Optima and Equilibria for Traffic Flow on Networks with Backward Propagating Queues

Alberto Bressan and Khai T. Nguyen

Department of Mathematics, Penn State University.
University Park, PA 16802, USA.

e-mails: bressan@math.psu.edu , ktn2@psu.edu

January 10, 2015

Abstract

The paper studies an optimal decision problem for several groups of drivers on a network of roads. Drivers have different origins and destinations, and different costs, related to their departure and arrival time. On each road the flow is governed by a conservation law, while intersections are modeled using buffers of limited capacity, so that queues can spill backward along roads leading to a crowded intersection. Two main results are proved: (i) the existence of a globally optimal solution, minimizing the sum of the costs to all drivers, and (ii) the existence of a Nash equilibrium solution, where no driver can lower his own cost by changing his departure time or the route taken to reach destination.

1 Introduction

Optimal traffic assignment and dynamic user equilibria on networks have been widely discussed in the engineering literature [13, 15]. For conservation law models of traffic flow on a network of roads, these problems were recently studied in [4]. The basic setting comprises a network with nodes $A_1, \ldots, A_m$, and connecting arcs γ_{ij}. Drivers choose their time of departure and route to destination in order to minimize the sum of a departure cost $\varphi(\tau^d)$ and an arrival cost $\psi(\tau^a)$. The problem is highly nontrivial because the arrival time τ^a depends not only on the departure time τ^d but also on the overall traffic pattern.

As in [26, 27], along each arc we model the traffic flow in terms of the conservation law

$$\rho_t + [\rho\, v_{ij}(\rho)]_x \;=\; 0\,. \tag{1.1}$$

Here t is time and $x \in [0, L_{ij}]$ is the space variable along the arc γ_{ij}. The variable $\rho = \rho(t,x)$ describes the traffic density, i.e. the number of cars per unit length, while the map $\rho \mapsto v_{ij}(\rho)$ is the speed of cars as function of the density, along the arc γ_{ij}. We assume that v_{ij} is a continuous, nonincreasing function of the density ρ. At each node of the network, the conservation laws (1.1) must be supplemented by suitable boundary conditions, modeling traffic flow at an intersection. In the earlier paper [4] a buffer of unlimited capacity was assumed to be present at the beginning of each road. Arriving cars are placed in this buffer, waiting for their turn to enter the new road. With this model, roads never become congested and queues never propagate backwards.

Aim of the present paper is to prove the existence of global optima and Nash equilibria, for a more realistic model where queues can propagate backwards along roads leading to a crowded intersection. Starting with a definition of *Riemann Solver*, models describing traffic flow at an intersection were recently developed in [10, 18, 19]. Unfortunately, in the specific context of our optimization problems, they lead to ill posed Cauchy problems. The counterexamples in [7] motivated the introduction of new intersection models [6], where each node of the network contains a buffer of limited capacity. When the buffer is nearly full, cars can access the intersection only at a very slow rate, and queues propagate backwards along incoming roads. As proved in [6], for these models the Cauchy problem is well posed within the general class of $\mathbf{L}^\infty$ initial data. The solution can be constructed as the unique fixed point of a contractive transformation, defined in terms of a Lax-type variational formula. A key feature of these models is that the travel time between any two nodes of the network depends continuously on the data, w.r.t. the topology of weak convergence. These properties are precisely what is needed, in order to apply the topological arguments in [4] and establish the existence of global optima and equilibria.

Our optimal decision problems are formulated for n groups of drivers traveling on the network. Different groups are distinguished by the locations of departure and arrival, and by their cost functions. For $k \in \{1, \ldots, n\}$, let G_k be the total number of drivers in the k-th group. All these drivers depart from a node $A_{d(k)}$ and arrive at a node $A_{a(k)}$, but can choose different paths to reach destination. Of course, we assume that there exists at least one path (i.e., a concatenation of arcs)

$$\Gamma \doteq \Big(\gamma_{i(0),i(1)}\,,\ \gamma_{i(1),i(2)}\,,\ \ldots\ ,\gamma_{i(N-1),i(N)}\Big) \tag{1.2}$$

with $i(0) = d(k)$ and $i(N) = a(k)$, connecting the departure node $A_{d(k)}$ with the arrival node $A_{a(k)}$. We shall denote by

$$\mathcal{V} \doteq \Big\{\Gamma_1, \Gamma_2,\ \ldots\ , \Gamma_K\Big\}$$

the set of all paths which do not contain any closed loop. Since there are m nodes in the network, and each chain can visit each of them at most once, the set $\mathcal{V}$ contains finitely many elements. For a given $k \in \{1, \ldots, n\}$, let $\mathcal{V}_k \subset \mathcal{V}$ be the set of all paths available to k-drivers, connecting $A_{d(k)}$ with $A_{a(k)}$. By $u_{k,p}(\cdot)$ we denote the departure rate of drivers of the k-th group, traveling along the viable path Γ_p. Hence

$$U_{k,p}(t) \doteq \int_{-\infty}^{t} u_{k,p}(s)\,ds \tag{1.3}$$

is the total number of drivers of the k-th group, traveling along the path Γ_p, who have started their journey before time t.

Definition 1. *Let the group of k-drivers have size $G_k > 0$. We say that $\{u_{k,p}\,;\quad k = 1, \ldots, n,\ \ p \in \mathcal{V}_k\}$ is an* **admissible family of departure rates** *if each $u_{k,p}$ is a nonnegative* *integrable function, and moreover*

$$\sum_{p\in\mathcal{V}_k} \int_{-\infty}^{\infty} u_{k,p}(t)\,dt \;=\; G_k \qquad \textit{for each } k\in\{1,\ldots,n\}\,. \tag{1.4}$$

Here the admissibility condition (1.4) means that, sooner or later, every driver of the k-th group will depart, choosing some path $\Gamma_p \in \mathcal{V}_k$ to reach his destination.

As in [2, 3, 5], we consider a set of departure costs $\varphi_k(\cdot)$ and arrival costs $\psi_k(\cdot)$ for the various drivers. A driver of the k-th group departing at time τ^d and arriving at destination at time τ^a has total cost

$$\varphi_k(\tau^d) + \psi_k(\tau^a). \tag{1.5}$$

In this framework, the concepts of globally optimal solution and of Nash equilibrium solution considered in [2, 3] can be extended to traffic flow on a network of roads.

Definition 2. *An admissible family $\{u_{k,p}\}$ of departure rates is* **globally optimal** *if it minimizes the sum of the total costs of all drivers.*

Definition 3. *An admissible family $\{u_{k,p}\}$ of departure rates is a* **Nash equilibrium solution** *if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.*

From the above definition it follows that, in a Nash equilibrium, all drivers in a same group must bear the same total cost (1.5).

In this paper we prove the existence of a globally optimal solution and of a Nash equilibrium solution, extending the results in [4] to a model where queues can spill backward through several nodes of the network. Our existence proofs, worked out in Sections 4 and 5, are similar to the ones given in [4] for buffers of infinite size. However, a substantial amount of preliminary analysis is needed.

Indeed, for intersection models with buffers of finite size, the well posedness results proved in [6] refer to the initial-value problem. These results need to be adapted to the boundary-value problem, where departure rates are assigned for all times $t \in \mathbb{R}$. In addition, one has to study how the travel time of each driver depends on all the departure rates, w.r.t. the topology of weak convergence. In this paper, Section 2 recalls the main definitions and modeling assumptions, while Section 3 establishes the key continuity properties of our solutions.

In addition, the proof of the existence of a Nash equilibrium requires a uniform a priori bound on the travel time of every driver. In a realistic situation, this is largely expected. In the case of buffers of unlimited size considered in [4], such a bound is easy to prove. However, in models where queues can spill backward, it is hard to pinpoint a condition which guarantees that traffic will never get stuck, and all drivers arrive at destination in finite time. See [9] for a discussion of this issue. In Section 6 of the present paper we prove a partial result in this direction. Namely, if the network does not contain any closed cycle, then the traffic will never get stuck.

For the basic modeling of traffic flow we refer to [1, 26, 27]. Traffic flow on networks has been the topic of an extensive literature, see for example [10, 13, 16, 19, 23] and references therein.

More detailed results on optima and equilibria for traffic flow on a single road can be found in [2, 3, 5] Different kinds of optimization problem for network flows have been considered in [8, 12, 21, 22].

2 The traffic flow model

2.1 Basic assumptions.

In our model, $x \in [0, L_{ij}]$ is the space variable, describing a point along the arc γ_{ij} joining the node A_i to the node A_j. Here L_{ij} measures the length of this arc. The basic assumptions on the flux functions $f_{ij}(\rho) = \rho\, v_{ij}(\rho)$ and on the cost functions φ_k, ψ_k are as follows.

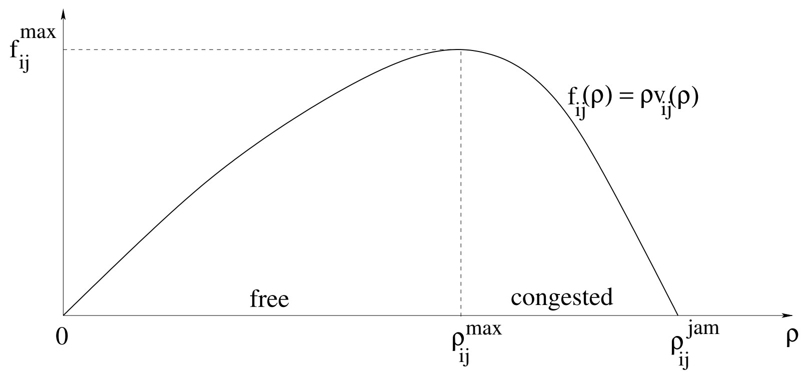


Figure 1: The flux f_{ij} as a function of the density ρ, along the arc γ_{ij} from the node A_i to A_j.

(A1) For every arc γ_{ij}, the flux function $\rho \mapsto f_{ij}(\rho) = \rho\, v_{ij}(\rho)$ is twice continuously differentiable, strictly concave down, and non-negative on some interval $[0, \rho_{ij}^{jam}]$, with $f_{ij}(0) = f_{ij}(\rho_{ij}^{jam}) = 0$. We shall denote by $\rho_{ij}^{max} \in]0, \rho_{ij}^{jam}[$ the unique value such that

$$f_{ij}(\rho_{ij}^{max}) \;=\; f_{ij}^{max} \doteq \max_{\rho \in [0, \rho_{ij}^{jam}]} f_{ij}(\rho)\,. \tag{2.1}$$

(A2) For every $k \in \{1, \ldots, n\}$ the cost functions φ_k, ψ_k are continuously differentiable and satisfy

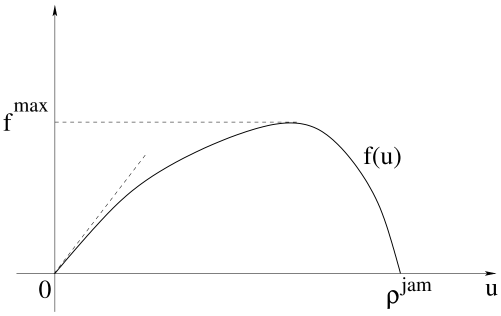
$$\begin{cases} \varphi_k'(t) \;<\; 0\,, \\ \psi_k'(t) \;>\; 0\,, \end{cases} \qquad \lim_{|t| \to \infty} \Big(\varphi_k(t) + \psi_k(t)\Big) \;=\; +\infty\,. \tag{2.2}$$

Remark 1. For each flux f_{ij}, consider the Legendre transform g_{ij} of f_{ij}

$$g_{ij}(v) \;\doteq\; \inf_{u \in [0, \rho_{ij}^{\mathrm{jam}}]} \big\{ uv - f_{ij}(u) \big\}\,. \tag{2.3}$$

Given a characteristic $t \mapsto x(t)$ with speed $\dot{x} = v$, the Legendre transform g_{ij} can be interpreted as

$$g_{ij}(v) \;=\; -\,[\text{flux of cars from left to right, across the characteristic}]. \tag{2.4}$$



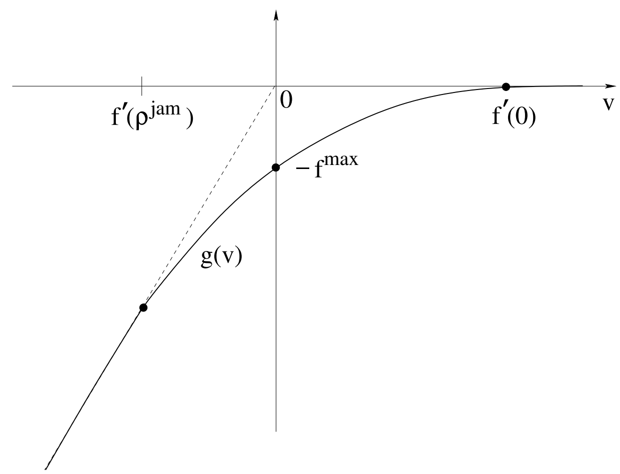


Figure 2: The flux function f_{ij} and its Legendre transform g_{ij} defined at (2.3). Notice that $g_{ij}(0) = -f_{ij}^{\max}$, while $g_{ij}(v) = 0$ for $v \geq f'_{ij}(0)$ and $g_{ij}(v) = \rho_{ij}^{jam} v$ for $v \leq f'_{ij}(\rho_{ij}^{jam})$.

For $v \in \left]f'_{ij}(\rho_{ij}^{jam}),\, f'_{ij}(0)\right[$, differentiating w.r.t. v, one obtains

$$g''_{ij}(v) \;=\; \frac{\partial}{\partial v} g'_{ij}(u^*(v)) \;=\; \frac{1}{f''_{ij}(u^*(v))} \;<\; 0\,, \tag{2.5}$$

showing that g_{ij} is strictly concave down on this open interval. As shown in Fig. 2, we also have the implications

$$\begin{cases} v \;\leq\; f'_{ij}(\rho_{ij}^{jam}) & \Longrightarrow \qquad g_{ij}(v) = \rho_{ij}^{jam}\, v, \\ \\ v \;\geq\; f'_{ij}(0) & \Longrightarrow \qquad g_{ij}(v) = 0. \end{cases} \tag{2.6}$$

2.2 Evolution of traffic density.

We now describe more in detail how the traffic flow on the entire network can be uniquely determined, given the departure rates $u_{k,p}$.

We assume that the set $\mathcal{N}$ of all nodes of the network can be partitioned as

$$\mathcal{N} \;=\; \mathcal{N}_d \cup \mathcal{N}_t \cup \mathcal{N}_a\,,$$

where the three sets on the right hand side denote the *departure nodes*, the *transit nodes* and the *arrival nodes*, respectively (Fig. 3).

I - Dynamics at departure nodes. Departure nodes have no incoming road and only one outgoing road. As in [4], let $u(t)$ be the rate of departures from a node $A_i \in \mathcal{N}_d$, and let $\rho(t,x)$, $x \in [0, L]$, be the density of traffic along this single outgoing road. We assume that ρ satisfies the conservation law

$$\rho_t + f(\rho)_x \;=\; 0.$$

Call $q(t)$ the length of the queue at the entrance of the road, and let

$$\bar{\rho}(t) = \lim_{x \to 0+} \rho(x,t)$$

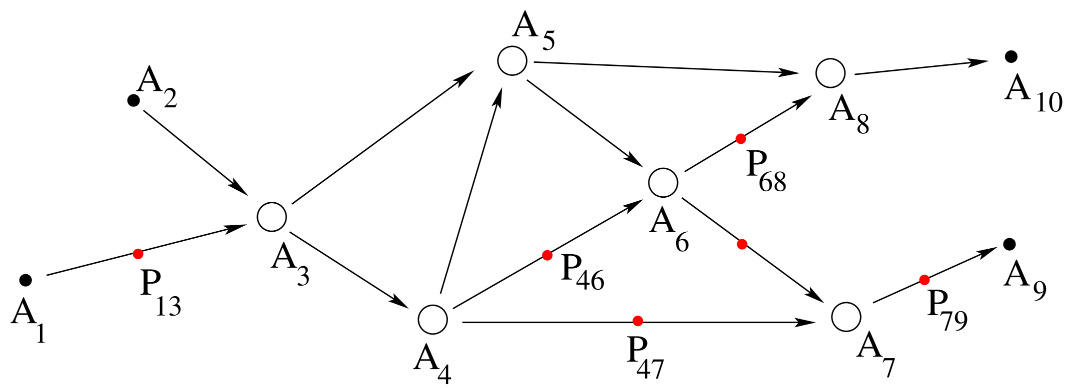


Figure 3: In this network, A_1, A_2 are entrance nodes, $A_3, \ldots, A_8$ are transit nodes, and A_9, A_{10} are exit nodes. We denote by P_{ij} the mid-point of the arc γ_{ij} from A_i to A_j.

be the boundary value for the density. Moreover, define

$$\omega(t) \;=\; \begin{cases} f^{max} & \text{if } \; \bar\rho(t) \le \rho^*, \\ f(\bar\rho(t)) & \text{if } \; \bar\rho(t) \ge \rho^*. \end{cases}$$

Notice that $\omega(t)$ is the maximum flux of cars that can enter the road at time t.

The boundary value for the flux and the length of the queue are then governed by the equations

$$f(\bar\rho(t)) \;=\; \begin{cases} \omega(t) & \text{if } \; q(t) > 0\,, \\ u(t) & \text{if } \; q(t) = 0\,, \end{cases} \tag{2.7}$$

$$\dot q(t) \;=\; u(t) - \omega(t) \qquad \text{if } \; q(t) > 0. \tag{2.8}$$

Here and throughout the sequel, an upper dot denotes a derivative w.r.t. time.

II - Dynamics at arrival nodes. We assume that each arrival node $A_j \in \mathcal{N}_a$ has one incoming road, say γ_{ij}, and no outgoing road. Cars exit instantly upon reaching the node A_j, and no backward queue is ever formed along the road leading to A_j. We can thus assume that the density ρ_{ij} satisfies

$$\rho_{ij} \;\le\; \rho^*_{ij}, \qquad\qquad f'_{ij}(\rho_{ij}) \;\ge\; 0,$$

for all $t \ge 0$, $x \in [0, L_{ij}]$. Since all characteristics have positive speed, the initial-boundary value problem along the arc γ_{ij} is well posed without assigning any condition at the terminal point $x = L_{ij}$.

III - Dynamics at transit nodes. Following [6], we assume that at each intersection there is a buffer of limited capacity. The incoming fluxes of cars toward the intersection are related to the current degree of occupancy of the buffer. To fix the ideas, consider an intersection with m incoming and n outgoing roads. To simplify the notation, we label with the index $i \in \mathcal{I}$ the incoming roads and $j \in \mathcal{O}$ the outgoing roads. As in [6], the space variable is $x < 0$ along incoming roads and $x > 0$ along outgoing roads. For $k \in \mathcal{I} \cup \mathcal{O}$, we denote by ρ_k the density of cars on the k-th road. Moreover, for $i \in \mathcal{I}$ and $j \in \mathcal{O}$, we denote by θ_{ij} the fraction of cars in road i who wish to turn into road j. The above functions evolve according to the conservation laws

$$(\rho_k)_t + f_k(\rho_k)_x \;=\; 0\,, \qquad\qquad k \in \mathcal{I} \cup \mathcal{O}, \tag{2.9}$$

and the linear transport equations

$$(\theta_{ij})_t + v_i(\rho_i)(\theta_{ij})_x \;=\; 0\,, \qquad\qquad i \in \mathcal{I}, \quad j \in \mathcal{O}\,. \tag{2.10}$$

The state of the buffer at the intersection is described by an n-vector

$$\mathbf{q} \;=\; (q_j)_{j\in\mathcal{O}}\,.$$

Here $q_j(t)$ is the number of cars at the intersection waiting to enter road $j \in \mathcal{O}$, i.e., the length of the queue in front of road j. Boundary values at the junction will be denoted by

$$\begin{cases} \bar\theta_{ij}(t) \;\doteq\; \lim_{x\to 0-}\theta_{ij}(t,x), & i\in\mathcal{I},\; j\in\mathcal{O}\,,\\[2mm] \bar\rho_i(t) \;\doteq\; \lim_{x\to 0-}\rho_i(t,x), & i\in\mathcal{I}\,,\\[2mm] \bar\rho_j(t) \;\doteq\; \lim_{x\to 0+}\rho_j(t,x), & j\in\mathcal{O}\,,\\[2mm] \bar f_i(t) \;\doteq\; f_i(\bar\rho_i(t)) \;=\; \lim_{x\to 0-} f_i(\rho_i(t,x)), & i\in\mathcal{I}\,,\\[2mm] \bar f_j(t) \;\doteq\; f_j(\bar\rho_j(t)) \;=\; \lim_{x\to 0+} f_j(\rho_j(t,x)), & j\in\mathcal{O}\,. \end{cases} \tag{2.11}$$

Conservation of the total number of cars implies

$$\dot q_j \;=\; \sum_{i\in\mathcal{I}} \bar f_i\bar\theta_{ij} - \bar f_j \qquad\qquad \text{for all } j\in\mathcal{O}\,. \tag{2.12}$$

Following [19], we say that a density $\rho \in [0, \rho_k^{jam}]$ along the k-th road is

- a *free state* if $\rho \in [0,\, \rho_k^{max}]$,
- a *congested state* if $\rho \in]\rho_k^{max}\,,\; \rho_k^{jam}]$.

We also define

$$\omega_i \;=\; \omega_i(\bar\rho_i) \;\doteq\; \begin{cases} f_i(\bar\rho_i) & \text{if } \bar\rho_i \text{ is a free state,}\\[2mm] f_i^{max} & \text{if } \bar\rho_i \text{ is a congested state,}\end{cases} \qquad\qquad i\in\mathcal{I}\,,$$

the maximum possible flux at the end of an incoming road. Notice that this is the largest flux $f_j(\rho)$ among all states ρ that can be connected to $\bar\rho_i$ with a wave of negative speed.

Similarly, we define

$$\omega_j \;=\; \omega_j(\bar\rho_j) \;\doteq\; \begin{cases} f_j(\bar\rho_j) & \text{if } \bar\rho_j \text{ is a congested state,}\\[2mm] f_j^{max} & \text{if } \bar\rho_j \text{ is a free state,}\end{cases} \qquad\qquad j\in\mathcal{O}\,,$$

the maximum possible flux at the beginning of an outgoing road. This is the largest flux $f_j(\rho)$ among all states ρ that can be connected to $\bar\rho_j$ with a wave of positive speed.

We consider two different sets of equations relating the incoming and outgoing fluxes $\bar f_i$ and $\bar f_j$, depending on the drivers' choices $\bar\theta_{ij}$ and on the lengths q_j of the queues in the buffer. As proved in [6], both models lead to well posed Cauchy problems within the general class of $\mathbf{L}^\infty$ data.

In the first model, the junction contains one single buffer of size M. Incoming cars are admitted at a rate depending of the amount of free space left in the buffer, regardless of their destination. Once they are within the intersection, cars flow out at the maximum rate allowed by the outgoing road of their choice. As usual, if the queue size q_j is nonzero, drivers respect their place in the queue: first-in-first-out.

Single Buffer Junction (SBJ). *Consider a constant $M > 0$, describing the maximum number of cars that can occupy the intersection at any given time, and constants $c_i > 0$, $i \in \mathcal{I}$, accounting for priorities given to different incoming roads.*

We then require that the incoming fluxes $\bar{f}_i$ satisfy

$$\bar{f}_i \;=\; \min\,\left\{\omega_i\,,\quad c_i\Big(M - \sum_{j\in\mathcal{O}} q_j\Big)\right\}, \qquad\qquad i\in\mathcal{I}\,, \tag{2.13}$$

while the outgoing fluxes $\bar{f}_j$ satisfy

$$\begin{cases} \text{if } q_j > 0, \text{ then } \bar{f}_j = \omega_j\,, \\[2ex] \text{if } q_j = 0, \text{ then } \bar{f}_j = \min\Big\{\omega_j,\ \ \sum_{i\in\mathcal{I}} \bar{f}_i\theta_{ij}\Big\}, \end{cases} \qquad\qquad j\in\mathcal{O}\,. \tag{2.14}$$

In our second model, there are n buffers, one for each outgoing road. Incoming drivers are admitted at a rate depending on the length of the queue at the entrance of the road of their choice. Once they are within the intersection, cars flow out at the maximum possible rate, respecting their place in the queue: first-in-first-out.

Multiple Buffer Junction (MBJ) *Consider constants M_j, $j \in \mathcal{O}$, describing the size of the buffer at the entrance of the j-th outgoing road, and constants $c_i > 0$, $i \in \mathcal{I}$, accounting for priorities given to different incoming roads.*

We then require that the incoming fluxes $\bar{f}_i$ satisfy

$$\bar{f}_i \;=\; \min\,\left\{\omega_i\,,\ \frac{c_i(M_j - q_j)}{\theta_{ij}},\quad j\in\mathcal{O}\right\}, \qquad\qquad i\in\mathcal{I}\,, \tag{2.15}$$

while the outgoing fluxes $\bar{f}_j$ satisfy (2.14).

We now consider the Cauchy problem for the system of equations (2.9), (2.10), (2.12), assuming that at each node the boundary conditions (2.13)-(2.14) or (2.15)-(2.14) are satisfied. We allow the possibility that the conditions **(SBJ)** hold at some nodes, while **(MBJ)** hold at other nodes. The initial data have the form

$$\begin{cases} \rho_k(0,x) & = & \rho_k^\diamondsuit(x) & k\in\mathcal{I}\cup\mathcal{O}\,, \\ \theta_{ij}(0,x) & = & \theta_{ij}^\diamondsuit(x) & i\in\mathcal{I},\quad j\in\mathcal{O}\,, \\ q_j(0) & = & q_j^\diamondsuit & j\in\mathcal{O}\,. \end{cases} \tag{2.16}$$

By an admissible solution of the above system we mean a family of functions $(\rho_k, \theta_{ij}, q_j)$, with

$$\rho_k \;\in\; [0, \rho_k^{jam}]\,, \qquad \theta_{ij}\in[0,1], \qquad \sum_{j\in\mathcal{O}}\theta_{ij} = 1\,, \tag{2.17}$$

$$q_j\geq 0, \qquad \begin{cases} \sum_{j\in\mathcal{O}} q_j < M, & \text{in case of } \textbf{(SBJ)}\,, \\ q_j < M_j \quad \text{for every } j\in\mathcal{O}, & \text{in case of } \textbf{(MBJ)}\,, \end{cases} \tag{2.18}$$

and with the following properties.

(i) The functions ρ_k provide entropy-weak solutions to the conservation laws in (2.9).

(ii) The functions θ_{ij} provide solutions to the linear transport equations in (2.10).

(iii) The functions q_j are Lipschitz continuous and satisfy the ODEs (2.12).

(iv) The initial values of ρ_k, θ_{ij} and q_j satisfy (2.16).

(v) The boundary values $\bar\rho_k(t)$, $\bar f_k(t)$, $\bar\theta_{ij}(t)$ in (2.11) are well defined in the sense of traces, and satisfy the boundary conditions (2.13)-(2.14) or (2.15)-(2.14) for a.e. $t\geq 0$.

IV - Dynamics on the entire network. The models studied in [6] dealt with one single intersection. In that case, the drivers' turning choices θ_{ij} at (2.10) had to be assigned only on incoming roads $i\in\mathcal{I}$. To model traffic flow on an entire network, we also need to keep track of how many drivers choose the path Γ_p to reach destination. For this purpose, we denote by

$$\rho_{i\ell,p} \;=\; \theta_p\cdot\rho_{i\ell} \tag{2.19}$$

the density of cars on road $\gamma_{i\ell}$ that follow path Γ_p. Clearly $\theta_p = 0$ if the arc $\gamma_{i\ell}$ is not part of the path Γ_p. Moreover, at every point (t,x) we have

$$\theta_p(t,x)\;\in\;[0,1]\,, \qquad \sum_p \theta_p(t,x) \;=\; 1\,.$$

Notice that the coefficients θ_p are passive scalars, transported along the flow. Along any arc $\gamma_{i\ell}$, they satisfy the linear transport equations

$$(\theta_p)_t + v_{i\ell}(\rho_{i\ell})\cdot(\theta_p)_x \;=\; 0\,. \tag{2.20}$$

Along the road $\gamma_{i\ell}$, the fraction $\theta_{ij}(t,x)$ of drivers traveling on road $\gamma_{i\ell}$ who will turn into road $\gamma_{\ell j}$ is recovered from the coefficients θ_p by

$$\theta_{ij} \;=\; \sum_{\gamma_{\ell j}\in\Gamma_p}\theta_p$$

Indeed a driver currently on the arc $\gamma_{i\ell}$, after reaching the intersection A_ℓ will turn into the road $\gamma_{\ell j}$ iff this road is part of the path Γ_p that he is using to reach destination.

Thanks to the finite propagation speed, all the equations can be solved iteratively in time. Indeed, the positive quantity

$$\Delta_{min} \doteq \frac{1}{2} \cdot \min_{ij} \frac{L_{ij}}{f'_{ij}(0) - f'_{ij}(\rho_{ij}^{jam})} \tag{2.21}$$

provides a lower bound on the time needed for characteristic to travel half way across any arc γ_{ij} of the network. Given the departure rates $u_{k,p}$, if the densities ρ_{ij} and the queues q_ℓ are known at time τ, one can uniquely determine the solution also for $t \in [\tau\,,\ \tau + \Delta_{min}]$, by solving separately the Cauchy problem in a neighborhood of each node. More precisely, let $P_{ij} = L_{ij}/2$ be the mid-point along the arc $\gamma_{ij} = [0, L_{ij}]$. Then for any given time $\tau \in \mathbb{R}$ the following holds.

1) Let A_ℓ be a departure node, with outgoing arc $\gamma_{\ell j}$. Let the departure rates $u_{k,p}(t)$ be given, for $t \in [\tau,\ \tau + \Delta_{min}]$. Moreover, let the traffic densities $\rho_{\ell j,p} = \theta_p \cdot \rho_{\ell j}$ be given at time τ, along the entire arc $\gamma_{\ell j}$. Then these initial and boundary data uniquely determine the traffic density $\rho_{\ell j,p}(t,x)$, for $t \in [\tau,\ \tau + \Delta_{min}]$ and $x \in [0,\ L_{\ell j}/2]$.

2) Let A_ℓ be an exit node, with incoming arc $\gamma_{i\ell}$. Let the traffic density $\rho_{i\ell}(\tau,\cdot)$ be given at time τ, along the entire arc $\gamma_{i\ell}$. Then these initial conditions uniquely determine the traffic density $\rho_{i\ell}(t,x)$, for $t \in [\tau,\ \tau + \Delta_{min}]$ and $x \in [L_{i\ell}/2\,,\ L_{i\ell}]$.

3) Let A_ℓ be a transit node, with incoming arcs $\gamma_{i\ell}$, $i \in \mathcal{I}_\ell$, and outgoing arcs $\gamma_{\ell j}$, $j \in \mathcal{O}_\ell$. Let the traffic densities $\rho_{i\ell,p}(\tau,\cdot)$ be given at time τ, along each of the the above arcs $\gamma_{i\ell}$, $\gamma_{\ell j}$, together with the sizes of the queues q_j, $j \in \mathcal{O}_\ell$. Then, for $t \in [\tau,\ \tau + \Delta_{min}]$, these initial conditions uniquely determine the traffic densities $\rho_{i\ell}(t,x)$ for $i \in \mathcal{I}_\ell$, $x \in [L_{i\ell}/2\,,\ L_{i\ell}]$, and $\rho_{\ell j}(t,x)$, for $j \in \mathcal{O}_\ell$, $x \in [0,\ L_{\ell j}/2]$.

To complete the inductive step, and determine the traffic densities $\rho_{\ell j,p} = \theta_p \cdot \rho_{\ell j}$ for all times, we still need a formula to determine the fraction θ_p of drivers following path Γ_p, along the outgoing roads $j \in \mathcal{O}_\ell$.

To fix the ideas, consider a node A_ℓ and let $\gamma_{i^*\ell}$ and $\gamma_{\ell j^*}$ be consecutive arcs in the path Γ_p (see Fig. 4, left). Let $q_{j^*}(t)$ be the length of the queue at the entrance of road $\gamma_{\ell j^*}$. To keep track of the composition of this queue, at each given time t let $\xi \in [0, q_{j^*}(t)]$ be a Lagrangian variable labeling drivers in this queue. Moreover, call $\Theta_{\ell,p} : [0, q_{j^*}(t)] \mapsto [0,1]$ the fraction of these drivers that follow the path Γ_p to reach their eventual destination (Fig. 4, right). Recalling (2.11) the function $\Theta_{\ell,p}$ can be determined by solving the linear boundary value problem

$$\Theta_{\ell,p}(t,0) = \frac{\bar f_{i^*}(t)\bar\theta_{i^*,p}(t)}{\sum_{i\in\mathcal{I}_\ell} \bar f_i(t)\bar\theta_{ij}(t)}\,, \tag{2.22}$$

$$\frac{\partial}{\partial t}\Theta_{\ell,p}(t,\xi) + \left(\sum_{i\in\mathcal{I}_\ell} \bar f_i(t)\bar\theta_{ij}(t)\right)\frac{\partial}{\partial \xi}\Theta_{\ell,p}(t,\xi) = 0\,. \tag{2.23}$$

Indeed, on the right hand side of (2.22) the numerator is the rate at which p-drivers (i.e., those following Γ_p to reach destination) join the queue q_{j^*}, while the denominator is the rate at which drivers of all types join this same queue. Call $\xi(t)$ the position of a particular driver inside this queue, i.e. the number of cars behind him, in the queue. Clearly, $\xi(t_0) = 0$ at the first time t_0 when this driver joins the queue, while $\xi(\tau) = q_{j^*}(\tau)$ at the time τ when he reaches the end of the queue. Since the map $t \mapsto \Theta_{\ell,p}(t, \xi(t))$ is constant, this yields (2.23).

On the outgoing road $\gamma_{\ell j^*}$, the boundary value for θ_p is now determined by

$$\theta_p(t, 0+) \;=\; \Theta_{\ell,p}(t, q_{j^*}(t)). \tag{2.24}$$

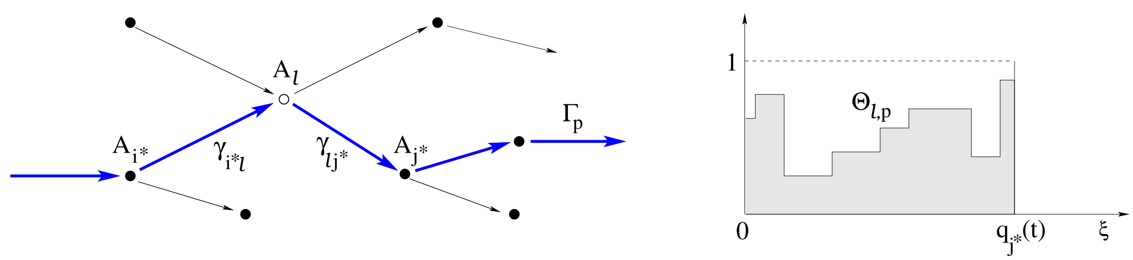


Figure 4: Left: a node A_ℓ along the path Γ_p. Right: the function $\xi \mapsto \Theta_{\ell,p}(t, \xi)$, determining the distribution of p-drivers within the queue q_{j^*}.

2.3 The optimal decision problems.

Let $G_{k,p}$ be the total number of drivers of the k-th group who travel along the path Γ_p. The admissibility condition implies $G_{k,1} + \cdots + G_{k,N} = G_k$. We use the Lagrangian variable $\beta \in [0, G_{k,p}]$ to label a particular driver in the subgroup $\mathcal{G}_{k,p}$ of k-drivers traveling along the path Γ_p. The departure and arrival time of this driver will be denoted by $\tau^d_{k,p}(\beta)$ and $\tau^a_{k,p}(\beta)$, respectively. Let $U^{depart}_{k,p}(t) = U_{k,p}(t)$ denote the amount of drivers of the subgroup $\mathcal{G}_{k,p}$ who have departed before time t. Similarly, let $U^{arrive}_{k,p}(t)$ be the amount of $\mathcal{G}_{k,p}$-drivers who have arrived at destination before time t. For a.e. $\beta \in [0, G_{k,p}]$ we then have

$$\tau^d_{k,p}(\beta) \;=\; \inf\Big\{t\,;\;\; U^{depart}_{k,p}(t) \;\geq\; \beta\Big\}, \qquad\qquad \tau^a_{k,p}(\beta) \;=\; \inf\Big\{t\,;\;\; U^{arrive}_{k,p}(t) \;\geq\; \beta\Big\}. \tag{2.25}$$

With this notation, the definition of globally optimal and of Nash equilibrium solution can be more precisely formulated.

Definition 2′. *An admissible family of departure distributions $\{U_{k,p}\}$ is a* **globally optimal solution** *if it provides a global minimum to the functional*

$$J \;\doteq\; \sum_{k,p} \int_0^{G_{k,p}} \Big(\varphi_k(\tau^d_{k,p}(\beta)) + \psi_k(\tau^a_{k,p}(\beta))\Big)\, d\beta\,. \tag{2.26}$$

Definition 3′. *An admissible family of departure distributions $\{U_{k,p}\}$ is a* **Nash equilibrium solution** *if there exist constants $c_1, \ldots, c_n$ such that:*

(i) For almost every $\beta \in [0, G_{k,p}]$ one has

$$\varphi_k(\tau^d_{k,p}(\beta)) + \psi_k(\tau^a_{k,p}(\beta)) \;=\; c_k\,. \tag{2.27}$$

(ii) For all $\tau \in \mathbb{R}$, there holds

$$\varphi_k(\tau) + \psi_k(T_{k,p}^{arrival}(\tau)) \;\geq\; c_k\,. \tag{2.28}$$

Here $T_{k,p}^{arrival}(\tau)$ is the arrival time of a driver that starts at time τ from the node $A_{d(k)}$ and reaches the node $A_{a(k)}$ traveling along the path Γ_p.

In other words, condition (i) states that all k-drivers bear the same cost c_k. Condition (ii) means that, regardless of the starting time τ, no k-driver can achieve a cost $< c_k$.

3 Continuity properties of the flow

The well posedness results proved in [6] apply to the Cauchy problem, where initial data are assigned on every road of the network at a given time $t = t_0$. On the other hand, to study optimal traffic assignment and users equilibria, we need to consider boundary conditions describing the departure rates $u_{k,p}(t)$, for any $t \in]-\infty, \;+\infty[$. The results in [6] on the well posedness of the Cauchy problem must therefore be adapted to this somewhat different situation. In particular, we need to study the continuous dependence of solutions w.r.t. weak convergence of the departure rates. Let $(u_{k,p})$ be an admissible family of departure rates. For every arc γ_{ij}, contained in some path Γ_p, we consider the following functions:

$V_{ij}(t,x)$ = total amount of cars that have crossed the point $x \in [0, L_{ij}]$ before time t.

$V_{ij,p}(t,x)$ = total amount of cars that have crossed the point $x \in [0, L_{ij}]$ before time t and follow the path Γ_p to reach their destination.

Clearly, $V_{ij}(t,x) = \sum_p V_{ij,p}(t,x)$. Given a sequence $(u_{k,p}^\nu)_{\nu\geq 1}$ of departure rates, the corresponding functions V_{ij}^ν, $V_{ij,p}^\nu$ are defined in the same way. In addition, we introduce

Definition 4. *A sequence $u^\nu = (u_{k,p}^\nu)$ of admissible departure rates is* **tight** *if, for every $\varepsilon > 0$, there exists T_ε such that the corresponding solutions satisfy*

$$\sum_{k,p} \left(\int_{-\infty}^{-T_\varepsilon} u_{k,p}^\nu(t)\,dt + \int_{T_\varepsilon}^{\infty} u_{k,p}^\nu(t)\,dt \right) \;<\; \varepsilon \qquad \textit{for all } \nu \geq 1. \tag{3.29}$$

According to (3.29), for every ν the total number of drivers departing before time $-T_\varepsilon$ or after T_ε is $< \varepsilon$.

The next lemma shows that, if the total number of cars traveling on the network is sufficiently small, all roads remain in a free state and no queue is formed at any intersection.

Lemma 3.1. *There exists $\varepsilon_0 > 0$ small enough and a travel time ΔT such that the following holds. Assume that the total amount of drivers departing before a given time T is*

$$\sum_{k,p} \int_{-\infty}^{T} u_{k,p}(t)\,dt \;\leq\; \varepsilon_0\,. \tag{3.30}$$

Then

(i) For $t < T$ all queues at all intermediate nodes are empty.

(ii) Every driver departing at a time $\tau^d \leq T - \Delta T$ reaches destination at time $\tau^q \leq T$.

(iii) Consider a second family of departure rates $\tilde{u}_{p,k}$, satisfying (3.30) together with

$$\tilde{u}_{p,k}(t) \;=\; u_{p,k}(t) \qquad \textit{for all } k,p \textit{ and all } t > T - \Delta T. \tag{3.31}$$

Then the corresponding densities and queues satisfy $\rho_{ij}(t,x) = \rho_{ij}(t,x)$, $q_j(t) = \tilde{q}_j(t)$ for all i,j, $x \in [0, L_{ij}]$, and $t > T$.

Proof. 1. Consider the quantities

$$\rho^\sharp \;\doteq\; \frac{1}{2}\min_{ij}\rho_{ij}^{max}, \qquad\qquad \delta^\sharp \;\doteq\; \min_{ij} f'_{ij}(\rho^\sharp) \;>\; 0\,.$$

Notice that, if the density satisfies $\rho(t,x) \leq \rho^\sharp$ then on any road γ_{ij}, the characteristic speed is

$$f'_{ij}(\rho) \;\geq\; f'_{ij}(\rho^\sharp) \;\geq\; \delta^\sharp.$$

We claim that, given $\rho^\flat > 0$, there exists $\varepsilon_0 > 0$ such that the following holds. If at any entrance node A_ℓ the total amount of drivers is $\leq \varepsilon_0$, on the first arc $[0, L_{\ell j}]$ the density satisfies

$$\rho_{\ell j}(t,x) \;\leq\; \rho^\flat \qquad \text{for all } t \text{ and all } \; x \in [L_{\ell j}/2\,,\; L_{\ell j}]. \tag{3.32}$$

Indeed, set

$$\delta_{\min} \;\doteq\; \frac{1}{2}\cdot\min_{ij}\;\frac{L_{ij}}{f'_{ij}(0)} \;<\; \Delta_{\min}.$$

For any $(t_0, x_0) \in \mathbb{R} \times [L_{\ell j}/2, L_{\ell j}]$, two cases are considered:

CASE 1: There exists $y_0 \in [0, L_{\ell j}[$ such that

$$V_{\ell j}(t_0,x_0) \;=\; V_{\ell j}(t_0 - \delta_{\min}, y_0) - \delta_{\min}\cdot g_{\ell j}\Big(\frac{y_0 - x_0}{\delta_{\min}}\Big)\,,$$

and

$$f'_{\ell j}(\rho_{\ell j}(t_0,x_0)) \;=\; \frac{y_0 - x_0}{\delta_{\min}}\,.$$

In this case we have

$$g_{\ell j}\Big(f'_{\ell j}(\rho_{\ell j}(t_0,x_0))\Big) \;=\; -\,\frac{V_{\ell j}(t_0,x_0) - V_{\ell j}(t_0 - \delta_{\min}, y_0)}{\delta_{\min}} \;\geq\; -\,\frac{\varepsilon_0}{\delta_{\min}}\,.$$

Choosing $\varepsilon_0 > 0$ sufficiently small, this yields (3.32).

CASE 2: There exists $\tau \in \;]t_0 - \delta_{\min}, t_0[$ such that

$$V_{\ell j}(t_0,x_0) \;=\; V_{\ell j}(\tau, L_{\ell j}) - (t_0 - \tau)\cdot g_{\ell j}\Big(\frac{x_0 - L_{\ell j}}{t_0 - \tau}\Big)\,.$$

Assume that A_{ℓ_0} is the exit node of road $\gamma_{\ell j}$. From [6], we have

$$V_{\ell j}(\tau, L_{\ell j}) \;\geq\; \delta_{\min}\cdot\min\,\Big\{f_{ij}^{\max}\;,\; c_j\Big(M - \sum_{k\in\mathcal{O}_\ell}\|q_k\|_{\mathbf{L}^\infty[t_0-\delta_{\min},t_0]}\Big)\Big\} \quad \text{in case of } \textbf{(SBJ)}\,,$$

and

$$V_{\ell j}(\tau, L_{\ell j}) \;\geq\; \delta_{\min} \cdot \min\, \left\{ f_{ij}^{\max}\;,\; c_j \cdot \frac{M_k - \|q_k\|_{\mathbf{L}^\infty[t_0-\delta_{\min}, t_0]}}{\theta_{jk}(t)}; k \in \mathcal{O}_\ell \right\} \quad \text{in case of } \textbf{(MBJ)}\,.$$

On the other hand, we also have that $V_{\ell j}(\tau, L_{\ell j}) \leq \varepsilon_0$ and $q_k(\tau) \leq K\varepsilon_0$ for all τ. Thus, it yields a contradiction if $\varepsilon_0 > 0$ is small enough.

Then by induction, if $\rho^\flat$ is chosen sufficiently small and (3.32) holds for every entrance arc, then no queue is ever formed and the maximum density is $\rho_{ij}(t,x) \leq \rho^\sharp$ one every other arc of the network. In particular, this achieves the proof of (i).

2. From step 1, for $0 < \varepsilon_0 < 1$ small enough, one can see that on any arc γ_{ij} the flow is always free at any time $t < T$, i.e. $\rho_{ij}(t,x) \leq \rho_{ij}^{max}$. This yields the uniform bound

$$v_{ij}(\rho_{ij}(t,x)) \;\geq\; v_{\min} \;\doteq\; \inf_{ij}\; v_{ij}(\rho_{ij}^{max}) \;>\; 0 \quad \text{for all } t < T\,.$$

On the other hand, the waiting time in the queue at any entrance node A_ℓ of each driver who departs before time T is less than $\frac{\varepsilon_0}{\min_{ij}\; f_{ij}^{\max}} < \frac{1}{\min_{ij}\; f_{ij}^{\max}}$. Define

$$\Delta T \;\doteq\; \frac{1}{\min_{ij}\; f_{ij}^{\max}} + \frac{\sum_{ij} L_{ij}}{v_{\min}}\,.$$

Then the total traveling time of any driver who departs before time $T - \Delta T$ is less than ΔT. This yields (ii).

3. Notice that

$$v_{ij}(\rho) \;\geq\; f'_{ij}(\rho)\,.$$

By (ii) and the non-crossing of backward characteristics, the value of $\rho_{ij}(T,x)$ for $x \in [0, L_{ij}[$ depends only on the value of $\{u_{k,p}\}$ in $[T-\Delta T, T]$. Thus, from (3.31), one has that

$$\tilde\rho_{ij}(T,x) \;=\; \rho_{ij}(T,x), \qquad x \in [0, L_{ij}[\,.$$

Recalling that for $t \in\,]-\infty, T]$ there is no queue at any transit node, we obtain (iii). □

In the following, given a node A_ℓ, the incoming arcs will be denoted by $\gamma_{i\ell}$, with $i \in \mathcal{I}_\ell$, while outgoing arcs are $\gamma_{\ell j}$, $j \in \mathcal{O}_\ell$. At the node A_ℓ, we denote by $q_{\ell j}(t)$ the length of the queue of cars waiting to enter the outgoing road $\gamma_{\ell j}$. Relying on the analysis in [6], we now prove

Lemma 3.2. *Consider a tight sequence of admissible departure rates $u^\nu = (u^\nu_{k,p})$ which satisfy the uniform bounds*

$$0 \;\leq\; u^\nu_{k,p}(t) \;\leq\; M_0 \qquad \textit{for all } \; t \in \mathbb{R}\,. \tag{3.33}$$

for some constants M_0 and all k, p, ν. Then, by possibly taking a subsequence, as $\nu \to \infty$ one has the weak convergence

$$u^\nu_{k,p} \;\rightharpoonup\; u^*_{k,p} \tag{3.34}$$

for some admissible family $(u^*_{k,p})$ of departure rates. In addition, one has *(i) For any $T>0$, as $\nu\to\infty$ one has the convergence*

$$V^\nu_{ij}(t,x)\to V^*_{ij}(t,x)\,, \tag{3.35}$$

$$V^\nu_{ij,p}(t,x)\to V^*_{ij,p}(t,x), \tag{3.36}$$

$$q^\nu_{\ell j}(t)\to q^*_{\ell j}(t), \tag{3.37}$$

uniformly for all $x\in[0,L_{ij}]$ and $t\in\,]-\infty,T]$. In turn, for every t one has

$$\rho^\nu_{ij}(t,\cdot)\;\to\;\rho^*_{ij}(t,\cdot)\qquad\qquad in\;\;\mathbf{L}^1([0,L_{ij}]). \tag{3.38}$$

(ii) If all drivers reach their destination before some fixed time $T^*>T$, then there exists a constant $v_{min}>0$ such that all velocities v^ν_{ij} on all roads satisfy the uniform lower bound

$$v^\nu_{ij}(t,x)\;\geq\;v_{min} \tag{3.39}$$

for all t,x,ν.

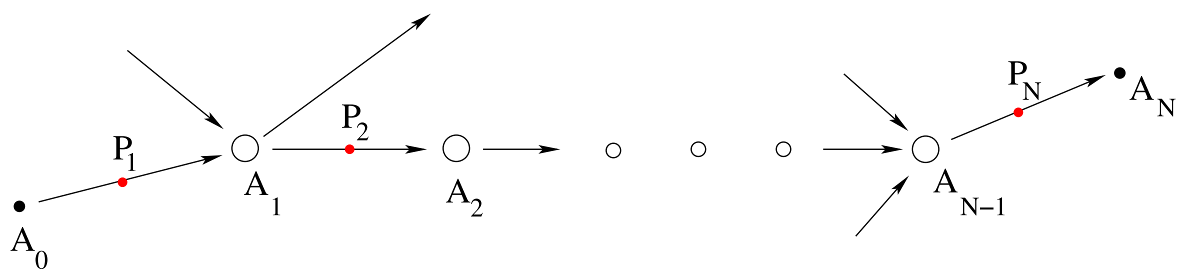


Figure 5:

Proof. Because of the tightness assumption, an entirely standard argument yields the existence of a subsequence converging to an admissible family of departure rates $(u^*_{k,p})$. For clarity of exposition, we first prove (i)-(ii) assuming that all $u^\nu_{k,p}(t)$ vanish for $t<-T$. At the end, we describe the modifications needed to cover the general case.

1. We start by proving (i), assuming that no driver departs before time $-T$. For $t\leq -T$ all functions ρ^ν_{ij}, V^ν_{ij}, $q^\nu_{\ell j}$ are thus identically zero and the result is trivially true. Recalling (2.21), consider the times

$$\tau_n\;=\;-T+n\,\Delta_{min}\,.$$

By induction, assume that the convergence in (3.35)-(3.38) holds on every arc γ_{ij} and every $t\leq\tau_n$.

2. For each departure node $A_\ell\in\mathcal{N}_d$, consider the initial-boundary value problem with initial data given at $t=\tau_n$ and boundary data at $x=0$ given for $t\in[\tau_n,\,\tau_{n+1}]$. By finite propagation speed these data uniquely determine the solution on the domain $t\in]\tau_n,\tau_{n+1}]$ and $x\in[0,\,L_{\ell j}/2]$. Call $u_{\ell,p}$ the rate of departures from node A_ℓ of drivers who follow the path Γ_p. The total number of departures up to time t is computed by

$$U_\ell(t)\;\doteq\;\sum_p U_{\ell,p}(t)\,,\qquad\qquad U_{\ell,p}(t)\;\doteq\;\int_{-\infty}^t u_{\ell,p}(\tau)\,d\tau\,.$$

For every $\nu \geq 1$, the Lax type formula derived in [6] yields

$$
\begin{aligned}
V^\nu_{\ell j}(t,x) \;=\; \min\Bigg\{ & \min_{y\geq 0}\left\{ V^\nu_{\ell j}(\tau_n,y) - (t-\tau_n)\cdot g_{\ell j}\left(\frac{x-y}{t-\tau_n}\right)\right\}, \\
& \min_{\tau_n\leq s\leq t}\left\{ U^\nu_\ell(s) - (t-s)\cdot g_{\ell j}\left(\frac{x}{t-s}\right)\right\}\Bigg\}.
\end{aligned}
\tag{3.40}
$$

We recall that $g_{\ell j}$ is the Legendre transform of the flux function $f_{\ell j}$, as in (2.3). The assumptions (3.33)-(3.34) imply the uniform convergence $U^\nu_\ell \to U^*_\ell$, while the inductive assumption yields the uniform convergence

$$
V_{\ell j}(\tau_n,x) \;\to\; V^*_{\ell j}(\tau_n,x) \qquad\qquad x\in[0,L_{\ell j}].
$$

By (3.40) this yields the convergence

$$
V^\nu_{\ell j}(t,x) \;\to\; V^*_{\ell j}(t,x), \qquad\qquad (t,x)\in[\tau_n,\,\tau_{n+1}]\times[0,L_{\ell j}/2]. \tag{3.41}
$$

In turn, since

$$
(V^\nu_{\ell j})_x \;=\; -\,\rho^\nu_{\ell j}\,, \qquad\qquad (V^*_{\ell j})_x \;=\; -\,\rho^*_{\ell j}\,,
$$

(3.41) implies the weak convergence

$$
\rho^\nu_{\ell j}(\tau_n,\cdot) \;\rightharpoonup\; \rho^*_{\ell j}(\tau_n,\cdot) \qquad\qquad \text{on} \quad [0,L_{\ell j}/2] \tag{3.42}
$$

for $t\in[\tau_n,\,\tau_{n+1}]$. We now observe that, by Oleinik's estimates, the functions $\rho^\nu_{\ell j}(\tau_n,\cdot)$ have uniformly bounded variation on any subinterval of the form $[\varepsilon,\,L_{\ell j}/2]$, with $\varepsilon>0$. Therefore, the weak convergence (3.42) implies the strong convergence

$$
\|\rho^\nu_{\ell j}(\tau_n,\cdot)-\rho^*_{\ell j}\|_{\mathbf{L}^1([0,L_{\ell j}/2])} \;\to\; 0. \tag{3.43}
$$

It remains to prove the uniform convergence in (3.36), for each path Γ_p. For this purpose, given $t\in[\tau_n,\tau_{n+1}]$ and $x\in[0,L_{\ell j}/2]$, consider the departure time of the driver who reaches point x on the road $\gamma_{\ell j}$ at time t, namely

$$
\begin{aligned}
\tau^\nu(t,x) &\;=\; \inf\big\{s\leq t\,;\quad U^\nu_\ell(s) \;=\; V^\nu_{j\ell}(t,x)\big\},\\
\tau^*(t,x) &\;=\; \inf\big\{s\leq t\,;\quad U^*_\ell(s) \;=\; V^*_{j\ell}(t,x)\big\}.
\end{aligned}
$$

By the uniform convergence $V^\nu_{j\ell}\to V^*_{j\ell}$ and $U^\nu_\ell\to U^*_\ell$ it follows

$$
\liminf_{\nu\to\infty}\tau^\nu(t,x) \;\geq\; \tau^*(t,x). \tag{3.44}
$$

Therefore

$$
\begin{aligned}
\liminf_{\nu\to\infty}\;\sum_p\int_{-\infty}^{\tau^*(t,x)} u^\nu_{\ell,p}(s)ds \;&\leq\; \liminf_{\nu\to\infty}\;\sum_p\int_{-\infty}^{\tau^\nu(t,x)} u^\nu_{\ell,p}(s)ds\\
&=\; \liminf_{\nu\to\infty}\;V^\nu_{\ell j}(t,x) \;=\; V^*_{\ell j}(t,x) \;\leq\; \sum_p\int_{-\infty}^{\tau^*(t,x)} u^*_{\ell,p}(s)ds\,.
\end{aligned}
$$

Recalling that $u^\nu_{\ell,p} \rightharpoonup u^*_{\ell,p}$, we have $\lim_{\nu\to\infty} \int_{-\infty}^{\tau^*(t,x)} u^\nu_{\ell,p}(s)ds = \int_{-\infty}^{\tau^*(t,x)} u^*_{\ell,p}(s)ds$. Therefore,

$$\lim_{\nu\to\infty} \sum_p \int_{-\infty}^{\tau^*(t,x)} u^\nu_{\ell,p}(s)ds ~=~ \lim_{\nu\to\infty} \sum_p \int_{-\infty}^{\tau^\nu(t,x)} u^\nu_{\ell,p}(s)ds ~=~ \sum_p \int_{-\infty}^{\tau^*(t,x)} u^*_{\ell,p}(s)ds\,.$$

For every p this implies

$$\lim_{\nu\to\infty} V^\nu_{\ell j,p}(t,x) ~=~ \lim_{\nu\to\infty} \int_{-\infty}^{\tau^\nu(t,x)} u^\nu_{\ell,p}(s)ds ~=~ \int_{-\infty}^{\tau^*(t,x)} u^*_{\ell,p}(s)ds ~=~ V^*_{\ell j,p}(t,x)\,,$$

proving the convergence (3.36). Since all functions $V^\nu_{\ell j,p}$ are uniformly Lipschitz continuous, the convergence holds uniformly on bounded sets.

3. A similar argument is valid for a terminal arc $\gamma_{i\ell}$, ending at some arrival node A_ℓ. Indeed, consider the Cauchy problem with initial data given at $t=\tau_n$ for $x\in[0,L_{i\ell}]$. By the Lax formula,

$$V_{j\ell}(t,x) ~=~ \min_{y\geq 0} \left\{ V_{j\ell}(\tau_n,y) - (t-\tau_n)\cdot g_{j\ell}\left(\frac{x-y}{t-\tau_n}\right)\right\}, \tag{3.45}$$

these data uniquely determine the solution on the domain $t\in[\tau_n,\tau_{n+1}]$ and $x\in[L_{i\ell}/2\,,~L_{i\ell}]$. The inductive assumption yields the uniform convergence

$$V^\nu_{j\ell}(\tau_n,x) ~\to~ V^*_{j\ell}(\tau_n,x)\,, \qquad\qquad x\in[0,~L_{i\ell}]. \tag{3.46}$$

Using (3.46) in (3.45) we obtain in turn the uniform convergence

$$V^\nu_{i\ell}(t,x) ~\to~ V^*_{i\ell}(t,x) \qquad\qquad (t,x)\in[\tau_n,\tau_{n+1}]\times[L_{i\ell}/2\,,~L_{i\ell}],$$

the weak convergence $\rho^\nu_{j\ell}(t,\cdot) ~\rightharpoonup~ \rho^*_{j\ell}(t,\cdot)$ on $[L_{i\ell}/2\,,~L_{i\ell}]$, and finally the strong convergence $\|\rho^\nu_{\ell j}(t,\cdot)-\rho^*_{\ell j}(t,\cdot)\|_{\mathbf{L}^1([0,L_{\ell j}/2])}\to 0$, for every $t\in[\tau_n,~\tau_{n+1}]$.

To prove the convergence (3.36) for every p, we argue as follows. Fix any $(t,x)\in[\tau_n,\tau_{n+1}]\times[L_{j\ell}/2,~L_{j\ell}]$ and define

$$\tau^\nu(t,x) ~=~ \inf\left\{s\leq t\,;\quad V^\nu_{j\ell}(s,0+)=V^\nu_{j\ell}(t,x)\right\},$$
$$\tau^*(t,x) ~=~ \inf\left\{s\leq t\,;~~ V^*_{j\ell}(s,0+)=V^*_{j\ell}(t,x)\right\}.$$

Observe that $\tau(t,x)\leq\tau_n$. Since $V^\nu_{j\ell}(t,x) ~\to~ V^*_{j\ell}(t,x)$ and $V^\nu_{j\ell}(\cdot,0+) ~\to~ V^*_{j\ell}(\cdot,0+)$ uniformly for $t\leq\tau_n$, the inequality (3.44) again holds. Therefore,

$$\begin{aligned}\liminf_{\nu\to\infty} \sum_p V^\nu_{j\ell,p}(\tau^*(t,x),0+) ~&\leq~ \liminf_{\nu\to\infty} \sum_p V^\nu_{j\ell,p}(\tau^\nu(t,x),0+)\\ &=~ \liminf_{\nu\to\infty} \sum_p V^\nu_{j\ell,p}(t,x) ~=~ \liminf_{\nu\to\infty} V^\nu_{j\ell}(t,x) ~=~ V^*_{j\ell}(t,x) ~=~ \sum_p V^*_{j\ell,p}(t,x)\,.\end{aligned}$$

On the other hand, by the inductive assumption, we also have

$$\liminf_{\nu\to\infty} \sum_p V^\nu_{j\ell,p}(\tau^*(t,x),0+) ~=~ \sum_p V^*_{j\ell,p}(\tau^*(t,x),0+) ~=~ \sum_p V^*_{j\ell,p}(t,x)\,.$$

Therefore, for all p it holds

$$\lim_{\nu\to\infty} V^\nu_{j\ell,p}(t,x) \;=\; V^*_{j\ell,p}(t,x)\,.$$

4. Next, consider a transit node A_ℓ. By the inductive assumption, at time τ_n we have the strong convergence

$$\begin{cases} \|\rho^\nu_{i\ell}(\tau_n,\cdot)-\rho^*_{i\ell}(\tau_n,\cdot)\|_{\mathbf{L}^1([0,L_{i\ell}])} \;\to\; 0 \qquad & i\in\mathcal{I}_\ell\,,\\[2mm] \|\rho^\nu_{\ell j}(\tau_n,\cdot)-\rho^*_{\ell j}(\tau_n,\cdot)\|_{\mathbf{L}^1([0,L_{\ell j}])} \;\to\; 0 \qquad & j\in\mathcal{O}_\ell\,,\end{cases} \tag{3.47}$$

together with the convergence of the queue sizes

$$q^\nu_{\ell j}(\tau_n) \;\to\; q^*_{\ell j}(\tau_n)\,, \tag{3.48}$$

and, for every p, the uniform convergence

$$\begin{cases} V^\nu_{i\ell,p}(\tau_n,x) \;\to\; V^*_{i\ell,p}(\tau_n,x) \qquad \hbox{on}\quad [0,L_{i\ell}] \qquad & i\in\mathcal{I}_\ell\,,\\[2mm] V^\nu_{\ell j,p}(\tau_n,x) \;\to\; V^*_{\ell j,p}(\tau_n,x) \qquad \hbox{on}\quad [0,L_{\ell j}] \qquad & j\in\mathcal{O}_\ell\,.\end{cases} \tag{3.49}$$

By (3.49) we also have the weak convergence

$$\rho^\nu_{i\ell}(\tau_n,\cdot)\cdot\theta^\nu_{ij}(\tau_n,\cdot) \;\rightharpoonup\; \rho^*_{i\ell}(\tau_n,\cdot)\cdot\theta^*_{ij}(\tau_n,\cdot) \qquad \hbox{on}\quad [0,L_{i\ell}] \qquad i\in\mathcal{I}_\ell\,. \tag{3.50}$$

Notice, however, that here the strong convergence in $\mathbf{L}^1$ may not hold, because the coefficients θ_{ij} satisfy a linear transport equation and can have unbounded variation. For all $t\in[\tau_n,\,\tau_{n+1}]$, according to Theorem 2 in [6] one has the weak convergence

$$\begin{cases} \rho^\nu_{i\ell}(t,\cdot) \;\rightharpoonup\; \rho^*_{i\ell}(t,\cdot) \qquad \hbox{on}\quad [L_{i\ell}/2,\,L_{i\ell}] \qquad & i\in\mathcal{I}_\ell\,,\\[2mm] \rho^\nu_{\ell j}(t,\cdot) \;\rightharpoonup\; \rho^*_{\ell j}(t,\cdot) \qquad \hbox{on}\quad [0,L_{\ell j}/2] \qquad & j\in\mathcal{O}_\ell\,,\end{cases} \tag{3.51}$$

together with the uniform convergence of the queue sizes

$$q^\nu_{\ell j}(t) \;\to\; q^*_{\ell j}(t). \tag{3.52}$$

For any $i\in\mathcal{I}_\ell$ and $(t,x)\in[\tau_n,\tau_{n+1}]\times[L_{i\ell}/2\,,\;L_{i\ell}]$, let

$$\tau^*(t,x) \;=\; \inf\bigl\{s\leq t\,;\;\; V^*_{i\ell}(s,0+)=V^*_{i\ell}(t,x)\bigr\}\,.$$

With the same argument in step 3, one can show that

$$\lim_{\nu\to\infty}\sum_p V^\nu_{i\ell,p}(\tau^*(t,x),0+) \;=\; \lim_{\nu\to\infty}\sum_p V^\nu_{i\ell,p}(\tau^\nu(t,x),0+) \;=\; \sum_p V^*(\tau^*(t,x),0+)\,.$$

Thus, for all p and $i\in\mathcal{I}_\ell$, it holds

$$\lim_{\nu\to\infty} V^\nu_{i\ell,p}(t,x) \;=\; V^*_{i\ell,p}(t,x) \qquad \hbox{for all } (t,x)\in[\tau_n,\tau_{n+1}]\times[L_{i\ell}/2\,,\;L_{i\ell}]\,.$$

To complete this step, we need to show that, for every $j \in \mathcal{O}_\ell$ and every p one has

$$\lim_{\nu\to\infty} V^\nu_{\ell j,p}(t,x) \;=\; V^*_{\ell j,p}(t,x) \qquad \text{for all } (t,x) \in [\tau_n, \tau_{n+1}] \times [0, L_{\ell j}/2]\,. \tag{3.53}$$

Indeed, for any $(t,x) \in [\tau_n, \tau_{n+1}] \times [0, , L_{\ell j}/2]]$ and $i \in \mathcal{I}_\ell$, let

$$\tau(t,x) \;=\; \inf\left\{ \tau \le t \;\middle|\; V_{i\ell}(\tau, P_{i\ell}) = V_{\ell j}(t,x)\right\},$$

one has

$$\lim_{\nu\to\infty} \sum_p V^\nu_{i\ell,p}(\tau^*(t,x), P_{i\ell}) \;=\; \lim_{\nu\to\infty} \sum_p V^\nu_{i\ell,p}(\tau^\nu(t,x), P_{i\ell}) \;=\; \sum_p V^*(\tau^*(t,x), P_{i\ell})\,.$$

This implies (3.53).

5. The proof of the convergence (3.35) is now achieved by induction on n. Since all functions V_{ij} are uniformly Lipschitz continuous w.r.t. both t, x, it is clear that the convergence is uniform for t, x in bounded sets.

In turn, this implies the weak convergence

$$\rho^\nu_{ij}(t,\cdot) \;\rightharpoonup\; \rho^*_{ij}(t,\cdot) \tag{3.54}$$

for every time t. Since the flux function f_{ij} is strictly concave, by Oleinik's estimate the restriction of $\rho^\nu_{ij}(t,\cdot)$ to each compact subinterval of $]0, L_{ij}[$ has uniformly bounded variation. Therefore, the weak convergence (3.54) implies the strong convergence (3.38).

6. In this step we remove the assumption that $u^\nu_{k,p}(t) = 0$ for all k, p and $t < -T$. For this purpose, for each integer $N \ge 1$ consider the truncated functions

$$u^{\nu,N}_{k,p}(t) \;=\; \begin{cases} u^\nu_{k,p}(t) & \text{if } t \ge -N, \\ 0 & \text{if } t < -N. \end{cases}$$

Let $V^{\nu,N}_{ij}$, $V^{*,N}_{ij}$, etc... be the corresponding functions, obtained by replacing $u^\nu_{k,p}$ with $u^{\nu,N}_{k,p}$. Recalling Lemma 3.1, consider any $\varepsilon \in]0, \varepsilon_0]$ and consider any two integers N, N' such that $T_\varepsilon + \Delta T < N < N'$. By Lemma 3.1, for every $t > T_\varepsilon$ we have

$$\rho^{\nu,N}_{ij}(t,x) \;=\; \rho^{\nu,N'}_{ij}(t,x)$$

for every road γ_{ij}. Hence, the position of any driver departing after time T_ε will be exactly the same in the two solutions with departure rates $u^{\nu,N}_{k,p}$ and $u^{\nu,N'}_{k,p}$. This implies

$$|V^{\nu,N}_{ij}(t,x) - V^{\nu,N'}_{ij}(t,x)| \;\le\; \varepsilon, \qquad |V^{\nu,N}_{ij,p}(t,x) - V^{\nu,N'}_{ij,p}(t,x)| \;\le\; \varepsilon,$$

for all i, j, p, t, x, provided that $T_\varepsilon < N < N'$. Since $\varepsilon > 0$ was arbitrary, letting $N \to \infty$ we obtain the convergence in (3.35)-(3.36).

7. Toward a proof of (ii), consider any transit node $A_\ell \in \mathcal{N}_t$. By assumption, for $t < -T$ all roads and all buffers q_j are empty. For $t \in [-T, T^*]$, the queues $q_{\ell,j}$ may be strictly positive. However, the buffers never become completely full.

More precisely, assume first that the flow at the node A_ℓ is governed by **(SBJ)**, where M is the size of the single buffer. Then, by Remark 1 in [6], there exists a constant $C > 0$ such that, for all $\nu, j \in \mathcal{O}_\ell$ and $t \in [-T, T^*]$ one has

$$M - \sum_{j \in \mathcal{O}_\ell} q_j^\nu(t) \ \geq C\,. \tag{3.55}$$

Next, assume that the flow at the node A_ℓ is governed by **(MBJ)**, with buffers of sizes M_j, $j \in \mathcal{O}_\ell$. Again by Remark 1 in [6], there exists a constant $C > 0$ such that, for all $\nu, j \in \mathcal{O}_\ell$ and $t \in [-T, T^*]$ one has

$$M_j - q_j^\nu(t) \ \geq \ C\,. \tag{3.56}$$

8. On an exit arc γ_{ij} the flow is always free, i.e. $\rho_{ij}(t,x) \leq \rho_{ij}^{max}$. This yields the uniform bound $v_{i,j} \geq v_{ij}(\rho_{ij}^{max}) \ > \ 0$.

Next, consider any arc $\gamma_{i\ell}$ ending at the transit node A_ℓ. Fix any $x \in]0, L_{i\ell}[\,,\ t \in [-T, T^*]$. Two cases can be considered.

CASE 1: The backward characteristic though (t,x) has positive speed: $f'_{i\ell}(\rho_{i\ell}(t,x)) \geq 0$. In this case $\rho_{i\ell}(t,x)) \in [0, \rho_{i\ell}^{max}]$ and we conclude as before: $v_{i\ell} \geq v_{i\ell}(\rho_{i\ell}^{max}) > 0$.

CASE 2: The backward characteristic though (t,x) has negative speed: $f'_{i\ell}(\rho_{i\ell}(t,x)) < 0$. In this case, this characteristic originates at some point $(\tau, L_{i,\ell})$, for some $\tau < t \leq T^*$. By (3.55) or (3.56), the flux $\bar f_{i\ell}(\tau) \doteq f_{i\ell}(\tau, L_{i\ell}-)$ exiting from road $\gamma_{i\ell}$ is strictly positive. Indeed, for a single buffer we have

$$\bar f_{i\ell}(\tau) \ = \ c_i \Big(M - \sum_{j \in \mathcal{O}_\ell} q_j(\tau) \Big) \ \geq \ c_i\, C\,,$$

while in the case of multiple buffer we have

$$\bar f_{i\ell}(\tau) \ = \ \min_{j \in \mathcal{O}_\ell} \ \frac{c_i(M_j - q_j(\tau))}{\theta_{ij}} \ \geq \ c_i\, C\,.$$

Since both the density and the flux are constant along characteristics, this implies

$$f_{i\ell}(\rho_{i\ell}(t,x)) \ \geq \ c_i C\,.$$

Observing that $v(\rho) = f(\rho)/\rho$, we obtain the uniform lower bound

$$v_{i\ell}(\rho_{i\ell}(t,x)) \ \geq \ \frac{c_i C}{\rho_{i\ell}^{jam}} \ > \ 0\,.$$

□

Assuming that the vehicle speed $v(\rho)$ remains uniformly positive, the following lemma shows that the arrival time of any car depends Hölder continuously on the departure time.

Lemma 3.3. *Let all departure rates $u_{k,p}(t)$ be uniformly bounded as in (3.33), and assume that the speed $v(\rho)$ remains uniformly positive, on all roads. Then, for every viable path Γ,* *there exist constants K, α such that the following holds. For any two cars departing at times $\tau < \tilde{\tau}$ and traveling along Γ, the arrival times $T^a(\tau) < T^a(\tilde{\tau})$ satisfy*

$$T^a(\tilde{\tau}) - T^a(\tau) \;\leq\; K(\tilde{\tau} - \tau)^\alpha. \tag{3.57}$$

Proof. 1. Consider two drivers, joining the queue at the entrance of a given road γ at times $T_{queue} < \widetilde{T}_{queue}$. Call $T_{depart} < \widetilde{T}_{depart}$ the times where they clear the queue and start traveling along γ. Since the total flux of cars joining the queue is uniformly bounded, and the rate at which cars flow out of the queue is uniformly positive, the difference between the departure times can be bounded as

$$\widetilde{T}_{depart} - T_{depart} \;\leq\; C' \cdot (\widetilde{T}_{queue} - T_{queue})\,, \tag{3.58}$$

for some uniform constant C'.

Next, consider two drivers traveling along the road γ, departing at times $T_{depart} < \widetilde{T}_{depart}$. Call $T_{arrive} < \widetilde{T}_{arrive}$ the times when they arrive at the end of road γ. To estimate the difference between these arrival times, let L be the length of the road and call $p(t), \tilde{p}(t) \in [0, L]$ respectively the positions of the two cars at time t. Observe that $p, \tilde{p}$ satisfy the ODE with discontinuous right hand side

$$\dot{p}(t) \;=\; v\Big(\rho(t, p(t))\Big) \qquad\qquad t \in [T_{depart}\,,\; T_{arrive}]\,. \tag{3.59}$$

By assumption, v is bounded and uniformly positive. Hence the distance between the two drivers at the time when the second one departs is bounded by

$$p(\widetilde{T}_{depart}) - \tilde{p}(\widetilde{T}_{depart}) \;=\; p(\widetilde{T}_{depart}) \;\leq\; v^{max} \cdot (\widetilde{T}_{depart} - T_{depart}).$$

Since the time difference $T_{arrive} - T_{depart}$ is a priori bounded, by Theorem 2.2 in [11] the distance between the two drivers at time $t = T_{arrive}$ when the first one arrives can be estimated as

$$p(T_{arrive}) - \tilde{p}(T_{arrive}) \;\leq\; C \cdot \Big(p(\widetilde{T}_{depart}) - \tilde{p}(\widetilde{T}_{depart})\Big)^\alpha, \tag{3.60}$$

for some constants $C, \alpha > 0$. Since the second driver travels with uniformly positive speed $v \geq v_{min} > 0$, his arrival time will satisfy

$$\begin{aligned} \widetilde{T}_{arrive} - T_{arrive} \;&\leq\; \frac{p(T_{arrive}) - \tilde{p}(T_{arrive})}{v_{min}} \\ \leq\; \frac{C}{v_{min}} \cdot \Big(p(\widetilde{T}_{depart}) - \tilde{p}(\widetilde{T}_{depart})\Big)^\alpha \;&\leq\; \frac{C}{v_{min}} \cdot \Big(v_{max} \cdot (\widetilde{T}_{depart} - T_{depart})\Big)^\alpha. \end{aligned} \tag{3.61}$$

2. After a relabeling, it is not restrictive to assume that Γ is the concatenation of $N-1$ arcs, joining the nodes $A_1, A_2, \ldots, A_N$. Namely,

$$\Gamma \;=\; \Big(\,\gamma_{12}\,,\;\; \ldots\;,\, \gamma_{N-1,N}\Big). \tag{3.62}$$

Consider a driver starting his journey at A_1 at time τ. For $k = 1, 2, \ldots, N$, define:

$T^1_{queue} = \tau =$ time when the car joins the queue at the entrance of the first arc γ_{12},

$T^{k-1}_{arrive} = T^k_{queue} =$ time when the car arrives at the node A_k, joining the queue to enter $\gamma_{k,k+1}$,

$T^k_{depart} =$ time when the queue at node A_k is cleared, and the car starts moving along $\gamma_{k,k+1}$,

$T^N_{arrive} = T^a(\tau) =$ time when the car arrives at the final node A_N.

Define the corresponding times $\widetilde{T}^{k-1}_{arrive} = \widetilde{T}^k_{queue}$, $\widetilde{T}^k_{depart}$, for a driver starting at time $\tilde{\tau}$.

By (3.58), for every k there exists a constant C'_k such that

$$\widetilde{T}^k_{depart} - T^k_{depart} \;\leq\; C'_k \cdot (\widetilde{T}^k_{queue} - T^k_{queue})\,. \tag{3.63}$$

By (3.61), for every k there exist constants C_k, α_k such that

$$\widetilde{T}^k_{arrive} - T^k_{arrive} \;\leq\; C_k\, (\widetilde{T}^k_{depart} - T^k_{depart})^{\alpha_k}. \tag{3.64}$$

Since the composition of Hölder continuous maps is Hölder continuous, by induction on $k = 1,\ldots,N$ we obtain (3.57). $\square$

The next lemma states the uniform convergence of the travel times along any path Γ.

Lemma 3.4. *Consider a sequence of departure rates $u^\nu = (u^\nu_{k,p})$ satisfying the uniform bounds (3.33). Assume that, as $\nu \to \infty$, one has the weak convergence $u^\nu_{k,p} \rightharpoonup u^*_{k,p}$ for all k, p. In addition, assume that the car speed remains uniformly positive, on all roads, say $v^\nu_{ij}(\rho(t,x)) \geq v_{min} > 0$.*

Let Γ be any viable path, and call $\tau^\nu(t), \tau^*(t)$ the corresponding arrival times of a driver who departs at time t and travels along Γ. One then has the convergence

$$\lim_{\nu\to\infty} \tau^\nu(t) \;=\; \tau^*(t), \tag{3.65}$$

uniformly for t in bounded sets.

Proof. 1. By taking a subsequence, we can assume:

(i) The uniform convergence of the queue sizes at each node A_ℓ

$$q^\nu_{\ell,j}(t) \;\to\; q^*_{\ell,j}(t) \qquad\qquad j \in \mathcal{O}_\ell\,. \tag{3.66}$$

(ii) The uniform convergence of the functions V_{ij}, namely

$$V^\nu_{ij}(t,x) \;\to\; V^*_{ij}(t,x) \qquad\qquad x \in [0, L_{ij}]. \tag{3.67}$$

(iii) The $\mathbf{L}^1$ convergence of the densities on each arc γ_{ij}:

$$\rho^\nu_{ij}(t,\cdot) \;\to\; \rho^*_{ij}(t,\cdot) \qquad\quad \text{in} \quad \mathbf{L}^1([0, L_{ij}]). \tag{3.68}$$

2. After a relabeling, we can assume that Γ has the form (3.62). For each $k = 1,\ldots,N-1$, consider a driver arriving at the node A_k at time t. Denote by $T^{\nu,k}_{depart}(t) \geq t$ the time when this driver starts moving on the following road $\gamma_{k,k+1}$, possibly after spending some time in the queue. Notice that these functions $T^{\nu,k}_{depart}$ are uniformly Lipschitz continuous, for t in bounded sets. By (i)-(ii), as $\nu \to \infty$ we have the convergence

$$T^{\nu,k}_{depart}(t) \ \to \ T^{*,k}_{depart}(t), \tag{3.69}$$

uniformly for t in bounded sets.

3. Next, consider a driver starting to move along the road $\gamma_{k,k+1}$ at time t. Denote by $T^{\nu,k}_{arrive}(t) \geq t$ the time when this driver reaches the end of this road. By (iii), using Theorem 2.2 in [11], we obtain the pointwise convergence

$$T^{\nu,k}_{arrive}(t) \ \to \ T^{*,k}_{arrive}(t)\,. \tag{3.70}$$

Since all functions $T^{\nu,k}_{arrive}$ are uniformly Hölder continuous, the convergence is uniform for t in bounded sets.

4. We now observe that, with the previous notation, the arrival time of a driver starting at time t and traveling along the path Γ in (3.62) can be written as the composition

$$\tau^\nu(t) \ = \ T^{\nu,N}_{arrive} \circ T^{\nu,N}_{depart} \circ \cdots \circ T^{\nu,k}_{arrive} \circ T^{\nu,k}_{depart} \circ \cdots \circ T^{\nu,1}_{arrive}(t).$$

The convergence $\tau^\nu(t) \to \tau^*(t)$ thus follows from (3.69)-(3.70), by an inductive argument. □

4 Globally optimal solutions

In this section we establish the existence of a globally optimal solution. The proof follows the direct method of the Calculus of Variations, constructing a minimizing sequence of solutions and showing that a subsequence converges to the optimal one.

Theorem 4.1. (existence of a globally optimal solution). *Let the flux functions f_{ij} and the cost functions φ_k, ψ_k satisfy the assumptions (A1)-(A2). Then, for any n-tuple $(G_1, \ldots, G_n)$ of positive numbers, there exists an admissible family of departure rates $u_{k,p}$ which yield a globally optimal solution of the traffic flow problem. These rates are uniformly bounded.*

Proof. 1. By possibly adding a constant, because of (A2) it is not restrictive to assume that $\varphi_k(t) + \psi_k(t) \geq 0$ for every time t. Calling m_0 the infimum of the total costs in (2.26), taken among all admissible departure rates $\{u_{k,p}\}$, this implies $m_0 \geq 0$.

We first claim that $m_0 < +\infty$. Indeed, let $G = \sum_k G_k$ be the total number of drivers, and choose an integer N large enough so that $G/N \leq \varepsilon_0$. We can then partition the set of all drivers into N subgroups, each with size $\leq \varepsilon_0$. We let all drivers of the first group start at time $t_0 = 0$. By Lemma 3.1, these drivers will all arrive at destination within time $t_1 = \Delta T$. We then let all drivers of the second group depart at time t_1. In turn, they will all arrive within time $t_2 = 2\,\Delta T$. Continuing by induction, we let the drivers of the N-th group depart at time $t_{N-1} = (N-1)\cdot\Delta T$. By Lemma 3.1, all of these drivers will arrive within time $t_N = N\cdot\Delta T$. The total cost of this strategy is bounded by

$$G\cdot\max_k\left(\max_{t\in[0,N\cdot\Delta T]}\varphi_k(t)+\max_{t\in[0,N\cdot\Delta T]}\psi_k(t)\right)\ \le\ G\cdot\max_k\Big(\varphi_k(0)+\psi_k(N\,\Delta T)\Big)\ <\ \infty.$$

Recalling Definitions 1 and 4, consider a minimizing sequence of departure rates $u^\nu_{k,p}$, and let

$$U^\nu_{k,p}(t)\ \doteq\ \int_{-\infty}^t u^\nu_{k,p}(t)\,dt\,.$$

$$G^\nu_{k,p}\ =\ \int_{-\infty}^\infty u^\nu_{k,p}(t)\,dt.$$

By choosing a subsequence, we can assume

$$\lim_{\nu\to\infty}G^\nu_{k,p}\ =\ G_{k,p}\qquad\text{with}\qquad\sum_p G_{k,p}\ =\ G_k\,.\tag{4.1}$$

2. Fix $\varepsilon>0$. By (A2), there is $T_\varepsilon>0$ such that for all $k\in\{1,2,...,n\}$ it holds

$$\varphi_k(t)+\psi_k(t)\ \ge\ \frac{m_0+1}{\varepsilon}\qquad\text{for all } t\in\,]-\infty,-T_\varepsilon[\ \cup\ ]T_\varepsilon,\infty[\,.\tag{4.2}$$

For $\beta\in[0,G^\nu_{k,p}]$, let

$$\beta\ \mapsto\ \tau^{d,\nu}_{k,p}(\beta)\quad\text{and}\quad\beta\ \mapsto\ \tau^{a,\nu}_{k,p}(\beta)$$

describe the departure and arrival time of the β-driver, in the subgroup $G_{k,p}$. From (A2), we have

$$\varphi_k(\tau^{d,\nu}_{k,p}(\beta))+\varphi_k(\tau^{a,\nu}_{k,p}(\beta))\ \ge\ \varphi_k(\tau^{d,\nu}_{k,p}(\beta))+\varphi_k(\tau^{d,\nu}_{k,p}(\beta))\,.$$

Thus,

$$\begin{aligned}&\int_0^{U^\nu_{k,p}(-T_\varepsilon)}\varphi_k(\tau^{d,\nu}_{k,p}(\beta))+\varphi_k(\tau^{a,\nu}_{k,p}(\beta))\ d\beta\\&\qquad\ge\ \int_0^{U^\nu_{k,p}(-T_\varepsilon)}\varphi_k(\tau^{d,\nu}_{k,p}(\beta))+\varphi_k(\tau^{d,\nu}_{k,p}(\beta))\ d\beta\ \ge\ U^\nu_{k,p}(-T_\varepsilon)\cdot\frac{m_0+1}{\varepsilon}\,.\end{aligned}$$

Since the total cost approaches the infimum m_0, there exists $N_0>0$ sufficiently large such that for all $\nu>N_0$

$$U^\nu_{k,p}(-T_\varepsilon)\ =\ \int_{-\infty}^{-T_\varepsilon}u^\nu_{k,p}(t)\,dt\ \le\ \varepsilon\,.\tag{4.3}$$

On the other hand, from (A2), we also have

$$\varphi_k(\tau^{d,\nu}_{k,p}(\beta))+\varphi_k(\tau^{a,\nu}_{k,p}(\beta))\ \ge\ \varphi_k(\tau^{a,\nu}_{k,p}(\beta))+\varphi_k(\tau^{a,\nu}_{k,p}(\beta))\,.$$

Set

$$T_{\varepsilon^{a,\nu}}\ \doteq\ \sup\ \big\{t\in\mathbb{R}\,;\ \ \tau^{a,\nu}_{k,p}(U^\nu_{k,p}(t))\le T_\varepsilon\big\}\,.\tag{4.4}$$

Thus,

$$\int_{U^\nu_{k,p}(T_{\varepsilon^{a,\nu}})}^{G^\nu_{k,p}} \varphi_k(\tau^{d,\nu}_{k,p}(\beta)) + \varphi_k(\tau^{a,\nu}_{k,p}(\beta))\ d\beta$$
$$\geq \int_{U^\nu_{k,p}(T_{\varepsilon^{a,\nu}})}^{G^\nu_{k,p}} \varphi_k(\tau^{a,\nu}_{k,p}(\beta)) + \varphi_k(\tau^{a,\nu}_{k,p}(\beta))\ d\beta \ \geq\ (G^\nu_{k,p} - U^\nu_{k,p}(T_{\varepsilon^{\nu,a}})) \cdot \frac{m_0+1}{\varepsilon}\,.$$

Since the total cost approaches the infimum m_0, there exists $N_0 > 0$ sufficiently large such that for all $\nu > N_0$

$$G^\nu_{k,p} - U^\nu_{k,p}(T_{\varepsilon^{\nu,a}}) \ \leq\ \varepsilon\,. \tag{4.5}$$

3. We claim that it is not restrictive to assume

$$u^\nu_{k,p}(t) \ \leq\ \max_{ij}\ f^{max}_{ij} \qquad \text{for a.e.} \ \ t\,. \tag{4.6}$$

Indeed, if one of these departure rates does not satisfy (4.6), then a queue is formed at some entrance node. But this is certainly not optimal. We can construct a second departure rate $\tilde{u}^\nu_{k,p}$ where each driver departs at exactly the same time where he would have cleared the queue in the original configuration. The departure time of each driver is later, while the arrival time is exactly the same. Hence the family $\{\tilde{u}^\nu_{k,p}\}$ yields a total cost which is no greater than $\{u^\nu_{k,p}\}$.

4. Choosing a subsequence, we can assume the weak convergence $u^\nu_{k,p} \rightharpoonup u_{k,p}$. This implies the uniform convergence $U^\nu_{k,p} \to U_{k,p}$. From (4.3) and (4.5), the limit family of departure rates $\{u_{k,p}\}$ is admissible.

To complete the proof, we show that the family of departure rates $\{u_{k,p}\}$ is optimal. From (4.3), (4.5) one has that

$$\tau^{d,\nu}_{k,p}(\beta), \tau^{a,\nu}_{k,p}(\beta) \in [-T_\varepsilon, T_\varepsilon] \qquad \text{for all } \beta \in [\varepsilon, G_{k,p} - \varepsilon]\,,$$

for a constant $T_\varepsilon > 0$. Moreover, the map $\beta \mapsto \tau^{d,\nu}_{k,p}(\beta)$ is nondecreasing. By Helly's compactness theorem, we can assume that

$$\tau^{d,\nu}_{k,p}(\beta) \ \to\ \tau^{d}_{k,p}(\beta) \qquad \text{for a.e. } \beta \in [\varepsilon, G_{k,p} - \epsilon]\,.$$

On the other hand, by Lemma (3.2), for $\varepsilon > 0$ sufficiently small, we have that $V^\nu_{ij,p}(t,x) \to V^*_{ij,p}(t,x)$ uniformly for all $x \in [0, L_{ij}]$ and $t \in]-\infty, T_\varepsilon[$. Thus, by using again Helly's theorem we obtain

$$\tau^{a,\nu}_{k,p}(\beta) \ \to\ \tau^{a}_{k,p}(\beta) \qquad \text{for a.e. } \beta \in [\varepsilon, G_{k,p} - \varepsilon]\,.$$

Therefore,

$$\sum_{k,p} \int_\varepsilon^{G_{k,p}-\varepsilon} \varphi_k(\tau^d_{k,p}(\beta)) + \psi_{k,p}(\tau^a_{k,p}(\beta))\ d\beta \ =\ \lim_{\nu\to\infty} \sum_{k,p} \int_\varepsilon^{G_{k,p}-\varepsilon} \varphi_k(\tau^{d,\nu}_{k,p}(\beta)) + \psi_{k,p}(\tau^{a,\nu}_{k,p}(\beta))\ d\beta$$
$$\leq\ \lim_{\nu\to\infty} \sum_{k,p} \int_0^{G_{k,p}} \varphi_k(\tau^{d,\nu}_{k,p}(\beta)) + \psi_{k,p}(\tau^{a,\nu}_{k,p}(\beta)) d\beta \ =\ m_0\,.$$

Since $\varepsilon > 0$ was arbitrary, this implies

$$\sum_{k,p} \int_0^{G_{k,p}} \varphi_k(\tau^d_{k,p}(\beta)) + \psi_{k,p}(\tau^a_{k,p}(\beta)) d\beta \ \leq \ m_0 \,,$$

completing the proof. □

5 Nash equilibria

In this section we prove the existence of a Nash equilibrium solution for traffic flow on a network. Toward this goal, in addition to (A1)-(A2), we need an additional assumption ruling out the possibility that drivers remain stuck in traffic for an arbitrarily large time.

(A3) Given the n-tuple $(G_1, \ldots, G_n)$, there exists a sufficiently large constant K such that, for every admissible family of departure rates $\{u_{k,p}\}$ satisfying (1.4), the time spent on the road by every driver is $\leq K$.

Theorem 5.1. (existence of a Nash equilibrium). *Let the flux functions f_{ij} and the cost functions φ_k, ψ_k satisfy the assumptions (A1)-(A2). Fix any n-tuple $(G_1, \ldots, G_n)$ and assume that (A3) holds. Then*

(i) There exists at least one admissible family of departure rates $\{u^*_{k,p}\}$ which yields a Nash equilibrium solution.

(ii) In every Nash equilibrium solution, all departure rates are uniformly bounded and have compact support.

Proof. Thanks to the continuity results proved in the previous sections, the proof can rely on the same ideas used in [4].

1. There exists a time interval $I_0 = [-T_0, T_0]$ so large that, in any Nash equilibrium, no driver will depart or arrive at a time $t \notin I_0$. Indeed, assume that a k-driver departs at time $t = 0$. Let T_1 be an upper bound on the time he needs to reach destination, under the worst possible traffic conditions. Then the total cost to this driver will be not larger than $\varphi_k(0) + \psi_k(T_1)$. By the assumptions (2.2) on the cost functions, there exists T_0 large enough so that

$$\varphi_k(-T_0) > \varphi_k(0) + \psi_k(T_1), \qquad\qquad \psi_k(T_0) > \varphi_k(0) + \psi_k(T_1). \tag{5.1}$$

Hence it is never convenient to depart at a time $t \notin [-T_0, T_0]$. This proves (ii).

2. Let $f_{max} \doteq \max_{i,j} f^{max}_{ij}$ be an upper bound for the fluxes over all arcs, and define

$$\varphi'_{max} \doteq \max_{1\leq k\leq n} \max_{t\in I_0} |\varphi'_k(t)| \,, \qquad\qquad \psi'_{min} \doteq \min_{1\leq k\leq n} \min_{t\in I_0} \psi'_k(t).$$

The same argument used in the proof of Theorem 2 in [4] shows that, in a Nash equilibrium, all departure rates $u_{k,p}$ must satisfy the priori bound

$$u_{k,p}(t) \ \leq \ \kappa \ \doteq \ \frac{\varphi'_{max} \cdot f_{max}}{\psi'_{min}} \qquad\qquad \text{for a.e. } t \,. \tag{5.2}$$

3. Let κ be as in (5.2) and let $G \doteq \sum_{k=1}^n G_k$ be total number of drivers. Choosing the time

$$T \;\doteq\; T_0 + \frac{G}{\kappa}\,, \tag{5.3}$$

we consider the family of admissible departure rates

$$\begin{array}{rl}\mathcal{U} \;\doteq\; \Bigg\{ (u_{k,p})_{1\le k\le n,\,1\le p\le N}\,; & u_{k,p} : \mathbb{R} \mapsto [0, 4\kappa]\,, \qquad u_{k,p}(t) = 0 \;\text{ for }\; t \notin [-T, T]\,, \\ & \displaystyle\sum_{p\in\mathcal{V}_k} \int u_{k,p}(t)\,dt \;=\; G_k \quad \text{for every } k \in \{1, 2, ..., n\} \Bigg\},\end{array} \tag{5.4}$$

which is a closed convex subset of $\mathbf{L}^1(\mathbb{R};\, \mathbb{R}^{n\times N})$.

For each fixed $\nu \ge 1$, we consider a finite dimensional subset $\mathcal{U}_\nu \subset \mathcal{U}$ consisting of all $u = (u_{k,p})$ which are piecewise constant on time intervals of length T/ν, i.e.,

$$\begin{array}{rl}\mathcal{U}_\nu \;\doteq\; \Big\{ u = (u_{k,p}) \in \mathcal{U}\,; & \\ & \text{every function } u_{k,p} \text{ is constant on each subinterval } I^\nu_\ell \;\doteq\;]t^\nu_{\ell-1},\, t^\nu_\ell] \Big\},\end{array} \tag{5.5}$$

where

$$t^\nu_\ell \;\doteq\; \frac{\ell}{\nu}\, T\,, \qquad\qquad -\nu \le \ell \le \nu\,.$$

4. Given $u = (u_{k,p}) \in \mathcal{U}$, let $\tau_q(t)$ be the arrival time of a driver starting at time t and traveling along the path Γ_q. Clearly, this arrival time depends on the overall traffic conditions, hence on all functions $u_{k,p}$. If this driver belongs to the j-th family, his total cost is

$$\Phi^{(u)}_{j,q}(t) \;=\; \varphi_j(t) + \psi_j(\tau_q(t))\,.$$

We now observe that, for each $\nu \ge 1$, the domain $\mathcal{U}_\nu$ is a finite dimensional, compact, convex subset of $\mathbf{L}^2([-T, T];\, \mathbb{R}^{n\times N})$. Moreover, by Lemma 3.4 the map $u \mapsto \Phi^{(u)}_{k,p}(\cdot)$ are continuous from $\mathcal{U}_\nu$ into $\mathbf{L}^2$. Hence, by the theory of variational inequalities [25], there exists a function $\bar u^\nu = (\bar u^\nu_{j,q}) \in \mathcal{U}_\nu$ which satisfies

$$\sum_{j,q} \int_{-T}^{T} \Phi^{(\bar u^\nu)}_{j,q}(t) \cdot \Big(v_{j,q}(t) - \bar u^\nu_{j,q}(t) \Big)\, dt \;\ge\; 0 \qquad\qquad \text{for all } v \in \mathcal{U}_\nu\,. \tag{5.6}$$

5. We now let $\nu \to \infty$. By the previous steps, there exists a sequence of piecewise constant functions $\bar u^\nu = (\bar u^\nu_{k,p}) \in \mathcal{U}_\nu$ such that (5.6) holds for every $\nu \ge 1$. Since all functions $\bar u^\nu_{k,p}$ are uniformly bounded and supported inside the interval $I = [-T, T]$, by taking a subsequence we can assume the weak convergence

$$(\bar u^\nu_{k,p}) \;\rightharpoonup\; (u^*_{k,p}) \tag{5.7}$$

for some function $u^* = (u^*_{k,p}) \in \mathcal{U}$. From Lemma 3.4 and Lemma 3.3, by Ascoli's compactness theorem we can assume that

$$\tau^\nu_{k,p}(t) \;\to\; \tau^*_{k,p}(t) \qquad\qquad \text{for all } k, p, \text{ uniformly for } t \in [-T, T], \tag{5.8}$$

By the continuity of $\varphi(\cdot)$ and $\psi(\cdot)$, we also get

$$\Phi^{(\bar u^\nu)}_{k,p}(\cdot) \;\to\; \Phi^{(u^*)}_{k,p}(\cdot) \qquad\qquad \text{for all } k,p \text{ uniformly for } t\in[-T,T]\,.$$

We claim that the departure rates $u^*_{k,p}$ yield a Nash equilibrium solution. More precisely:

(NE) Given any $k\in\{1,2,...,n\}, p\in\mathcal{V}_k$, any $t_1\in Supp(u^*_{k,p})$, $t_2\in\mathbb{R}$ and any path Γ_q with the same initial and final nodes as Γ_p, one has

$$\Phi^{(u^*)}_{k,p}(t_1) \;\leq\; \Phi^{(u^*)}_{k,q}(t_2). \tag{5.9}$$

Indeed, (5.9) implies that no k-driver can lower his own cost by switching to the time t_2 or choosing the alternative path Γ_q to reach destination. We recall that t is in the support of a function $f\in\mathbf{L}^1$ if and only if $\int_{t-\varepsilon}^{t+\varepsilon}|f(s)|\,ds\neq 0$ for every $\varepsilon>0$.

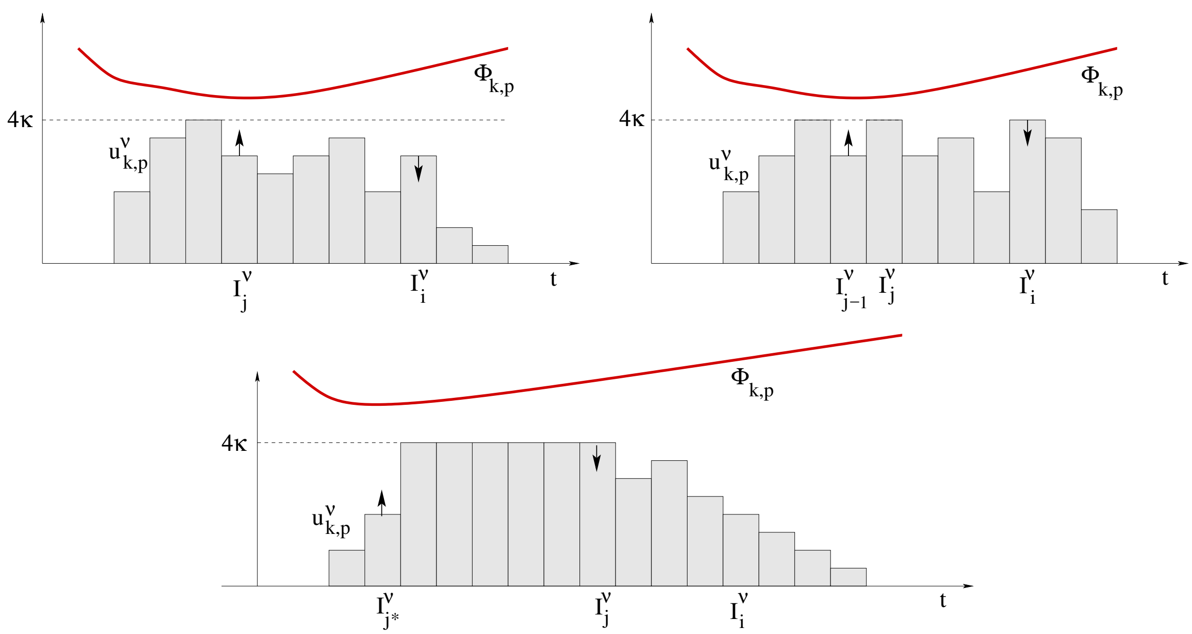


Figure 6: Proving that the limit departure rates $\{u^*_{k,p}\}$ provide a Nash equilibrium. If (NE) fails, different cases are considered. In Case 1 (top left) the average of the cost $\Phi_{k,p}$ on the interval I_i^ν is higher than on the interval I_j^ν. To obtain a contradiction with (5.6) we simply move some of the mass from I_i^ν to I_j^ν. In Case 2a (top right) one cannot increase the value of $u^\nu_{k,p}$ on the interval I_j^ν because of the constraint $u\leq 4\kappa$. However, some mass can be moved from I_i^ν to the previous interval I^ν_{j-1}. In Case 2b (bottom) there are several adjacent intervals where $u^\nu_{k,p}\equiv 4\kappa$. In this case, if $I^\nu_{j^*}$ is the first interval to the left of I^ν_j where $u^\nu_{k,p}<4\kappa$, we argue that (i) $t^\nu_{j^*}>-T$, and (ii) the average of the cost $\Phi_{k,p}$ on $I^\nu_{j^*}$ is strictly less than on I^ν_j. In this last case, to obtain a contradiction with (5.6) we move some mass from I^ν_j to $I^\nu_{j^*}$.

6. If (5.9) fails, then by continuity there exists $\delta>0$ such that

$$\Phi^{(u^*)}_{k,p}(t) \;>\; \Phi^{(u^*)}_{k,q}(t')+2\delta \qquad\qquad \text{whenever } |t-t_1|\leq 2\delta,\quad |t'-t_2|\leq 2\delta\,. \tag{5.10}$$

By uniform convergence, for all ν large enough we have

$$\Phi_{k,p}^{(\bar u^\nu)}(t) \ > \ \Phi_{k,q}^{(\bar u^\nu)}(t') + \delta \qquad\qquad \text{whenever } |t-t_1|\le 2\delta, \quad |t'-t_2|\le 2\delta\,. \tag{5.11}$$

Observe that it is not restrictive to assume that $t_2\in[-T_0,T_0]$. Indeed, if (5.9) fails for some $t_2\notin[-T_0,T_0]$, then (5.1) implies

$$\Phi_{k,p}^{(u^*)}(t_1) \ > \ \Phi_{k,q}^{(u^*)}(t_2) \ > \ \Phi_{k,q}^{(u^*)}(0),$$

and we can simply replace t_2 with zero.

The weak convergence (5.7), together with the assumption on the support of the function $u^*_{k,p}$, now implies

$$\lim_{\nu\to\infty}\int_{t_1-\delta}^{t_1+\delta}\bar u^\nu_{k,p}(t)\,dt \ = \ \int_{t_1-\delta}^{t_1+\delta} u^*_{k,p}(t)\,dt \ > \ 0\,.$$

Therefore, for every ν sufficiently large we can find two intervals

$$I_i^\nu \ = \ ]t^\nu_{i-1},\, t^\nu_i] \ \subset \ [t_1-\delta\,,\ t_1+\delta], \qquad\qquad I_j^\nu \ = \ ]t^\nu_{j-1},\, t^\nu_j] \ \subset \ [t_2-\delta\,,\ t_2+\delta] \tag{5.12}$$

with $t^\nu_j > t_2$ and $\bar u^\nu_{k,p}(t)>0$ for $t\in I^\nu_i$.

7. We now derive a contradiction, showing that, for ν sufficiently large, the departure rates $\bar u^\nu_{k,p}$ do not satisfy the variational inequality (5.6). Two possibilities can arise

CASE 1: $\bar u^\nu_{k,q,j} \ < \ 4\kappa$. In this case we define a new set of departure rates $v^\varepsilon=(v^\varepsilon_{k,p})\in\mathcal{U}_\nu$ by setting

$$\begin{aligned} v^\varepsilon_{k,p}(t) \ &= \ u^\nu_{k,p}(t)-\varepsilon \qquad \text{if} \quad t\in I^\nu_i\,,\\ v^\varepsilon_{k,q}(t) \ &= \ u^\nu_{k,q}(t)+\varepsilon \qquad \text{if} \quad t\in I^\nu_j\,, \end{aligned}$$

and setting $v^\varepsilon_{h,r}(t)=u^\nu_{h,r}(t)$ in all other cases. Notice that, if $\varepsilon=\min\{u^\nu_{k,p,i},\ 4\kappa-u^\nu_{k,q,j}\}$ then these new departure rates are still admissible. By (5.11) and (5.12), this construction yields

$$\sum_{h,r}\int_{-T}^{T}\Phi_{h,r}^{(\bar u^\nu)}(t)\cdot\Big(v^\varepsilon_{h,r}(t)-\bar u^\nu_{h,r}(t)\Big)\,dt \ = \ \varepsilon\int_{I^\nu_j}\Phi_{k,q}^{(\bar u^\nu)}(t)\,dt-\varepsilon\int_{I^\nu_i}\Phi_{k,p}^{(\bar u^\nu)}(t)\,dt \ \le \ -2\varepsilon\delta\,, \tag{5.13}$$

providing a contradiction with (5.6).

CASE 2: $\bar u^\nu_{k,q,j} \ = \ 4\kappa$. If this equality holds, consider the index

$$j^* \ \doteq \ \max\,\{i<j\,;\ \ \bar u^\nu_{k,q,i} \ < \ 4\kappa\}\,.$$

Notice that $t^\nu_{j^*} > -T$. Indeed, by construction $t_2>-T_0$. If $t^\nu_{j^*}\le -T$, by (5.3) this would imply

$$\bar u^\nu_{k,q}(t) = \ 4\kappa \qquad\qquad \text{for all} \ \ t\in[t^\nu_{j^*}\,,t^\nu_j] \ \supseteq \ [-T,\,-T_0]\,,$$

$$\int\bar u^\nu_{k,q}(t)\,dt \ \ge \ 4\kappa(t^\nu_j-t^\nu_{j^*}) \ \ \ge \ 4\kappa(T-T_0) > \ G\,,$$

reaching a contradiction. We consider two subcases.

CASE 2a: $j^* = j-1$. In this case, since it is not restrictive to assume $\frac{T}{\nu} < \frac{\delta}{4}$, we have $I^\nu_{j-1} = [t^\nu_{j-2}, t^\nu_{j-1}] \subset [t_2 - \delta,\ t_2 + \delta]$. We can thus derive a contradiction as in CASE 1, simply replacing j by $j-1$.

CASE 2b: $j^* \leq j-2$. Observe that, for all $s_1 < s_2$,

$$\tau^\nu_{k,q}(s_2) - \tau^\nu_{k,q}(s_1) \ \geq\ \frac{1}{f_{max}} \int_{s_1}^{s_2} \bar{u}^\nu_{k,q}(\xi)\, d\xi\,. \tag{5.14}$$

In particular, for any $s_1 \in I^\nu_{j^*}$ and $s_2 \in I^\nu_j$ we have

$$\tau^\nu_{k,q}(s_2) - \tau^\nu_{k,q}(s_1) \ \geq\ \frac{1}{f_{max}} \int_{s_1}^{s_2} \bar{u}^\nu_{k,q}(\xi)\, d\xi \ \geq\ \frac{1}{f_{max}}\, 4\kappa\, [t_{j-1} - t_{j^*}] \ \geq\ \frac{4\kappa(s_2 - s_1)}{3 f_{max}}\,. \tag{5.15}$$

This yields the estimate

$$\psi_k\big(\tau^\nu_{k,q}(s_2)\big) - \psi_k\big(\tau^\nu_{k,q}(s_1)\big) \ \geq\ \psi'_{min}\big(\tau^\nu_{k,q}(s_2) - \tau^\nu_{k,q}(s_1)\big) \ \geq\ \psi'_{min} \cdot \frac{4\kappa(s_2 - s_1)}{3 f_{max}}\,. \tag{5.16}$$

On the other hand, we have

$$\varphi_k(s_2) - \varphi_k(s_1) \ \geq\ -\,\varphi'_{max}(s_2 - s_1)\,. \tag{5.17}$$

Recalling the definition of the constant κ in (5.2), from (5.16)-(5.17) we obtain

$$\Phi^{\bar{u}^\nu}_{k,q}(s_2) - \Phi^{\bar{u}^\nu}_{k,q}(s_1) \ \geq\ \left(\frac{4\kappa\psi'_{min}}{3 f_{max}} - \varphi'_{max}\right)(s_2 - s_1) \ =\ \frac{1}{3}\varphi'_{max} \cdot (s_2 - s_1) \tag{5.18}$$

for all $s_1 \in I^\nu_{j^*}$ and $s_2 \in I^\nu_j$.

We now choose the departure rates $v^\varepsilon = (v^\varepsilon_{k,p}) \in \mathcal{U}_\nu$ by setting

$$\begin{array}{rcll} v^\varepsilon_{k,p}(t) & = & u^\nu_{k,q}(t) - \varepsilon & \quad \text{if} \quad t \in I^\nu_j\,, \\ v^\varepsilon_{k,q}(t) & = & u^\nu_{k,q}(t) + \varepsilon & \quad \text{if} \quad t \in I^\nu_{j^*}\,, \end{array} \tag{5.19}$$

and setting $v^\varepsilon_{h,r}(t) = u^\nu_{h,r}(t)$ in all other cases. Notice that, if $\varepsilon = \min\{u^\nu_{k,q,j},\ 4\kappa - u^\nu_{k,q,j^*}\}$ then these new departure rates are still admissible.

Using (5.18) with $s_1 = t$, $s_2 = t + t^\nu_j - t^\nu_{j^*}$ we compute

$$\begin{array}{l} \displaystyle\sum_{h,r} \int_{-T}^{T} \Phi^{(\bar{u}^\nu)}_{h,r}(t) \cdot \Big(v^\varepsilon_{h,r}(t) - \bar{u}^\nu_{h,r}(t)\Big)\, dt \ =\ \varepsilon \left(\int_{I^\nu_{j^*}} \Phi^{(\bar{u}^\nu)}_{k,q}(t)\, dt - \int_{I^\nu_j} \Phi^{(\bar{u}^\nu)}_{k,q}(t)\, dt\right) \\ \displaystyle =\ \varepsilon \int_{I^\nu_{j^*}} \Big(\Phi^{(\bar{u}^\nu)}_{k,q}(t) - \Phi^{(\bar{u}^\nu)}_{k,q}(t + t^\nu_j - t^\nu_{j^*})\Big)\, dt \ \leq\ -\frac{\varepsilon}{3}\varphi'_{max} \cdot (t^\nu_j - t^\nu_{j^*}) \ <\ 0\,. \end{array} \tag{5.20}$$

Once again we reached a contradiction with (5.6), completing the proof. □

6 Stuck traffic

In this section we discuss the key assumption (A3) used in Theorem 5.1. Namely, regardless of the departure rates $\{u_{k,p}\}$, the travel time of every driver should remain uniformly bounded.

We begin with an example showing that, in some cases, (A3) can fail. A similar situation was considered in [9] in connection with a traffic circle.

Example 1. Consider a network with 9 arcs, as in Fig. 7, left. Assume there are three groups of drivers:

- $\mathcal{G}_1$, departing at node 1, arriving at node 7,
- $\mathcal{G}_2$, departing at node 2, arriving at node 8,
- $\mathcal{G}_3$, departing at node 3, arriving at node 9.

Assume that, at each of the transit nodes 4,5,6, the two incoming arcs are given equal priority. In other words, if both incoming roads are congested, then cars are admitted to the intersection at equal rate from the two roads.

Assume that the maximum flux on every road is $f^{max} = 1$, and assume that for $t > 0$, cars depart from nodes $1, 2, 3$ at unit rate. Call Δ the triangle of roads joining the intermediate nodes 4,5,6. At each time $t > 0$, at each intermediate node the rate at which cars enter Δ is at least twice as the rate at which cars exit from Δ. We thus conclude that

> *For each $t > 0$, the total number of drivers that have reached their destination within time t is smaller than the number of drivers that at time t are still located within the triangle of roads Δ, including the buffers at the nodes $4, 5, 6$.*

Since the total amount of cars that can be contained in the triangle of roads Δ and in the buffers at the nodes 4,5,6, is finite, this implies that only a fixed number of drivers will reach their destination in finite time, while all the others will be stuck in traffic forever.

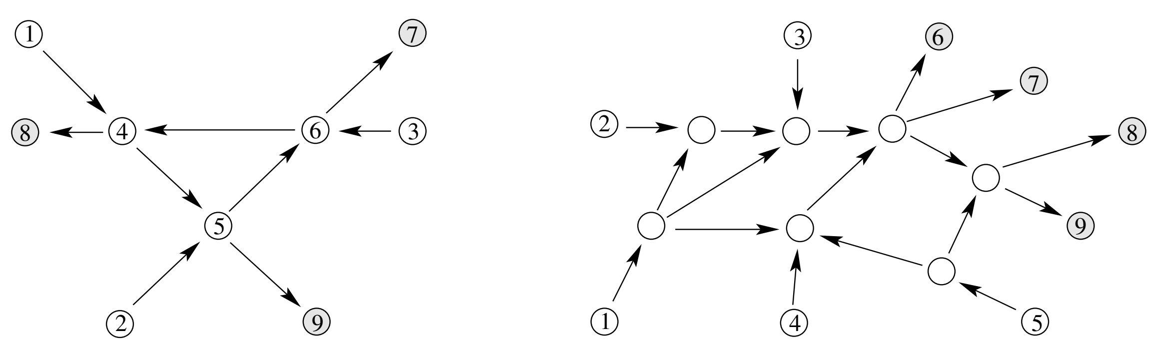


Figure 7: Left: a network where the traffic can become completely stuck. For some departure rates, part of the drivers never reach their destination. Right: a network that does not contain cycles. In this case, for any departure rates, all drivers will eventually reach their destinations.

In the following, by a *cycle* we mean a path such as (1.2) with $i(0) = i(N)$, so that the initial and terminal nodes coincide.

Lemma 6.1. *If the network of roads does not contain any cycle, then for every n-tuple $(G_1, \ldots, G_n)$ the assumption (A3) is satisfied.*

Proof. 1. Call $G = G_1 + \cdots + G_n$ the total number of drivers. Let $\mathcal{A} = \{\gamma_\alpha\,; \quad \alpha \in I\}$ be the set of all the arcs in the network. We say that γ_α precedes γ_β, and write $\gamma_\alpha \prec \gamma_\beta$, if there exist a chain of arcs Γ where γ_α is the first arc and γ_β is the last one. In other words, $\gamma_\alpha \prec \gamma_\beta$ if it is possible to drive first along γ_α and then along γ_β.

If the network has no cycles, the ordering "$\prec$" is strict. We can thus partition the set $\mathcal{A}$ of all arcs into disjoint classes, say

$$\mathcal{A} \;=\; \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m\,,$$

in such a way that

$$\gamma_\alpha \in \mathcal{A}_i\,, \qquad \gamma_\beta \in \mathcal{A}_j\,, \qquad i \leq j$$

implies that γ_β does not precede γ_α.

2. To prove an upper bound on the travel time, it suffices to prove a uniformly positive lower bound on the speed v, on all roads. This will be achieved by backward induction on $i = m, m-1, \ldots, 2, 1$.

Exit arcs $\gamma_\alpha \in \mathcal{A}_m$ are never congested. Hence on these arcs the speed is $v \geq v_\alpha(\rho_\alpha^{max}) > 0$.

By induction, assume that for $h \in \{k, k+1, \ldots, m\}$ the speed of cars over every arc $\gamma_\alpha \in \mathcal{A}_h$ satisfies a uniform lower bound:

$$v_\alpha \;\geq\; v_h^{min} \;>\; 0. \tag{6.1}$$

In particular, this implies that, if the road γ_α is in a congested state, then the flux f_α is bounded below. Namely, there exists a constant $F_k > 0$ such that

$$\rho > \rho_\alpha^{max} \qquad \Longrightarrow \qquad f_\alpha \;=\; \rho\, v_\alpha(\rho) \;\geq\; F_k \tag{6.2}$$

for all roads $\gamma_\alpha \in \mathcal{A}_k \cup \mathcal{A}_{k+1} \cup \cdots \cup \mathcal{A}_m$

Consider an arc $\gamma_\beta \in \mathcal{A}_{k-1}$. To fix the ideas, let $\gamma_\beta = \gamma_{i\ell}$ be an arc reaching the node A_ℓ. Then for each $j \in \mathcal{O}_\ell$, the outgoing arc $\gamma_{\ell j}$ lies in $\mathcal{A}_k \cup \mathcal{A}_{k+1} \cup \cdots \cup \mathcal{A}_m$. By the inductive assumption, on all these outgoing arcs the speed $v_{\ell j}$ remains uniformly positive, as in (6.1).

3. We now derive an upper bound on the length of the queue q_j at the entrance of $\gamma_{\ell j}$. Two cases must be considered.

(i) Assume that the node A_ℓ is governed by **(SBJ)**, where M is the size of the single buffer. Then, by (2.13) the flux of cars at the end of road $\gamma_{i\ell}$ is

$$\bar{f}_{i\ell}(t) \;\leq\; c_i \cdot \Big(M - \sum_{j\in\mathcal{O}_\ell} q_j(t)\Big), \quad \text{for all } i \in \mathcal{I}_\ell\,.$$

If at least one of the queues is nonempty, i.e. if $\sum_{j\in\mathcal{O}_\ell} q_j(t) > 0$, then (2.12) implies

$$\frac{d}{dt}\sum_{j\in\mathcal{O}_\ell} q_j(t) \;=\; \sum_{i\in\mathcal{I}_\ell} \bar{f}_{i\ell}(t) - \sum_{j\in\mathcal{O}_\ell} \bar{f}_{\ell j}(t) \;\leq\; \Big(\sum_{i\in\mathcal{I}_\ell} c_i\Big)\cdot\Big(M - \sum_{j\in\mathcal{O}_\ell} q_j(t)\Big) - F_k\,.$$

Indeed, if some $q_j > 0$, then the corresponding outgoing road $\gamma_{\ell j}$ is in a congested state. Hence the outgoing flux is $\bar f_{\ell j}(t) \geq F_k$. In turn, this implies

$$\frac{d}{dt}\Big(M - q_j(t)\Big) \;\geq\; F_k - \Big(\sum_{i\in\mathcal{I}_\ell} c_i\Big)\cdot\Big(M - \sum_{j\in\mathcal{O}_\ell} q_j(t)\Big).$$

Therefore, for every time t we have the uniform lower bound

$$M - \sum_{j\in\mathcal{O}_\ell} q_j(t) \;\geq\; \frac{F_k}{\sum_{i\in\mathcal{I}_\ell} c_i} \;>\; 0. \tag{6.3}$$

(ii) Next, assume that the node A_ℓ is governed by **(MBJ)**, with buffers of sizes M_j, $j\in\mathcal{O}_\ell$. Then, by (2.15),

$$\bar f_{i\ell}(t)\bar\theta_{ij}(t) \;\leq\; c_i\cdot(M_j - q_j(t)), \quad i\in\mathcal{I}_\ell, j\in\mathcal{O}_\ell.$$

Thus, by recalling (2.12), we obtain that for all $j\in\mathcal{O}_\ell$,

$$\dot q_j(t) \;=\; \sum_{i\in\mathcal{I}} \bar f_{i\ell}(t)\bar\theta_{ij}(t) - \bar f_{\ell j}(t) \;\leq\; \Big(\sum_{i\in\mathcal{I}_\ell} c_i\Big)\cdot(M_j - q_j(t)) - F_k.$$

Hence,

$$\frac{d}{dt}\Big(M_j - q_j(t)\Big) \;\geq\; F_k - \Big(\sum_{i\in\mathcal{I}_\ell} c_i\Big)\cdot(M_j - q_j(t)).$$

For every time t this yields the uniform lower bound

$$M_j - q_j(t) \;\geq\; \frac{F_k}{\sum_{i\in\mathcal{I}_\ell} c_i} \;>\; 0. \tag{6.4}$$

4. From (6.3)-(6.4), a lower bound on the speed $v_{i\ell}$ on the incoming road $\gamma_{i\ell}\in\mathcal{A}_{k-1}$ is obtained as follows. At a point (t,x) where the road is not congested we have the trivial bound

$$v_{i\ell}(\rho(t,x)) \;\geq\; v_{i,\ell}(\rho^{max}_{i,\ell}) \;>\; 0.$$

Next, consider a point $(\bar t,\bar x)$ where the road is congested, i.e. $\rho(\bar t,\bar x) > \rho^{max}_{i,\ell}$. Then the characteristic $t\mapsto x(t)$ through this point has negative speed. Let $\tau\leq t$ be the time when this characteristic reaches the end of the arc $\gamma_{i\ell}$, so that $x(\tau) = L_{i\ell}$. Since the flux is constant along characteristics, we have

$$f_{i\ell}(\rho(\bar t,\bar x)) \;=\; \rho(\bar t,\bar x)\, v_{i\ell}(\rho(\bar t,\bar x)) \;=\; \bar f_{i\ell}(\tau) \;\geq\; c_i\cdot\frac{F_k}{\sum_{i\in\mathcal{I}_\ell} c_i} \;>\; 0.$$

This provides a uniform lower bound on $v_{i\ell}(\rho(\bar t,\bar x))$. Since $\gamma_{i\ell}$ was an arbitrary arc in the set $\mathcal{A}_{k-1}$, the induction step is complete.

5. By backwards induction on $i = m, m-1,\ldots,2,1$, the above arguments show that the speed of cars remains uniformly positive at all times on all roads. In addition, if a queue is present, the flux at the entrance of each road remains strictly positive. This yields a uniform a priori bound on the time that every driver needs to reach destination. □

References

[1] N. Bellomo, M. Delitala, and V. Coscia, On the mathematical theory of vehicular traffic flow. I. Fluid dynamic and kinetic modelling. *Math. Models Methods Appl. Sci.* **12** (2002), 1801–1843.

[2] A. Bressan and K. Han, Optima and equilibria for a model of traffic flow. *SIAM J. Math. Anal.* **43** (2011), 2384–2417.

[3] A. Bressan and K. Han, Nash equilibria for a model of traffic flow with several groups of drivers, *ESAIM; Control, Optim. Calc. Var.*, **18** (2012), 969–986.

[4] A. Bressan and K. Han, Existence of optima and equilibria for traffic flow on networks, *Networks & Heter. Media* **8** (2013), 627–648.

[5] A. Bressan, C. J. Liu, W. Shen, and F. Yu, Variational analysis of Nash equilibria for a model of traffic flow, *Quarterly Appl. Math.* **70** (2012), 495–515.

[6] A. Bressan and K. Nguyen, Conservation law models for traffic flow on a network of roads, to appear.

[7] A. Bressan and F. Yu, Continuous Riemann solvers for traffic flow at a junction. *Discr. Cont. Dyn. Syst.*, submitted.

[8] A. Cascone; C. D'Apice, B. Piccoli, and L. Rarità, Optimization of traffic on road networks. *Math. Models Methods Appl. Sci.* **17** (2007), 1587–1617.

[9] Y. Chitour and B. Piccoli, Traffic circles and timing of traffic lights for cars flow. *Discrete Contin. Dyn. Syst. B*, **5** (2005), 599–630.

[10] G. M. Coclite, M. Garavello, and B. Piccoli, Traffic flow on a road network. *SIAM J. Math. Anal.* **36** (2005), 1862–1886.

[11] R. M. Colombo and A. Marson, A Hölder continuous ODE related to traffic flow. *Proc. Roy. Soc. Edinburgh* **A 133** (2003), 759–772.

[12] C. D'Apice, P. I. Kogut, and R. Manzo, Efficient controls for traffic flow on networks. *J. Dyn. Control Syst.* **16** (2010), 407–437.

[13] T. L. Friesz, *Dynamic Optimization and Differential Games*, Springer, New York, 2010.

[14] L. C. Evans, *Partial Differential Equations. Second edition.* American Mathematical Society, Providence, RI, 2010.

[15] T. Friesz and K. Han. *Dynamic Network User Equilibrium.* Springer-Verlag, to appear.

[16] T. L. Friesz, T. Kim, C. Kwon, and M. A. Rigdon, Approximate network loading and dual-time-scale dynamic user equilibrium. *Trans. Res.* **B 45** (2010), 176–207.

[17] M. Garavello and P. Goatin, The Cauchy problem at a node with buffer. *Discrete Contin. Dyn. Syst.* **32** (2012), 1915–1938.

[18] M. Garavello and B. Piccoli, *Traffic Flow on Networks. Conservation Laws Models.* AIMS Series on Applied Mathematics, Springfield, Mo., 2006.

[19] M. Garavello and B. Piccoli, Conservation laws on complex networks. *Ann. Inst. H. Poincaré* **26** (2009) 1925–1951.

[20] M. Garavello and B. Piccoli, A multibuffer model for LWR road networks, in *Advances in Dynamic Network Modeling in Complex Transportation Systems, Complex Networks and Dynamic Systems 2,*, S V. Ukkusuri and K. Ozbay eds., Springer, New York, 2013.

[21] M. Gugat, M. Herty, A. Klar, and G. Leugering. Optimal control for traffic flow networks. *J. Optim. Theory Appl.* **126** (2005), 589–616.

[22] M. Herty, C. Kirchner, and A. Klar, Instantaneous control for traffic flow. *Math. Methods Appl. Sci.* **30** (2007), 153–169.

[23] M. Herty, J. P. Lebacque, and S. Moutari, A novel model for intersections of vehicular traffic flow. *Netw. Heterog. Media* **4** (2009), 813–826.

[24] M. Herty, S. Moutari, M. Rascle, Optimization criteria for modelling intersections of vehicular traffic flow, *Netw. Heter. Media* **1**, 2006.

[25] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications.* Reprint of the 1980 original. SIAM, Philadelphia, PA, 2000.

[26] M. Lighthill and G. Whitham, On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London: Series A,* **229** (1955), 317–345.

[27] P. I. Richards, Shock waves on the highway, *Oper. Res.* **4** (1956), 42–51.