(i) If the instantaneous interest rate $I = I(X, A)$ on new loans is known, then the optimal feedback control $u^* = u^*(X, A)$ can be recovered in terms of the value function by the formula (7.2).

(ii) If the optimal feedback u^* is known, then for every initial data (X_0, A_0) one obtains a solution $t \mapsto (X(t), A(t))$ of the Cauchy problem (4.5). In turn, by (4.13) this determines the instantaneous interest rate I, namely

$$I(X_0, A_0) \;=\; I(\beta) \;=\; r + (1-\sigma)(\lambda + r)\frac{\beta}{1-\beta}, \tag{7.4}$$

where

$$\beta(X_0, A_0) \;\doteq\; \int_0^\infty e^{-(r+\lambda)t} \cdot \rho(X(t), A(t)) \exp\left\{-\int_0^t \rho(X(s), A(s))\, ds\right\} dt\,. \tag{7.5}$$

Combining (i) and (ii), we obtain a system of nonlinear PDEs for the two functions $V(X_0, A_0)$ and $\beta(X_0, A_0)$. Indeed, if u^* is an optimal feedback control defined at (7.2), differentiating w.r.t. t the two expressions

$$\begin{aligned} V\big(X(t), A(t)\big) \;&=\; \int_t^\infty e^{-r(\tau - t)} \exp\left\{-\int_t^\tau \rho\big(X(s), A(s)\big)\, ds\right\} \\ &\qquad\qquad \cdot\Big[\rho\big(X(\tau), A(\tau)\big)\, B + L(u^*(\tau))\Big]\, d\tau\,, \\ \beta\big(X(t), A(t)\big) \;&=\int_t^\infty e^{-(r+\lambda)(\tau-t)} \exp\left\{-\int_t^\tau \rho\big(X(s), A(s)\big)\, ds\right\} \cdot \rho\big(X(\tau), A(\tau)\big)\, d\tau\,, \end{aligned}$$

one obtains

$$\left\{\begin{aligned} \frac{d}{dt} V(X(t), A(t)) \;&=\; (r + \rho(X, A))V - L(u^*) - \rho(X, A)\, B, \\ \frac{d}{dt} \beta(X(t), A(t)) \;&=\; \big(r + \lambda + \rho(X, A)\big)\beta - \rho(X, A)\,. \end{aligned}\right. \tag{7.6}$$

By (4.5), the functions V, β thus satisfy the system of first order PDEs

$$\left\{\begin{aligned} \big(AX - u^*\big)V_X + (I - A)\Big(\lambda + A - \frac{u^*}{X}\Big)\, V_A \;&=\; (r + \rho)V - L(u^*) - \rho\, B, \\ \big(AX - u^*\big)\beta_X + (I - A)\Big(\lambda + A - \frac{u^*}{X}\Big)\, \beta_A \;&=\; (r + \lambda + \rho)\, \beta - \rho\,. \end{aligned}\right. \tag{7.7}$$

Here X, A are the independent variables, r, λ, B are positive constants, ρ is a given function of X, A, while $I = I(\beta)$ is defined at (7.4). Finally, u^* can be recovered from X, A, V_X, V_A and $I(\beta)$ in terms of (7.2).

It seems natural to solve (7.7) with boundary conditions at $X = 0$ and at $X = M$, namely

$$\left\{\begin{aligned} V(0, A) \;&=\; 0, \\ \beta(0, A) \;&=\; 0\,, \end{aligned}\right. \qquad\qquad \left\{\begin{aligned} V(M, A) \;&=\; B, \\ \beta(M, A) \;&=\; 1\,. \end{aligned}\right. \tag{7.8}$$

To get some additional insight, we shall transform the system (7.7) into a second order scalar equation. Taking the directional derivative in the direction of the characteristic vector

$$\mathbf{b} \;=\; (b_1, b_2) \;\doteq\; \left(AX - u^* \,,\; (I-A)\Big(\lambda + A - \frac{u^*}{X}\Big)\right),$$

from (7.7) and (7.4) it follows

$$\begin{cases} \mathbf{b}\cdot\nabla V \;=\; F(X,A,I,V_X,V_A) \;\doteq\; (r+\rho)V - L\big(u^*\big) - \rho B\,, \\[2mm] \mathbf{b}\cdot\nabla I \;=\; G(X,A,I) \;\doteq\; \dfrac{[I-r-(1-\sigma)\cdot\rho]\cdot[I-r+(1-\sigma)\cdot(r+\lambda)]}{1-\sigma}\,. \end{cases} \tag{7.9}$$

Observe that the components b_1, b_2 depend also on u^*, and hence on X, A, I, V_X, V_A. Assuming that the minimum in (7.2) is attained at an interior point $\omega = u^*$, the necessary conditions for optimality yield

$$L'(u^*) \;=\; V_X + \frac{I-A}{X} V_A\,. \tag{7.10}$$

We assume that, by the implicit function theorem, (7.10) and the first identity in (7.9) can be used to uniquely determine the functions

$$u^* \;=\; u^*(X,A,I,V_X,V_A), \qquad\qquad I \;=\; I(X,A,V,V_X,V_A). \tag{7.11}$$

Differentiating the first equation in (7.10) the direction of the vector $\mathbf{b}$, we obtain

$$\begin{aligned} 0 \;&=\; \mathbf{b}\cdot\nabla\Big[\mathbf{b}\cdot\nabla V - (r+\rho)V + L(u^*) + \rho B\Big] \\ &=\; \big(AX-u^*\big)\cdot\frac{\partial}{\partial X}\Big[(AX-u^*)V_X + (I-A)\Big(\lambda + A - \frac{u^*}{X}\Big)V_A - (r+\rho)V + L(u^*) + \rho B\Big] \\ &\quad\; + (I-A)\Big(\lambda + A - \frac{u^*}{X}\Big)\frac{\partial}{\partial A}\Big[(AX-u^*)V_X \\ &\qquad\qquad + (I-A)\Big(\lambda + A - \tfrac{u^*}{X}\Big)V_A - (r+\rho)V + L(u^*) + \rho B\Big] \\ &=\; \big(AX-u^*\big)^2\cdot V_{XX} + 2(AX-u^*)(I-A)\Big(\lambda + A - \frac{u^*}{X}\Big)\cdot V_{XA} \\ &\quad\; + (I-A)^2\Big(\lambda + A - \frac{u^*}{X}\Big)^2\cdot V_{AA} - \Phi(X,A,V,V_X,V_A), \end{aligned} \tag{7.12}$$

where the function Φ collects all the remaining terms. We claim that Φ depends only on the first derivatives V_X, V_A, i.e., its expression does not involve second derivatives. Indeed, by the second equation in (7.9) the quantity

$$\big(AX-u^*\big)\cdot\frac{\partial I}{\partial X} + (I-A)\Big(\lambda + A - \frac{u^*}{X}\Big)\frac{\partial I}{\partial A} \;=\; G(X,A,I)$$

can be expressed in terms of X, A, and $I(X,A,V,V_X,V_A)$. Moreover, the optimality condition (7.10) implies

$$\left[-V_X - \frac{I-A}{X} + L'(u^*)\right]\frac{\partial}{\partial X}u^* \;=\; \left[-V_X - \frac{I-A}{X} + L'(u^*)\right]\frac{\partial}{\partial A}u^* \;=\; 0\,.$$

This proves our claim.

We observe that (7.12) is a quasilinear degenerate elliptic equation for the value function $V = V(X, A)$, on the domain

$$\Omega \doteq \Big\{(X, A)\,; \quad X \in [0, M]\,, \quad A \geq 0\Big\}, \tag{7.13}$$

with boundary data as in (7.8). In more compact notation, it takes the form

$$a^2\, V_{XX} + 2ab\, V_{XA} + b^2\, V_{AA} \;=\; \Phi\,, \tag{7.14}$$

where the coefficients a, b, Φ are highly nonlinear functions of X, A, V, V_X, V_A.

Acknowledgments. This research was partially supported by NSF, with grant DMS-1108702: "Problems of Nonlinear Control". The authors would also like to thank the anonymous referees, whose suggestions and comments helped to improve many sections of the paper.

References

[1] M. Bardi and I. Capuzzo Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, 1997.

[2] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2^d Edition, Academic Press, London 1995.

[3] A. Brace, D. Gatarek, and M. Musiela, The market model of interest rate dynamics. *Math. Finance* **7** (1997), 127–147.

[4] A. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control*, AIMS Series in Applied Mathematics, Springfield Mo. 2007.

[5] D. Brigo and F. Mercurio. *Interest Rate Models - Theory and Practice with Smile, Inflation and Credit.* Springer, Berlin, 2nd edition, 2006.

[6] M. Burke and K. Prasad, An evolutionary model of debt. *J. Monetary Economics* **49** (2002) 1407–1438.

[7] A. Cairns. *Interest Rate Models: An Introduction.* Princeton University Press, Princeton, 2004.

[8] D. A. Carlson, A. B. Haurie, and A. Leizarowitz, *Infinite Horizon Optimal Control. Deterministic and Stochastic Systems.* Springer, Berlin, 1991.

[9] J. C. Cox, J. E. Ingersoll, and S. A. Ross, A theory of the term structure of interest rates. *Econometrica*, **53** (1985), 385–407.

[10] L. C. Evans, *Partial Differential Equations.* Second Edition, American Mathematical Society, Providence, 2010.

[11] Z. He and W. Xiong, Dynamic debt runs. *Rev. Financ. Stud.* **25** (2012), 1799–1843.